Tôhoku Math. Journ.
20 (1968), 244-247.

AN ELEMENTARY PROOF OF A THEOREM OF HENRY HELSON

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(Received January 12, 1968)

Let G be a locally compact abelian group with dual group X, and L the group algebra of G, consisting of the functions summable on G for Haar measure dx. A denotes, then, the function algebra on X of the Fourier transforms of all functions in L, the norm of a function f in A being

$$\|f\| = \int_G |F|\, dx,$$

where F is the function in L whose Fourier transform is f. The zero-set of an ideal I in A is defined to be the set

$$Z(I) = \bigcap_{f \in I} \{p \in X : f(p) = 0\} .$$

Suppose that A contains two closed ideals I and J such that

$$I \subset J,\ I \neq J \text{ and } Z(I) = Z(J). \tag{1}$$

A theorem of Malliavin [3, p. 172] states that: if X is non-discrete, then A contains two closed ideals as in (1). Henry Helson showed in [2] that, if (1) is the case, there exists a closed ideal K in A such that

$$I \subset K \subset J \quad \text{and} \quad I \neq K \neq J. \tag{2}$$

It seems that his methods chiefly depend upon the use of a theorem of Godement on unitary representations of abelian groups [1]. In this paper we give an elementary proof of his theorem, without using Godement's theorem.

THEOREM. *Let I and J be two closed ideals in A as in* (1), *and put $E = Z(I) = Z(J)$. Then we have*:

(i) *A contains a closed ideal K as in* (2);

(ii) *If, in addition, each point of E has a countable open basis in the relative topology of E, we can find a strictly decreasing sequence of closed ideals K_n of A such that $K_n \subset J$ for all $n = 1, 2, 3, \cdots$, and such that*

$$\bigcap_{n=1}^{\infty} K_n = I.$$

PROOF. By the assumptions, there exists an f of J which is not in I. Let Q be the set of all p in X at which f does not belong to I locally (for the definition, see [3; p. 133]). We claim that Q is not empty. In fact, we can find a sequence $\{v_n\}$ in A such that each v_n has compact support and such that

$$\lim_{n\to\infty} \|f - fv_n\| = 0;$$

if Q were empty, each fv_n would belong to I locally at every point of X including the point at infinity, and hence would be in I (see [3, p. 134]); since I is closed in A, $f \in I$, which contradicts the choice of f. Since Q is not empty, it follows that Q is a perfect subset of E (see [3; p. 160]).

To prove (i), take any two points p and q in Q, and let U and V be disjoint open sets in X which contain, respectively, p and q. We can find a function k in A such that $k=1$ on some neighborhood of p and $k=0$ outside U. Define K as the closed ideal in A generated by fk and the functions in I. Since fk is not in I by the choice of k, it follows that I is properly contained in K. Since $f \in J$, and since J is a closed ideal which contains I, it also follows that $K \subset J$. To show that $K \neq J$, take any function g in K and any point r in X which is not in the closure of U. We can find a sequence $\{h_n\}$ in A and a sequence $\{i_n\}$ in I such that

$$\lim_{n\to\infty} \|fkh_n + i_n - g\| = 0.$$

Choose a function j in A so that $j=0$ on U and $j=1$ on some neighborhood of r. Since $kj=0$, it follows that

$$\lim_{n\to\infty} \|i_n j - gj\| = 0.$$

Since I is a closed ideal in A, $gj \in I$. This means that g belongs to I locally at r. Since g is any function in K and since r is any point in the complement of the closure of U, we conclude that every function of K belongs to I locally at every point in the complement of the clousure of U. On the other hand, f does not belong to I locally at q; since q is not in the closure of U, it follows that f is not in K, and we get the desired conclusion that $K \neq J$. This completes the proof of (i).

To prove (ii), let f, Q and p be as above, and take a compact neighborhood U_0 of p. Since we are now assuming that each point of E has a countable open basis in the relative topology of E and since Q is a perfect subset of E, we can find a sequence of compact neighborhoods U_n of p, subject to these conditions:

(1′) For each $n=1, 2, 3, \cdots$, U_n lies in the interior of U_{n-1}: U_{n-1} contains an open set V_n, which is disjoint from U_n and contains at least one point of Q.

$$(2') \qquad \left(\bigcap_{n=1}^{\infty} U_n\right) \cap Q = \{p\}.$$

For $n=1, 2, 3, \cdots$, take a function k_n in A so that $k_n=1$ on U_{2n} and $k_n=0$ outside U_{2n-1}, and define K_n to be the closed ideal in A associated to fk_n as before. Fix any natural number n; repeating the preceding arguments, we see that $K_n \subset J$, $K_{n+1} \subset K_n$ and $K_{n+1} \neq K_n$. To prove that $\bigcap_{n=1}^{\infty} K_n = I$, it suffices to show that every function in $\bigcap_{n=1}^{\infty} K_n$ belongs to I locally at any point of X except for p. But we can easily show this from the assumption (2′) on $\{U_n\}$. This completes the proof of (ii).

The Theorem is now established.

REMARK. The Theorem implies, in particular, that the quotient space J/I is infinite dimensional as a linear space, provided that I and J are as in (1). But it tells us nothing about the problem whether J contains two functions f and g such that $fg \notin I$. We do not know the answer. This question is interesting in the following sence: if the answer is affirmative, it follows that finite unions of S-sets are S-sets (for the definition, see [3; p. 158]). Indeed, let E_1 and E_2 be two S-sets in X, and let E be the union of E_1 and E_2. If it happens that E is not an S-set, we can find two closed ideals I and J of A as in the Theorem. Suppose that J contains two functions f and g such that $fg \notin I$. Since f vanishes on E_1, and since E_1 is an S-set, we can find a sequence $\{f_n\}$ in A such that $\|f_n - f\| \to 0$ as $n \to \infty$ and each f_n vanishes on an open set containing E_1 (depending on n). Similarly, we can find a sequence $\{g_n\}$ in A such that $\|g_n - g\| \to 0$ and each g_n vanishes on an open set containing E_2. Since each $f_n g_n$ vanishes on an open set containing E, $f_n g_n \in I$; since I is closed in A, and since $\|f_n g_n - fg\| \to 0$, it follows that $fg \in I$, a contradiction.

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