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SOME REMARKS OF SATURATION PROBLEM IN THE LOCAL APPROXIMATION

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1. Introduction. Let $f(x)$ be integrable in $(-\pi, \pi)$ and periodic with period 2π, and let its Fourier series be

$$(1)\qquad S[f]=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) = \sum_{n=0}^{\infty} A_n(x).$$

If the positivity of $f(x)$ implies the positivity of a linear operator $L_n(f, x)$, the operator is called a linear positive operator.

Let $\rho_k^{(n)}(k=0, 1, 2, \cdots, \rho_0^{(n)}=1)$ be the "summating" function with the properties

$$(2)\qquad \lim_{n\to\infty} \rho_1^{(n)}=1.$$

Let us consider a family of linear positive operators

$$(3)\qquad L_n(f, x)=\sum_{k=0}^{\infty}\rho_k^{(n)}A_k(x),$$

and suppose that the linear positive operators (3) can be represented in the following form:

$$L_n(f, x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)U_n(t)dt,$$

where

$$(4)\qquad U_n(t)=\frac{1}{2}+\sum_{k=1}^{\infty}\rho_k^{(n)}\cos kt \geqq 0.$$

The purpose of the present paper lies in determining the class of local saturation by some linear positive operators. Throughout the paper the norm should be taken with respect to the variable x and the interval $[c, d]$ is an arbitrarily fixed subinterval of a given interval $[a, b]$ which is situated in $(-\pi, \pi)$. For the sake of simplicity, we consider only uniform approximation and the norm means uniform norm. But another norm may be treated by the analogous method. Also, let us write

$$\|L_n(f, x)-f(x)\|_{(a,b)} \equiv \max_{x\in[a,b]} |L_n(f, x)-f(x)|$$

and

$$\text{Lip}(1; a, b) \equiv \{f(x) \mid \sup_{|h|\leqq\delta} \|f(x+h)-f(x)\|_{(a,b)} = O(\delta)\}.$$

In the preceding paper (Y.Suzuki [13]), the following Theorem A was proved to determine the class of local saturation for some positive methods of summation in the theory of Fourier series.

THEOREM A. *Suppose that*

$$\lim_{n\to\infty} \frac{1-\rho_k^{(n)}}{1-\rho_1^{(n)}} = k^2 \qquad (k=1, 2, \cdots\cdots). \tag{5}$$

Then for any function $f(x) \in C[a,b] \cap L(-\pi, \pi)$, *we get the following* (1°), (2°) *and* (3°).

(1°) *If*

$$\|L_n(f, x) - f(x)\|_{(a,b)} = o(1 - \rho_1^{(n)}),$$

then $f(x)$ *is a linear function in* $[c, d]$.

(2°) *If*

$$\|L_n(f, x) - f(x)\|_{(a,b)} = O(1 - \rho_1^{(n)}),$$

then $f'(x)$ *belongs to the class* Lip(1; c, d).

(3°) *If* $f'(x)$ *belongs to the class* Lip(1; a, b), *then*

$$\|L_n(f, x) - f(x)\|_{(c,d)} = O(1 - \rho_1^{(n)}).$$

Now, we shall give a direct proof for the following fact that was pointed out by P.P.Korovkin.

PROPOSITION 1. (P. P. Korovkin [4]).
$\lim_{n\to\infty}\frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}}=4$ *is equivalent to* $\lim_{n\to\infty}\frac{1-\rho_k^{(n)}}{1-\rho_1^{(n)}}=k^2$ $(k=1,2,\cdots\cdot)$.

PROOF. We have only to prove the necessity. Let us set

$$\psi_k(x)=1-\cos kx \quad (k=0,1,2,\cdots),$$

then we get

$$L_n(\psi_k,0)=\frac{1}{\pi}\int_{-\pi}^{\pi}(1-\cos kt)U_n(t)dt=1-\rho_k^{(n)}.$$

Since

$$\lim_{t\to 0}\frac{\psi_k(t)}{\psi_1(t)}=\lim_{t\to 0}\frac{1-\cos kt}{1-\cos t}=k^2,$$

for any $\varepsilon>0$, there is a positive constant $\delta(\varepsilon)$ such that

$$|\psi_k(t)-k^2\psi_1(t)|<\frac{\varepsilon}{2}\psi_1(t), \text{ for all } |t|<\delta.$$

On the other hand, we have

$$L_n(\psi_k,0)-k^2L_n(\psi_1,0)=\frac{1}{\pi}\int_{-\pi}^{\pi}\{\psi_k(t)-k^2\psi_1(t)\}U_n(t)\,dt$$

$$=\frac{1}{\pi}\int_{|t|<\delta}\{\psi_k(t)-k^2\psi_1(t)\}U_n(t)dt+\frac{1}{\pi}\int_{\delta\le|t|\le\pi}\{\psi_k(t)-k^2\psi_1(t)\}U_n(t)\,dt$$

$=I_1+I_2$, say.

Then

$$|I_1|\leqq\frac{1}{\pi}\int_{|t|<\delta}|\psi_k(t)-k^2\psi_1(t)|U_n(t)\,dt \tag{6}$$

$$<\frac{\varepsilon}{2}\int_{|t|<\delta}\psi_1(t)U_n(t)dt<\frac{\varepsilon}{2}\int_{-\pi}^{\pi}\psi_1(t)U_n(t)\,dt$$

$$=\frac{\varepsilon}{2}L_n(\psi_1,0),$$

(7)
$$
\begin{aligned}
|I_2| &\leqq \frac{1}{\pi}\int_{\delta\leqq|t|\leqq\pi} |\psi_k(t)-k^2\psi_1(t)|\,U_n(t)dt \\
&\leqq \frac{2(k^2+1)}{\pi}\int_{\delta\leqq|t|\leqq\pi} U_n(t)dt \\
&\leqq \frac{2(k^2+1)}{\pi(1-\cos\delta)^2}\int_{\delta\leqq|t|\leqq\pi} (1-\cos t)^2 U_n(t)dt \\
&\leqq \frac{2(k^2+1)}{\pi(1-\cos\delta)^2}\int_{-\pi}^{\pi} (1-\cos t)^2 U_n(t)dt \\
&= \frac{(k^2+1)(1-\rho_1^{(n)})}{(1-\cos\delta)^2}\left(4-\frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}}\right).
\end{aligned}
$$

Since

$$\lim_{n\to\infty}\frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}}=4,$$

there exists a positive integer $N(\varepsilon)$ for any $\varepsilon>0$ such that

$$\left|4-\frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}}\right|<\frac{\varepsilon(1-\cos\delta)^2}{2(k^2+1)}.$$

Hence, from (7), we have

(8)
$$
\begin{aligned}
|I_2| &< \frac{\varepsilon}{2}(1-\rho_1^{(n)}) \\
&= \frac{\varepsilon}{2}L_n(\psi_1, 0).
\end{aligned}
$$

Hence, by (6) and (8) we obtain

$$|L_n(\psi_k, 0)-k^2L_n(\psi_1, 0)|<\varepsilon L_n(\psi_1, 0), \text{ for all } n>N(\varepsilon).$$

That is,

$$\lim_{n\to\infty}\frac{1-\rho_k^{(n)}}{1-\rho_1^{(n)}}=\lim_{n\to\infty}\frac{L_n(\psi_k,0)}{L_n(\psi_1,0)}=k^2. \qquad \text{q.e.d.}$$

PROPOSITION 2. *If the condition* (5) *of Theorem A is satisfied only* *in the case of $k=2$, the conclusion of Theorem A is true.*

PROOF. Combining Theorem A with Proposition 1, it is trivial.

2. Korovkin's operator. We know that in the general case $\rho_1^{(n)} \leqq \cos \pi/(n+2)$. P.P.Korovkin gave an example of linear positive operator which has the extremal property: $\rho_1^{(n)} = \cos \frac{\pi}{n+2}$. Korovkin's operator is defined by

$$K_n(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) U_n(t) dt,$$

with the kernel:

$$U_n(t) = \frac{\left| \sum_{k=1}^{n+1} \sin \frac{k\pi}{n+2} e^{i(k-1)t} \right|^2}{2 \sum_{k=1}^{n+1} \sin^2 \frac{k\pi}{n+2}} = \frac{1}{2} + \sum_{k=1}^{n} \rho_k^{(n)} \cos kt,$$

where

$$\rho_1^{(n)} = \cos \frac{\pi}{n+2}, \quad \rho_2^{(n)} = \frac{1}{n+2} \left[1 + (n+1) \cos \frac{2\pi}{n+2} \right].$$

THEOREM 1. (R.G.Mamedov [7]). *Let $f(x)$ belong to the class $C[a, b] \cap L[-\pi, \pi]$.* (1°) *If*

$$\|K_n(f, x) - f(x)\|_{(a,b)} = o\left(\frac{1}{n^2}\right),$$

then $f(x)$ is a linear function in $[c, d]$.

(2°) *If*

$$\|K_n(f, x) - f(x)\|_{(a,b)} = O\left(\frac{1}{n^2}\right),$$

then $f'(x)$ belongs to the class Lip$(1; c, d)$.

(3°) *If $f'(x)$ belongs to the class* Lip$(1; a, b)$, *then*

$$\|K_n(f, x) - f(x)\|_{(c,d)} = O\left(\frac{1}{n^2}\right).$$

PROOF. Since

$$1-\rho_1^{(n)}=1-\cos\frac{\pi}{n+2}\sim\frac{1}{n^2}$$

we have only to verify the equality (2). In fact, by easy calculation, we get

$$\lim_{n\to\infty}\frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}}=\lim_{n\to\infty}\frac{n+1}{n+2}\cdot\frac{1-\cos\dfrac{2\pi}{n+2}}{1-\cos\dfrac{\pi}{n+2}}$$

$$=\lim_{n\to\infty}\frac{n+1}{n+2}\cdot\frac{2\left(1-\cos^2\dfrac{\pi}{n+2}\right)}{1-\cos\dfrac{\pi}{n+2}}$$

$$=4.$$

Hence, from Proposition 2, the proof of Theorem 1 is completed.

We may write this result by the notation:

$$\text{L.Sat.}\,[K_n]=[\{f\,|\,f'\in\text{Lip}(1\,;a,b)\},n^{-2},f=\text{linear}].$$

REMARK 1. M.G.Mamedov [7] states that in Theorem 1

$$\|K_n(f,x)-f(x)\|_{(a,b)}=o(1-\rho_1^{(n)})$$

implies $f(x)$=constant, but a careful inspection of his method will suggest that $f(x)$ is linear.

3. The generalized Korovkin's operator. P.P.Korovkin [5] introduced a linear positive operator of general character which is defined in the following.

Let $\varphi(x)$ be continuous on $[0,1]$ and have the properties:

(i) $\varphi(x)$ is differentiable on $(0,1)$,

(ii) $\varphi'(x)$ is integrable in the sense of Riemann,

(iii) $\int_0^1\varphi^2(x)dx\neq 0$.

Now, the generalized Korovkin's operator is defined by

$$(G)K_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)U_n(t)dt,$$

with the kernel:

$$U_n(t) = \frac{1}{2A_n}\left|\sum_{k=0}^{n}\varphi\left(\frac{k}{n}\right)e^{ikt}\right|^2 = \frac{1}{2} + \sum_{k=1}^{n}\rho_k^{(n)}\cos kt,$$

where

$$A_n = \sum_{s=0}^{n}\varphi^2\left(\frac{s}{n}\right) \neq 0.$$

Then, we have easily

$$\rho_k^{(n)} = \frac{A_{k,n}}{A_n}, \quad A_{k,n} = \sum_{s=0}^{n-k}\varphi\left(\frac{s}{n}\right)\varphi\left(\frac{s+k}{n}\right).$$

3.1. The case $\varphi(0)=\varphi(1)=0$.

THEOREM 2. *If $\varphi(0)=\varphi(1)=0$, then*

$$\text{L.Sat.}[(G)K_n] = [\{f \mid f' \in \text{Lip}(1; a, b)\}, n^{-2}, f=\textit{linear}].$$

PROOF. We get

$$(9) \quad A_n - A_{k,n} = \sum_{s=0}^{n}\varphi^2\left(\frac{s}{n}\right) - \sum_{s=0}^{n-k}\varphi\left(\frac{s}{n}\right)\varphi\left(\frac{s+k}{n}\right)$$
$$= \frac{1}{2}\sum_{s=0}^{n-k}\left\{\varphi\left(\frac{s}{n}\right) - \varphi\left(\frac{s+k}{n}\right)\right\}^2 + \frac{1}{2}\sum_{s=0}^{k-1}\left\{\varphi^2\left(\frac{s}{n}\right) + \varphi^2\left(\frac{n-s}{n}\right)\right\}.$$

Let M be an absolute constant. From the property (ii), we have

$$(10) \quad \begin{cases} \left|\varphi\left(\frac{s}{n}\right)\right| = \left|\varphi\left(\frac{s}{n}\right) - \varphi(o)\right| \leqq \frac{Ms}{n}, \\ \left|\varphi\left(1-\frac{s}{n}\right)\right| = \varphi\left(1-\frac{s}{n}\right) - \varphi(1)\right| \leqq \frac{Ms}{n}. \end{cases}$$

Hence, by (9) and (10), we obtain

$$n(A_n - A_{k,n}) = \frac{1}{2}\sum_{s=0}^{n-k}\left\{\frac{\varphi\left(\frac{s}{n}\right)-\varphi\left(\frac{s+k}{n}\right)}{\frac{1}{n}}\right\}^2 \frac{1}{n} + O\left(\frac{1}{n}\right)$$

$$= \frac{k^2}{2}\sum_{s=0}^{n-k}\varphi'^2(\xi_{s,n})\,\frac{1}{n} + O\left(\frac{1}{n}\right)$$

$$\rightarrow \frac{k^2}{2}\int_0^1 \varphi'^2(x)dx \qquad (n\rightarrow\infty),$$

where the point $\xi_{s,n}$ is between s/n and $(s+k)/n$. Observing this fact, we have

$$(11) \qquad n^2(1-\rho_k^{(n)}) = \frac{n(A_n - A_{k,n})}{\frac{1}{n}A_n}$$

$$\rightarrow \frac{k^2}{2}\frac{\int_0^1 \varphi'^2(x)dx}{\int_0^1 \varphi^2(x)dx} \qquad (n\rightarrow\infty).$$

Since, by (11)

$$\lim_{n\rightarrow\infty}\frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}} = 4, \qquad 1-\rho_1^{(n)} \sim n^{-2},$$

using Proposition 2, we obtain the required result.

REMARK 2. If $\varphi(x)=\sin\pi x$, $(G)K_n(f, x)$ is Korovkin's operator defined in §2.

3.2. The case $\varphi^2(0)+\varphi^2(1)>0$.

THEOREM 3. *If* $\varphi^2(0)+\varphi^2(1)>0$, *then*

(1°) $\|(G)K_n(f,x)-f(x)\|_{(a,b)} = o\left(\frac{1}{n}\right) \rightarrow \tilde{f} = constant$ *on* $[c,d]$,

(2°) $\|(G)K_n(f,x)-f(x)\|_{(a,b)} = O\left(\frac{1}{n}\right) \rightarrow \tilde{f} \in \mathrm{Lip}(1; c, d)$.

For the proof of Theorem 3, we need the following Theorems.

THEOREM B. (G.Sunouchi [11]). *A necessary and sufficient condition for $\tilde{f}'(x)$ to exist and belong to the class L^∞ over $[a,b]$ is the uniform boundedness of $\sigma_m[x,\tilde{S}']$ over $[a,b]$, where $\sigma_m[x,\tilde{S}']$ means the* $(C,1)$*-means of the first derived conjugate series of* (1).

THEOREM C. (P.P.Korovkin [5]). *If $\varphi^2(0)+\varphi^2(1)>0$, then*

$$\lim_{n\to\infty}\frac{1-\rho_k^{(n)}}{1-\rho_1^{(n)}}=k,\quad 1-\rho_1^{(n)}\sim\frac{1}{n}.$$

PROOF. Obviously, from (9), we have

$$\begin{aligned}A_n-A_{k,n}&=\frac{1}{2}\sum_{s=0}^{n-k}\left\{\varphi\left(\frac{s}{n}\right)-\varphi\left(\frac{s+k}{n}\right)\right\}^2+\frac{1}{2}\sum_{s=0}^{k-1}\left\{\varphi^2\left(\frac{s}{n}\right)+\varphi^2\left(\frac{n-s}{n}\right)\right\}\\&=I_1+I_2,\text{ say.}\end{aligned}$$

Then

$$I_1\leqq\frac{1}{2}\,\omega\left(\frac{k}{n}\right)\sum_{s=0}^{n-k}\left|\varphi\left(\frac{s}{n}\right)-\varphi\left(\frac{s+k}{n}\right)\right|\leqq\frac{k}{2}\,\omega\left(\frac{k}{n}\right)V,$$

where $\omega(\delta)$ is the modulus of continuity of $\varphi(x)$ and V is the total variation of $\varphi(x)$ on $[0,1]$.

$$\begin{aligned}I_2\leqq&\frac{1}{2}\sum_{k=0}^{k-1}\{\varphi^2(0)+\varphi^2(1)\}+\frac{1}{2}\sum_{s=0}^{k-1}\left|\varphi\left(\frac{s}{n}\right)-\varphi(0)\right|\left|\varphi\left(\frac{s}{n}\right)+\varphi(0)\right|\\&+\frac{1}{2}\sum_{s=0}^{k-1}\left|\varphi\left(1-\frac{s}{n}\right)-\varphi(1)\right|\left|\varphi\left(1-\frac{s}{n}\right)+\varphi(1)\right|\\=&\frac{k}{2}\{\varphi^2(0)+\varphi^2(1)\}+kO(1)\omega\left(\frac{k}{n}\right).\end{aligned}$$

Thus we have

$$A_n-A_{k,n}=\frac{k}{2}\left\{\varphi^2(0)+\varphi^2(1)+O(1)\omega\left(\frac{k}{n}\right)\right\}.$$

Hence

$$1-\rho_k^{(n)}=1-\frac{A_{k,n}}{A_n}=\frac{k}{2A_n}\left\{\varphi^2(0)+\varphi^2(1)+O(1)\omega\left(\frac{k}{n}\right)\right\}.$$

Consequently

$$n(1-\rho_k^{(n)})=\frac{k}{2A_n\dfrac{1}{n}}\left\{\varphi^2(0)+\varphi^2(1)+O(1)\omega\left(\frac{k}{n}\right)\right\}$$

$$\to\frac{k\{\varphi^2(0)+\varphi^2(1)\}}{2\int_0^1\varphi^2(x)dx}\qquad(n\to\infty).$$

From (12), we infer that

$$\lim_{n\to\infty}\frac{1-\rho_k^{(n)}}{1-\rho_1^{(n)}}=k,\ 1-\rho_1^{(n)}\sim\frac{1}{n}\qquad\text{q.e.d.}$$

PROOF OF THEOREM 3. The proofs of the proposition (1^o) and (2^o) in the theorem are almost the same. So we shall only give the proof of the proposition that

$$\|(G)K_n(f,x)-f(x)\|_{(a,b)}=O\left(\frac{1}{n}\right)\to\tilde{f}(x)\in\mathrm{Lip}(1;c,d).$$

Since

$$(G)K_n(f,x)-f(x)=O\left(\frac{1}{n}\right),\ \text{uniformly over } [a,b],$$

we have

$$\sigma_m[x,n\{f(x)-(G)K_n(f,x)\}]=O(1),$$

for every m and uniformly in x in any fixed subinterval of $[a,b]$ because

$$f(x)-(G)K_n(f,x)\sim\sum_{k=1}^{n}(1-\rho_k^{(n)})A_k(x)+\sum_{k=n+1}^{\infty}A_k(x),$$

(see, Zygmund [15, p.367, Th.9.20]). Letting $n\to\infty$, we have by Theorem C

$$\sigma_m[x, \tilde{S}'(f)] = O(1).$$

Hence we have $\tilde{f}' \in L^\infty$ over $[c, d]$ from Theorem B. q.e.d.

If $\varphi(x) \equiv C(\neq 0)$, the operator $(G)K_n(f)$ is Fejér's operator. Hence,

THEOREM D. (G.Sunouchi [11],[12]). *If $\varphi(x)$ is a constant $C(\neq 0)$, then*

$$\text{L. Sat.}[(G)K_n] = [\{f \mid \tilde{f} \in \text{Lip}(1; a, b)\}, n^{-1}, \tilde{f} = constant].$$

That is, we know the local saturation theorem of $(G)K_n(f)$ in the case of $\varphi(x)$=constant. In this connection, the question arises as to whether there exists another function $\varphi(x)$ having such a property. It seems to the authors that this problem remains open.

4. **Matsuoka's operator**. Y.Matsuoka [8] introduced a linear positive operator which is a further generalization of the generalized Jackson's operator. Let p and q be positive integers such $p \geqq q \geqq 2$ and let

$$M_n^{(p,q)}(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)U_n(t)dt$$

with the kernel:

$$U_n(t) = \frac{1}{A_{p,q}(n)} \frac{\sin^{2p}\frac{nt}{2}}{\sin^{2q}\frac{t}{2}} = \frac{1}{2} + \sum_{k=1}^{pn-q} \rho_k^{(n)} \cos kt,$$

where

$$A_{p,q}(n) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin^{2p}\frac{nt}{2}}{\sin^{2q}\frac{t}{2}} dt$$

$$= \frac{2^{-2p+2q+1}}{(2q-1)!} \sum_{v=1}^{p} (-1)^{q+v} \binom{2p}{p-v} (vn+q-1)(vn+q-2)\cdots$$

$$\cdots\cdots(vn-q+2)(vn-q+1).$$

For the local saturation of this Matsuoka's operator, we show the following theorem.

THEOREM 4.

$$\text{L. Sat.}[M_n^{(p,q)}]=[\{f|f'\in \text{Lip}(1;a,b)\},n^{-2},f(x)=linear].$$

PROOF. From the results of Y.Matsuoka [8], we have

$$\lim_{n\to\infty}\frac{1-\rho_2^{(n)}}{1-\rho_1^{(n)}}=4$$

and

$$1-\rho_1^{(n)}=-(2q-1)(q-1)\frac{S(p,2q-3)}{S(p,2q-1)}\cdot\frac{1}{n^2}+O\left(\frac{1}{n^4}\right),$$

where

$$S(p,\lambda)=\sum_{v=0}^{p}(-1)^v\binom{2p}{p-v}v^{\lambda}.$$ q.e.d.

REMARK 3. If $p=q$, this is generalized Jackson's operator. (see [13]).

5. Generalized Gauss-Weierstrass singular integral. This is

$$T_\xi(f,x)=\frac{p}{2\Gamma\left(\frac{1}{p}\right)}\cdot\frac{1}{\xi}\int_{-\pi}^{\pi}f(x+t)e^{-|t/\xi|^p}\,dt\quad(p>0)$$

THEOREM 5.
L.Sat $[T_\xi]=[\{f|f'\in \text{Lip}(1;a,b)\},\ \xi^2, f(x)=linear]$.

PROOF. For each $\varepsilon>0$, there exist $\xi_0(\varepsilon)$ and $N_0(\varepsilon)$ such that

(13) $$\frac{1}{\xi^2}\int_{\pi/\xi}^{\infty}(1-\cos k\xi u)e^{-u^p}du<\varepsilon,\ \text{for}\ 0<\xi<\xi_0(\varepsilon)$$

and

(14) $$\frac{1}{\xi^2}\int_{N}^{\infty}(1-\cos k\xi u)e^{-u^p}du<\varepsilon,\ \text{for}\ N>N_0(\varepsilon).$$

We have for large $N>N_0(\varepsilon)$

$$(15)\qquad \frac{1}{\xi^2}(1-\rho_k{}^{(\xi)})=\frac{p}{\xi^2\Gamma\left(\frac{1}{p}\right)}\int_0^{\pi/\xi}(1-\cos k\xi u)e^{-u^p}du$$

$$=\frac{p}{\xi^2\Gamma\left(\frac{1}{p}\right)}\left(\int_0^N+\int_N^\infty-\int_{\pi/\xi}^\infty\right)(1-\cos k\xi u)e^{-u^p}du$$

$$=o(\xi)+\frac{pk^2}{2\Gamma\left(\frac{1}{p}\right)}\int_0^N u^2e^{-u^p}du+\frac{p}{\xi^2\Gamma\left(\frac{1}{p}\right)}\left(\int_N^\infty-\int_{\pi/\xi}^\infty\right)(1-\cos k\xi u)e^{-u^p}du.$$

Then we get

$$\left|\frac{1-\rho_k{}^{(\xi)}}{\xi^2}-\frac{k^2\Gamma\left(\frac{3}{p}\right)}{2\Gamma\left(\frac{1}{p}\right)}\right|=\left|\frac{1-\rho_k{}^{(\xi)}}{\xi^2}-\frac{pk^2}{2\Gamma\left(\frac{1}{p}\right)}\int_0^\infty u^2e^{-u^p}du\right|$$

$$\leqq\frac{pk^2}{2\Gamma\left(\frac{1}{p}\right)}\int_N^\infty u^2e^{-u^p}du+o(\xi)$$

$$+\frac{p}{\xi^2\Gamma\left(\frac{1}{p}\right)}\left(\int_N^\infty+\int_{\pi/\xi}^\infty\right)(1-\cos k\xi u)e^{-u^p}du$$

and from (13), (14) and (15)

$$\left|\frac{1-\rho_k{}^{(\xi)}}{\xi^2}-\frac{k^2\Gamma\left(\frac{3}{p}\right)}{2\Gamma\left(\frac{1}{p}\right)}\right|<4\varepsilon,\ \text{for}\ 0<\xi<\xi_0(\varepsilon).$$

Consequently we obtain

$$1-\rho_k{}^{(\xi)}=\frac{k^2\Gamma\left(\frac{3}{p}\right)}{2\Gamma\left(\frac{1}{p}\right)}\xi^2+o(\xi^2).\qquad\text{q.e.d.}$$

REMARK 4. If $p=1$, this is Picard's singular integral and if $p=2$, this is Gauss-Weierstrass singular integral.

REMARK 5. The whole interval case has been investigated by G.Sunouchi [10].

6. Ostrowski's operator. This is

$$A_\xi(f, x)=\frac{1}{2\xi^2\Gamma(\xi)}\int_{-\pi}^{\pi} f(x+t)e^{-\left(\frac{|t|}{\xi}\right)^{\frac{1}{\xi}}}\,dt, \text{ (see, [2] and [9]).}$$

THEOREM 6.

$$\text{L.Sat.}[A_\xi]=[\{f \mid f' \in \text{Lip}(1; a, b)\}, \xi^2, f(x)=linear].$$

For the proof of Theorem 6, we need a lemma.

LEMMA. *The following identity holds:*

$$\lim_{\xi\to+0}\frac{\Gamma(\xi)}{\Gamma(3\xi)}=3.$$

PROOF. We know the relations:

$$(16)\quad \begin{cases} \prod_{j=0}^{n-1}\Gamma\left(z+\frac{j}{n}\right)=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}\,\Gamma(nz), \\ \Gamma(z)\Gamma(1-z)=\dfrac{\pi}{\sin\pi z}. \end{cases}$$

Since, from (16)

$$\Gamma(3\xi)=\frac{\prod_{j=0}^{2}\Gamma\left(\xi+\frac{j}{3}\right)}{2\pi 3^{1/2-3\xi}}=\frac{\Gamma(\xi)\Gamma\left(\xi+\frac{1}{3}\right)\Gamma\left(\xi+\frac{2}{3}\right)27^{\xi}}{2\sqrt{3}\,\pi},$$

We have

$$\frac{\Gamma(\xi)}{\Gamma(3\xi)}=\frac{2\sqrt{3}\,\pi}{\Gamma\left(\xi+\frac{1}{3}\right)\Gamma\left(\xi+\frac{2}{3}\right)27^{\xi}}$$

$$\to\frac{2\sqrt{3}\,\pi}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)}=\frac{2\sqrt{3}\,\pi}{\dfrac{\pi}{\sin\frac{\pi}{3}}}=3 \qquad (\xi\to+0).$$

Thus, we complete the proof of lemma.

PROOF OF THEOREM 6. For each $\varepsilon > 0$, there exist $\xi_1(\varepsilon)$ and $N_1(\varepsilon)$ such that

$$\frac{1}{\xi^3\Gamma(\xi)}\int_{\pi/\xi}^{\infty}(1-\cos k\xi u)e^{-u^{1/\xi}}du < \varepsilon, \text{ for } 0<\xi<\xi_1(\varepsilon) \tag{17}$$

and

$$\frac{1}{\xi^3\Gamma(\xi)}\int_{N}^{\infty}(1-\cos k\xi u)e^{-u^{1/\xi}}du < \varepsilon, \text{ for } N > N_1(\varepsilon). \tag{18}$$

We have for large $N > N_1(\varepsilon)$

$$\begin{aligned}\frac{1-\rho_k^{(\xi)}}{\xi^2} &= \frac{1}{\xi^2\Gamma(\xi)}\int_0^{\pi/\xi}(1-\cos k\xi u)e^{-u^{1/\xi}}du \\ &= \frac{1}{\xi^3\Gamma(\xi)}\left(\int_0^{N}+\int_N^{\infty}-\int_{\pi/\xi}^{\infty}\right)(1-\cos k\xi u)e^{-u^{1/\xi}}du \\ &= \frac{k^2}{2\xi\Gamma(\xi)}\int_0^{N}u^2e^{-u^{1/\xi}}du+o(\xi)+\frac{1}{\xi^3\Gamma(\xi)}\left(\int_N^{\infty}-\int_{\pi/\xi}^{\infty}\right)(1-\cos k\xi u)e^{-u^{1/\xi}}du.\end{aligned} \tag{19}$$

Then we get

$$\begin{aligned}&\left|\frac{1-\rho_k^{(\xi)}}{\xi^2}-\frac{k^2\Gamma(3\xi)}{2\Gamma(\xi)}\right| \\ &\qquad=\left|\frac{1-\rho_k^{(\xi)}}{\xi^2}-\frac{k^2}{2\xi\Gamma(\xi)}\int_0^{\infty}u^2e^{-u^{1/\xi}}du\right| \\ &\qquad\leqq\frac{k^2}{2\xi\Gamma(\xi)}\int_N^{\infty}u^2e^{-u^{1/\xi}}du+o(\xi) \\ &\qquad\qquad+\frac{1}{\xi^3\Gamma(\xi)}\left(\int_N^{\infty}+\int_{\pi/\xi}^{\infty}\right)(1-\cos k\xi u)e^{-u^{1/\xi}}du\end{aligned}$$

and from (17), (18) and (19)

$$\left|\frac{1-\rho_k^{(\xi)}}{\xi^2}-\frac{k^2\Gamma(3\xi)}{2\Gamma(\xi)}\right|<4\varepsilon, \text{ for } 0<\xi<\xi_1(\varepsilon).$$

Consequently we obtain

$$1-\rho_k^{(\xi)} = \frac{k^2\xi^2\Gamma(3\xi)}{2\Gamma(\xi)} + o(\xi^2)$$

$$= \frac{k^2\xi^2}{6} + o(\xi^2). \qquad \text{q.e.d.}$$

7. **A special form of Baskakov's operator**. V.A. Baskaov [1] gave an example of linear positive operator of which Bernstein polynomial and Szász's operator are particular cases, and which is defined in the following.

In the sequence of real functions $\varphi_n(x)$ $(n=1,2,\cdots)$, each function has the following properties:

1) $\varphi_n(x)$ can be expanded in Taylor's series in $[0,\infty)$,
2) $\varphi_n(0)=1$,
3) $(-1)^k\varphi_n^{(k)}(x)\geqq 0$ $(k=0,1,2,\cdots)$, for $x\in[0,\infty)$,
4) $-\varphi_n^{(k)}(x)=n\varphi_{n+c}^{(k-1)}(x)$ $(k=1,2,\cdots)$, for $x\in[0,\infty)$,

where c is an integer, Now we define for $x\in[0,\infty)$ the Baskakov's operator $B_n(f,x)$ by

$$B_n(f,x)=\sum_{k=0}^{\infty}(-1)^k\frac{\varphi_n^{(k)}(x)}{k!}x^k f\left(\frac{k}{n}\right), \quad (n=1,2,\cdots\cdot). \tag{20}$$

It has a meaning for each function $f(x)$ which is continuous on $[0,R]$ and zero on (R,∞), and it is positive on account of 3), where $R(\geqq 1)$ is an arbitrarily fixed positive number.

In this section, we consider only the case $\varphi_n(x)=(1+x)^{-n}$ and $c=1$. Then the operator (20) is represented in the following form.

$$V_n(f,x)=\frac{1}{(1+x)^n}\sum_{l=0}^{\infty}\frac{n(n+1)\cdots(n+l-1)}{l!}\left(\frac{x}{1+x}\right)^l f\left(\frac{l}{n}\right).$$

THEOREM E. (Y.Suzuki [14]). *Let us suppose that $B_n(f;x)$ satisfies the following conditions*:

$$\text{(i)} \qquad \frac{np_{n+c,l}}{p_{nl}}=\frac{lc+n}{cx+1},$$

$$\text{(ii)} \qquad \int_0^{E(c)}p_{nl}(x)dx=\frac{1}{n-c}, \qquad \int_0^{E(c)}xp_{nl}(x)dx=\frac{l+1}{(n-c)(n-2c)},$$

$$\text{(iii)} \qquad \sum_{n\alpha\leqq l\leqq n\beta}\int_R^{E(c)}x^i p_{nl}(x)dx=O\left(\frac{1}{n}\right) \quad (i=0,1),$$

where $0 < \alpha < \beta < R$ *and*

$$p_{nl}(x) = (-1)^l \frac{\varphi_n^{(l)}(x)}{l!} x^l, \quad E(c) = \frac{5c^2 - c - 2}{2c(c-1)} \quad (\textit{if } c \neq 0, 1),$$

and $E(1) = E(0) = \infty$. *Then Baskakov's operator is saturated locally with the weight function* $\psi(x) = x(1 + cx)$. *That is, we get the propositions*:

(I) *If*

$$|B_n(f, x) - f(x)| < \frac{M\psi(x)}{2n}, \quad x \in [a, b] \ (n = 1, 2, \cdots),$$

then $f(x)$ *has a derivative which belongs to* $\mathrm{Lip}_M 1$ *on* $[a, b]$;

(II) *If* $f'(x)$ *exists and belongs to* $\mathrm{Lip}_M 1$ *on* $[a_1, b_1]$, *then*

$$|B_n(f, x) - f(x)| < \frac{M\psi(x)}{2n} + o\left(\frac{1}{n}\right),$$

uniformly in $x \in [a_2, b_2]$;

(III) *If in addition to the assumption of* (I), *the relation*

$$B_n(f, x) - f(x) = o\left(\frac{1}{n}\right)$$

holds a.e. on $[a_1, b_1]$, *then* $f(x)$ *is linear on* $[a_1, b_1]$, *where* $0 \leqq a < a_1 < a_2 < b_2 < b_1 < b \leqq R$.

We write this result by the notation:

$$\text{L.Sat.}[B_n] = [\{f \mid f' \in \mathrm{Lip}_M 1\},\ n^{-1},\ f(x) = \textit{linear},\ \psi(x)].$$

Applying Theorem E to the special case: $\varphi_n(y) = (1 + y)^{-n}$, $c = 1$, we can prove the necessary conditions autonomously and we get the local saturation theorem by the operator $V_n(f, x)$.

THEOREM 7.

$$\text{L.Sat.}[V_n] = [\{f \mid f' \in \mathrm{Lip}_M 1\},\ n^{-1},\ f(x) = \textit{linear},\ x(1 + x)].$$

In order to prove the Theorem 7, we need the following theorem.

THEOREM F. (G. H. Hardy [3], p.201). *Suppose that* $0 < k < 1$ *and*

$$u_m = u_m(n) = k^{n+1}\binom{m}{n}(1-k)^{m-n} \quad (m \geqq n),$$

so that

$$\sum_{m=0}^{\infty} u_m = k^{n+1}\{1+(n+1)(1-k)+\frac{(n+1)(n+2)}{2!}(1-k)^2+\cdots\}=1.$$

Then, (i) *the largest* u_m *is* u_M, *where* $M=[n/k]$, *two terms,* u_{M-1} *and* u_M, *being equal if* n/k *is an integer;*

(ii) *if* $m=M+h$ *and* $0<\delta<1$, *then*

$$\sum_{|h|>\delta n} u_m = O(e^{-\gamma n})$$

where γ *is a positive constant depending on* k *and* δ.

PROOF OF THEOREM 7. We have only to verify that for the case $E(R)=\infty$ and

$$p_{nl}(x) = \frac{(n+l-1)!}{(n-1)!\,l!}\,\frac{x^l}{(1+x)^{n+l}},$$

the conditions (i), (ii) and (iii) are satisfied. From an easy calculation, we get

$$\frac{np_{n+1,l}(x)}{p_{nl}(x)} = \frac{n(n+l)!\,x^l}{n!\,l!\,(1+x)^{n+1-l}}\,\frac{(n-1)!\,l!\,(1+x)^{n+l}}{n(n+l-1)!\,x^l}$$
$$= \frac{n+l}{1+x},$$

and

$$\int_0^\infty p_{nl}(x)dx = \frac{n(n+1)\cdots(n+l-1)}{l!}\int_0^\infty \frac{x^l}{(1+x)^{n+l-1}}dx$$
$$= \int_0^\infty \frac{dx}{(1+x)^n} = \frac{1}{n-1}.$$

Similarly, we obtain

$$\int_0^\infty x p_{nl}(x)dx = \frac{l+1}{(n-1)(n-2)}.$$

Consequently, we may confine ourselves to the proof of satisfying the condition (iii). Let us rewrite

$$\int_R^\infty p_{nl}(x)dx = \frac{1}{(n-1)}\sum_{m=0}^{l} u_m(n), \tag{21}$$

where

$$u_m(n) = \frac{1}{(1+R)^{n-1}}\frac{1}{m!}(n-1)n(n+1)\cdots(n+m-2)\left(\frac{R}{1+R}\right)^m.$$

If we set $\varepsilon=\alpha$ and $\beta=R-\varepsilon$, then by (21), we have

$$n\sum_{n\alpha\leqq l\leqq n\beta}\int_R p_{nl}(x)dx = \frac{n}{n-1}\sum_{\varepsilon n\leqq l\leqq (R-\varepsilon)n}\sum_{m=0}^{l} u_m(n)$$
$$\leqq \frac{n}{n-1}\sum_{\varepsilon n\leqq l\leqq (R-\varepsilon)n}\sum_{m=0}^{[(R-\varepsilon)n]} u_m(n).$$

On the other hand, if we set $k=\dfrac{1}{1+R}$ and interchange n with $n-2$ in Theorem F, it follows that for the largest term $u_M(n)$,

$$M=\left[\frac{n-2}{\frac{1}{1+R}}\right]\geqq (n-2)(R+1)-1.$$

Since $\varepsilon n\leqq l\leqq (R-\varepsilon)n$, there is a positive integer N_2 such that

$$M-[(R-\varepsilon)n]\geqq (n-2)(R+1)-1-(R-\varepsilon)n$$
$$=\left(1+\varepsilon-\frac{2R+3}{n}\right)n$$
$$\geqq \delta n,\ for\ n\geqq N_2,$$

where $0<\delta<1$.
Hence, using Theorem F again, we obtain

$$n\sum_{n\alpha\leqq l\leqq n\beta}\int_R^\infty p_{nl}(x)dx\leqq \frac{n}{n-1}O(n)O(e^{-\gamma n})=o(1). \tag{22}$$

Similarly, we get

$$n \sum_{n\alpha \leqq l \leqq n\beta} \int_R^\infty x p_{nl}(x)\,dx \leqq \frac{n(l+1)}{(n-1)(n-2)} O(n)O(e^{-\gamma n}) = o(1). \tag{23}$$

Hence, from (22) and (23) the condition (iii) is satisfied. Thus, using Theorem E, we get the complete proof of Theorem 7.

REMARK 6. In the definition of Baskakov's operator, if $\varphi_n(x)=(1-x)^n$ and $c=-1$, it is Bernstein polynomial, and also if $\varphi_n(x)=e^{-nx}$ and $c=0$, it is Szasz's operator. (see, [6] and [14]).

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