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ON 1ST CHERN FORM AND HOLONOMY ALGEBRA OF A KÄHLER MANIFOLD

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1. Introduction. Let M be a $2n$-dimensional Kähler manifold. We consider a real coordinate neighborhood $U(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{2n})$ and natural frames $(\partial/\partial x^i)$[1] in the tangent space at each point of U. Let g_{ij} be the Kähler metric of M and $\left\{ {i \atop jk} \right\}$ be the Christoffel symbol of g_{ij}, then the curvature tensor is given by

$$R^i{}_{jkh} = \partial\left\{ {i \atop jk} \right\}\Big/\partial x^h - \partial\left\{ {i \atop jh} \right\}\Big/\partial x^k + \left\{ {i \atop ah} \right\}\left\{ {a \atop jk} \right\} - \left\{ {i \atop ak} \right\}\left\{ {a \atop jh} \right\}.$$

The Ricci tensor and the scalar curvature are

$$R_{ij} = R^a{}_{ija}, \qquad R = g^{ij}R_{ij}.$$

We denote the almost complex structure by $F^i{}_j$, then it is well known that

$$F^i{}_aF^a{}_j = -\delta^i_j\,, \ g_{ab}F^a{}_iF^b{}_j = g_{ij},\ F_{ij} \equiv g_{ia}F^a{}_j = -F_{ji},\ \nabla_kF^i{}_j = 0,$$

where ∇_k denotes the covariant differentiation with respect to $\left\{ {i \atop jk} \right\}$. Furthermore we know that

$$F^i{}_aR^a{}_{jkh} = F^a{}_jR^i{}_{akh}\,. \tag{1}$$

Now if we put

$$R^i{}_{jab}F^{ab} = K^i{}_j\,, \tag{2}$$

then

$$K^i{}_j = -2F^i{}_aR^a{}_j \qquad (R^a{}_j = g^{ab}R_{bj}) \tag{3}$$

1) Throughout this paper, the indices $a, b, c, \cdots, i, j, k, \cdots$ run from 1 to $2n(=\dim M)$ and for doubly used indices the summation convention is adopted.

and $K_{ij}=g_{ia}K^a{}_j$ is the coefficient of the *first Chern form* except a constant factor.

On the other hand, it is known ([1]) that if the metric g_{ij} is of class C^∞, the set of matrices at a point P

$$(4)\qquad \mathfrak{h}: R^i{}_{jkh},\ \nabla_{a_1}R^i{}_{jkh},\ \nabla_{a_2a_1}R^i{}_{jkh},\cdots,\ \nabla_{a_p\cdots a_1}R^i{}_{jkh},\cdots\quad (\nabla_{a_p\cdots a_1}\equiv\nabla_{a_p}\cdots\nabla_{a_1})$$

spans the infinitesimal holonomy algebra of M, where i and j designate the row and the column of the matrices. $\mathfrak{h}$ generates the infinitesimal holonomy group h' at P. Taking account of (1) and the covariant constancy of $F^i{}_j$, we see that $h'\subseteq U(n)$. If g_{ij} is analytic, h' coincides with the restricted homogeneous holonomy group h^0 and $\mathfrak{h}$ is the homogeneous holonomy algebra of M.

Contracting F^{kh} to each of (4) and taking account of the covariant constancy of F^{kh}, we see that the set at P

$$(5)\qquad \mathfrak{h}^*: K^i{}_j,\ \nabla_{a_1}K^i{}_j,\ \nabla_{a_2a_1}K^i{}_j,\cdots,\ \nabla_{a_p\cdots a_1}K^i{}_j,\cdots$$

spans a subspace of $\mathfrak{h}$.

In this paper, we study this $\mathfrak{h}^*$. It is essentially determined by the Ricci tensor and its successive covariant derivatives.

2.

THEOREM 1. *Let M be a Kähler manifold with metric of class C^∞. Then the set $\mathfrak{h}^*$ spans a Lie subalgebra of $\mathfrak{h}$ and it is an ideal in $\mathfrak{h}$.*

PROOF. It is analoguous to [1]. According to the Ricci's identity for $\nabla_{a_p\cdots a_1}K^i{}_j$, we have

$$(6)\qquad \nabla_{hka_p\cdots a_1}K^i{}_j-\nabla_{kha_p\cdots a_1}K^i{}_j=R^i{}_{akh}(\nabla_{a_p\cdots a_1}K^a{}_j)-R^a{}_{jkh}(\nabla_{a_p\cdots a_1}K^i{}_a)$$
$$-\sum_{\lambda=1}^{p}R^a{}_{a_\lambda kh}(\nabla_{a_p\cdots \underset{\lambda}{\check{a}}\cdots a_1}K^i{}_j).$$

We denote by $R^{(p)}$ and $K^{(p)}$ the subspaces spanned by $\nabla_{a_p\cdots a_1}R^i{}_{jkh}$ and $\nabla_{a_p\cdots a_1}K^i{}_j$ respectively ($R^{(0)}$ and $K^{(0)}$ are spanned by $R^i{}_{jkh}$ and $K^i{}_j$), then (6) means that

$$(6')\qquad [R^{(0)},K^{(p)}]\subset K^{(p)}+K^{(p+2)},$$

where p is an arbitrary non-negative integer. By a contraction of F^{kh} to (6′) or (6), we can easily see that

$$[K^{(0)}, K^{(p)}] \subset K^{(p)} + K^{(p+2)}.$$

Now we will proceed inductively. Assume that for an arbitrary non negative integer p and for a non-negative integer q, the following equation holds:

$$(\nabla_{b_q \cdots b_1} R^i{}_{akh})(\nabla_{a_p \cdots a_1} K^a{}_j) - (\nabla_{a_p \cdots a_1} K^i{}_a)(\nabla_{b_q \cdots b_1} R^a{}_{jkh}) \tag{7}$$
$$= \Big[\sum (\nabla_{b_q \cdots b_1} R^a{}_{a_\lambda kh})(\nabla_{a_p \cdots a \cdots a_1} K^i{}_j) + \sum \pm (\nabla_{j_{q-1} \cdots j_1} R^a{}_{i_\lambda kh})(\nabla_{i_{p+1} \cdots a \cdots i_1} K^i{}_j)$$
$$+ \cdots + \sum \pm R^a{}_{i_\lambda kh}(\nabla_{i_{p+q} \cdots a \cdots i_1} K^i{}_j) \Big]$$
$$+ \sum \pm (\nabla_{j_q \cdots j_{\mu+1} kh j_\mu \cdots j_1 a_p \cdots a_1} K^i{}_j - \nabla_{j_q \cdots j_{\mu+1} hk j_\mu \cdots j_1 a_p \cdots a_1} K^i{}_j),$$

where $(j_{q-1} \cdots j_1 i_{p+1} \cdots i_1)$ and so on in Σ's run over some permutations of $(b_q \cdots b_1 a_p \cdots a_1)$ and the summations with respect to λ runs over all or a part of $1, \cdots, p+q$ while that of μ runs over a part of $1, \cdots, q$.[2)]

The above assumption is true for p=arbitrary and $q=0$, since (6) actually holds. And (7) means that

$$[R^{(q)}, K^{(p)}] \subset K^{(p)} + K^{(p+1)} + \cdots + K^{(p+q)} + K^{(p+q+2)}. \tag{8}$$

If we contract F^{kh} to (7), we see immediately that

$$[K^{(q)}, K^{(p)}] \subset K^{(p)} + K^{(p+1)} + \cdots + K^{(p+q)} + K^{(p+q+2)}. \tag{9}$$

We operate $\nabla_{b_{q+1}}$ to (7) and apply (7) for $\nabla_{b_{q+1} a_p \cdots a_1} K^i{}_j$ instead of $\nabla_{a_p \cdots a_1} K^i{}_j$. Then we have

$$(\nabla_{b_{q+1} b_q \cdots b_1} R^i{}_{akh})(\nabla_{a_p \cdots a_1} K^a{}_j) - (\nabla_{a_p \cdots a_1} K^i{}_a)(\nabla_{b_{q+1} \cdots b_1} R^a{}_{jkh})$$
$$= \Big[\sum (\nabla_{b_{q+1} \cdots b_1} R^a{}_{a_\lambda kh})(\nabla_{a_p \cdots a \cdots a_1} K^i{}_j) + \sum \pm (\nabla_{l_q \cdots l_1} R^a{}_{m_\lambda kh})(\nabla_{m_{p+1} \cdots a \cdots m_1} K^i{}_j)$$

2) For example, if $q=1$ this equation is as follows:

$$(\nabla_{b_1} R^i{}_{akh})(\nabla_{a_p \cdots a_1} K^a{}_j) - (\nabla_{a_p \cdots a_1} K^i{}_a)(\nabla_{b_1} R^a{}_{jkh})$$
$$= \sum_{\lambda=1}^{p} (\nabla_{b_1} R^a{}_{a_\lambda kh})(\nabla_{a_p \cdots a \cdots a_1} K^i{}_j) - R^a{}_{b k_1 h}(\nabla_{a a_p \cdots a_1} K^i{}_j)$$
$$+ (\nabla_{kh b_1 a_p \cdots a_1} K^i{}_j - \nabla_{hk b_1 a_p \cdots a_1} K^i{}_j) - (\nabla_{b_1 kh a_p \cdots a_1} K^i{}_j - \nabla_{b_1 hk a_p \cdots a_1} K^i{}_j).$$

$$+\cdots+\sum \pm R^{a}{}_{m_\lambda kh}(\nabla_{m_{p+q+1}\cdots a\cdots m_1}K^{i}{}_{j})\Big]$$

$$+\sum \pm(\nabla_{l_{q+1}\cdots l_{\mu+1}khl_\mu\cdots l_1 a_p\cdots a_1} K^{i}{}_{j}-\nabla_{l_{q+1}\cdots l_{\mu+1}hkl_\mu\cdots l_1 a_p\cdots a_1} K^{i}{}_{j}),$$

where $(l_q\cdots l_1 m_{p+1}\cdots m_1)$ and so on in Σ's run over some permutations of $(b_{q+1}\cdots b_1 a_p\cdots a_1)$ and the summation with respect to λ runs over all or a part of $1,\cdots,p+q+1$ while that of μ runs over a part of $1,\cdots,q+1$.

Consequently (7) is true for an arbitrary non-negative p and for $q+1$, hence by the induction it is valid for all $p,q\geqq 0$. Therefore (8) and (9) hold true for all non-negative integers p and q. This shows that $\mathfrak{h}^*$ is an ideal of $\mathfrak{h}$. Q.E.D.

3. In this section, we suppose that the Kähler metric g_{ij} is analytic, hence $\mathfrak{h}$ is the homogeneous holonomy algebra of M.

THEOREM 2. *Let M $(n>1)$ be an irreducible Kähler manifold with analytic Kähler metric. Then the ideal $\mathfrak{h}^*$ of* (5) *is proper if and only if*

(i) *M is Kähler-Einstein* $\left(R \begin{smallmatrix}\geqq\\<\end{smallmatrix} 0\right)$

or

(ii) $R=0$ *all over M (not Einstein).*

PROOF. If M is locally symmetric: $\nabla_l R^{i}{}_{jkh}=0$, then we have $\nabla_l R^{i}{}_{j}=0$. Hence $R^{i}{}_{j}$ is invariant under the restricted homogeneous holonomy group h^0. Since M is irreducible, that is h^0 is irreducible in real number field, we have $R^{i}{}_{j}=c\delta^i_j$ (Schur's lemma), which means that M is Einstein.

Assume that M is irreducible and not locally symmetric. In this case, it is known ([2]) that the restricted homogeneous holonomy group h^0 of M is one of the following types:

$$\psi\,,\ \psi\otimes T^1\,,\ \psi\otimes SU(2)\,,$$

where ψ is a simple Lie group $(\subseteq SO(2n))$ and T^1 is the one dimensional torus group. In our case $h^0=U(n)=SU(n)\otimes T^1$ or its subgroup. The third case $\psi\otimes SU(2)$ does not occur, since this is not a subgroup of $U(n)$.[3] Hence

$$h^0=\psi \quad \text{or} \quad \psi\otimes T^1,$$

3) This group is absolutely irreducible, i.e., irreducible even in complex number field (Cartan's 1st class).

and since ψ is simple, it is $SU(n)$ or its simple subgroup. This corresponds in the holonomy algebra to

$$\mathfrak{h}=\psi' \quad \text{or} \quad \psi'+\mathfrak{t} \text{ (direct sum)},$$

where ψ' is the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$ or its simple subalgebra. In the case $\mathfrak{h}=\psi'+\mathfrak{t}$, ψ' is an ideal in $\mathfrak{h}$ and $\mathfrak{t}$ is one dimensional subalgebra generated by the matrix $(F^i{}_j)$.

a) Case $\mathfrak{h}=\psi'$. If $\mathfrak{h}^*$ is a proper ideal of the simple ψ', $\mathfrak{h}^*=\{0\}$ and hence $K^i{}_j=R^i{}_j=0$ all over M. We remark that in this case any element $(\xi^i{}_j)$ of $\mathfrak{h}$ of (4) satisfies $F^j{}_i\xi^i{}_j=0$ and hence $\mathfrak{h}\subseteq\mathfrak{su}(n)$.

b) Case $\mathfrak{h}=\psi'+\mathfrak{t}$. If $\mathfrak{h}^*$ is a proper ideal of $\mathfrak{h}$ and is contained in ψ', then $\mathfrak{h}^*=\{0\}$ or ψ'. But the case $\mathfrak{h}^*=\{0\}$ is impossible, for if so, $\mathfrak{h}$ can not contain $\mathfrak{t}$ (as in the above remark, in this case $\mathfrak{h}\subseteq\mathfrak{su}(n)$). Hence $\mathfrak{h}^*=\psi'$. In this case, any element $(\xi^i{}_j)\in\mathfrak{h}^*$ satisfies $F^j{}_i\xi^i{}_j=0$. We have

$$F^j{}_iK^i{}_j=2R=0 \qquad \text{all over } M,$$

and M is not Einstein (if otherwise, $R^i{}_j=0$ hence $\mathfrak{h}^*=\{0\}$).

If $\mathfrak{h}^*=\psi'_1+\mathfrak{t}$ $(\psi'_1\subset\psi)$, then ψ'_1 is an ideal of ψ' because ψ' and $\mathfrak{h}^*$ are both ideals. Hence $\psi'_1=\{0\}$ and $\mathfrak{h}^*=\mathfrak{t}$. Then any $(\xi^i{}_j)\in\mathfrak{h}^*$ is proportional to $F^i{}_j$. We have $K^i{}_j=cF^i{}_j (c\neq0)$ at each point of M, which means that

$$R^i{}_j=\frac{c}{2}\delta^j_i \qquad (c\neq0). \tag{10}$$

M is therefore Einstein with $R\neq0$.

Conversely, suppose that M is Einstein. If $R=0$, i.e., $R^i{}_j=0$, then $\mathfrak{h}^*=\{0\}$. This is trivially a proper ideal of $\mathfrak{h}$. If $R\neq0$ then (10) and hence $K^i{}_j=cF^i{}_j$ (c=const.) holds. Therefore $\mathfrak{h}^*=\mathfrak{t}$. In this case if furthermore $\mathfrak{h}=\mathfrak{t}$, we have $R^i{}_{jkh}=F^i{}_j\varphi_{kh}$. And by a contraction with $F^j{}_i$ we see that $\varphi_{kh}=(1/2n)\,K_{kh}$, i.e., $R^i{}_{jkh}=(1/2n)\,F^i{}_jK_{kh}=(c/2n)\,F^i{}_jF_{kh}$. Contracting g^{jk} we have $R^i{}_h=-(c/2n)\,\delta^i_h$ which yields $c=0$ by virtue of (10). This is a contradiction,[4] and hence $\mathfrak{h}^*$ is a proper ideal of $\mathfrak{h}$.

Lastly suppose that $R=0$ all over M and M is not Einstein ($R_{kh}\neq0$, i.e., $F^j{}_iR^i{}_{jkh}\neq0$). Then $\mathfrak{h}^*\subseteq\mathfrak{su}(n)$ and $\mathfrak{h}\nsubseteq\mathfrak{su}(n)$, hence $\mathfrak{h}^*$ is a proper ideal of $\mathfrak{h}$. In this case, $\mathfrak{h}=\mathfrak{h}^*+\mathfrak{t}$ (since M is not locally symmetric, see the case b)).

Q.E.D.

COROLLARY. *Let M $(n>1)$ be an irreducible Kähler manifold with analytic Kähler metric. Then the holonomy algebra of M is spanned by $\mathfrak{h}^*$ of* (5), *except in the following cases*:

4) This also follows from the fact that M is reducible because $\mathfrak{h}$ or h^0 is 1-dimensional and hence solvable.

(i) *M is Kähler-Einstein* $\left(R \gtreqless 0\right)$

or

(ii) $R=0$ *all over* M (*not Einstein*).

From the proof of Theorem 2 and from the corollary, under the same assumption for M as in the Theorem 2, we can sum up as follows:

$$\begin{cases} M \text{ is Kähler-Einstein } (R=0) & \Leftarrow \quad \mathfrak{h}^*=\{0\} \\ M \text{ is Kähler-Einstein } (R\neq 0) & \Leftarrow \quad \mathfrak{h}^*=\mathfrak{t} \\ R=0 \text{ all over } M \text{ (not Einstein)} & \Longleftrightarrow \quad \mathfrak{h}=\mathfrak{h}^*+\mathfrak{t} \text{ (direct sum; } \mathfrak{h}^*\neq\{0\}) \\ \text{all the other cases} & \Longleftrightarrow \quad \mathfrak{h}=\mathfrak{h}^*. \end{cases}$$

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