Tôhoku Math. Journ.
20 (1968), 254-256.

RIEMANNIAN MANIFOLDS WITH DENSE ORBITS UNDER LIE GROUPS OF MOTIONS

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(Received November 24, 1967)

Let M be an n-dimensional connected complete differentiable[1] Riemannian manifold admitting an intransitive effective connected Lie group H of motions[2] on M. For any $p \in M$, the differentiable submanifold $H(p)=\{h(p): h \in H\}$ is usually called the orbit of p under H or the H-orbit of p. On the other hand, the group H can be regarded as an analytic subgroup of the Lie group $I(M)$ of all motions on M and the closure (in $I(M)$) of H forms a subgroup which is a connected Lie group. Such a Lie group we denote by $\overline{H}$. The closure of an H-orbit consists of one point or has the structure of a regularly imbedded[3] connected differentiable submanifold. This follows from the fact that the closure $\overline{H(p)}$, $p \in M$, coincides with the $\overline{H}$-orbit of p, i.e., $\overline{H(p)}=\overline{H}(p)$. We shall call such a manifold $\overline{H(p)}$ the *closure manifold* of $H(p)$. For any $q \in \overline{H(p)}$, we can see $\overline{H(p)}=\overline{H(q)}$.

In the following, suppose an H-orbit of M is dense in M[4]. The object of this note is to give geometrical structures of such a manifold M.

Now, as is easily seen, the group $\overline{H}$ acts on M transitively and effectively. It is shown that the group H is normal in $\overline{H}$. Hence, for any $g \in \overline{H}$ we have $g \cdot H(p)=H(g \cdot p)$, where $p \in M$. Thus the following theorem is obtained:

THEOREM 1. 1) *Every H-orbit is dense in M,*

2) *any element of $\overline{H}$ carries every H-orbit into an H-orbit, and*

3) *M has the structure of a foliated manifold*[5] *with H-orbits as its leaves.*

Goto proved the following theorem (see [1] or [2]), which plays an important role in this note: For a connected Lie group G and its analytic

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1) The word "differentiable" means "C^∞-differentiable".
2) By a motion, we means an isometry as usual.
3) This means that the topology of the submanifold coincides with the relative one.
4) An orbit is sometimes regarded as a subset of M, as is here the case.
5) For the definition, see [4].

subgroup S which is not closed in G, there exists a 1-parameter subgroup of S whose closure (in G) is not contained in S. We shall call such a subgroup a *Goto's 1-parameter subgroup.*

Next, let G be a connected abelian Lie group and let N be a complete differentiable Riemannian manifold. Then the following facts are well-known:

1) Any invariant differentiable Riemannian metric on G reduces to a euclidean metric (see [3]).

2) If G is a group of motions on N, then the isotropy subgroup G_p at $p \in N$ leaves the orbit $G(p)$ pointwise invariant.

3) If G is a 1-parameter group of motions on N, then the closure manifold of a G-orbit consists of one point, or is homeomorphic to a straight line or a torus of dimension $\geqq 1$ and the metric induced from N becomes euclidean (see [3]).

Let $\bar{H}_p$ be the isotropy subgroup of $\bar{H}$ at $p \in M$. The set $H \cdot \bar{H}_p$ forms a Lie subgroup of $\bar{H}$ which is also the minimal subgroup containing H and $\bar{H}_p$. We denote this group by $J_{(p)}$. Then $\bar{H} \supset J_{(p)} \supset H$ and $J_{(p)} = J_{(q)}$ for any $q \in H(p)$. $J_{(p)}$ consists of all the elements of $\bar{H}$ which leaves $H(p)$ invariant. Let $J^0_{(p)}$ denote the identity component of $J_{(p)}$. Then $J^0_{(p)} \supset H$. By referring to Theorem 1, we see that $J^0_{(p)}$ leaves not only $H(p)$ but also $H(x)$ for any $x \in M$ invariant. Hence $J^0_{(p)} = J^0_{(x)}$. So, we denote $J^0_{(p)}$ by J^0 hereafter. The closure (in $I(M)$) of J^0 coincides with $\bar{H}$ and J^0 is not closed in $\bar{H}$. Therefore we have a Goto's 1-parameter subgroup $\gamma \subset J^0$ such that the closure $\bar{\gamma}$ (in $\bar{H}$) of γ is not contained in J^0. For any $x \in M$, $\gamma(x) \subset H(x)$ and the closure manifold $\overline{\gamma(x)}$ (in M) of $\gamma(x)$ is not contained in $H(x)$. For, if $\overline{\gamma(x)} \subset H(x)$ we have $\bar{\gamma}(x) = \overline{\gamma(x)} \subset H(x)$. So $\bar{\gamma} \subset J_{(x)}$. This implies $\bar{\gamma} \subset J^0$ which is a contradiction. We can further see that the closure (in $H(x)$) of $\gamma(x)$ coincides with $\gamma(x)$. Otherwise, this closure is homeomorphic to a torus of dimension > 1, and so compact under the topology of $H(x)$. Hence it would be shown that $H(x)$ contains the closure manifold $\overline{\gamma(x)}$ (in M) of $\gamma(x)$. This is a contradiction as mentioned above. Thus, $H(x)$ has a structure of product bundle with $\gamma(y)$, $y \in H(x)$, as fibers. Summing up these facts, we have

THEOREM 2. *The group $\bar{H}$ has a 1-parameter subgroup γ with the following properties: for any $x \in M$,*

1) *$\gamma(x) \subset H(x)$, but the closure manifold $\overline{\gamma(x)}$ (in M) is not included in $H(x)$,*

2) *$\overline{\gamma(x)}$ is homeomorphic to a torus of dimension > 1 and a euclidean metric is induced from M, and*

3) *$H(x)$ has a structure of product bundle with $\gamma(y)$, $y \in H(x)$, as fibers.*

Further we suppose the group H is abelian. Then so also is the group $\overline{H}$. The isotropy subgroup $\overline{H}_p$, $p \in M$, leaves M pointwise invariant and so consists of the identity only, $\overline{H}$ being effective. Hence the group $\overline{H}$ is diffeomorphic to M and the metric on M must be euclidean. Thus we may prove the following

THEOREM 3. *Suppose the group H is abelian. Then,*

1) *M is diffeomorphic to the abelian Lie group $\overline{H}$,*

2) *The metric on M reduces to euclidean one,*

3) *every H-orbit is totally geodesic,*

and further, with respect to the 1*-parameter subgroup γ in Theorem* 2,

4) *M has a structure of fiber bundle with $\overline{\gamma(x)}$, $x \in M$, as fibers, where fibers have euclidean metrics as the induced ones and are isometric to one another, and*

5) *every $\overline{\gamma(x)}$ is totally geodesic.*

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