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A PROOF OF CARTAN'S THEOREMS A AND B

YUM-TONG SIU

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In this note we give a new proof of the following theorems of Cartan:

THEOREM A. *If* $\mathfrak{F}$ *is a coherent analytic sheaf on a (reduced) Stein space* X, *then* $\Gamma(X, \mathfrak{F})$ *generates* $\mathfrak{F}_x$ *for all* $x \in X$.

THEOREM B. *If* $\mathfrak{F}$ *is a coherent analytic sheaf on a Stein space* X, *then* $H^p(X, \mathfrak{F})=0$ *for* $p \geqq 1$.

The known proofs of these theorems depend on one of the following: (i) Cartan's Lemma of invertible holomorphic matrices ([2], [3]), (ii) methods of partial differential equations ([5]), and (iii) methods of differential geometry ([1]). In the proof here essentially we make use only of Dolbeault's Lemma (I.D.3, [4]) and Schwartz's Theorem (App.B, 12, [4]). Theorem B is first proved and then Theorem A is derived from it.

NOTATIONS. ${}_n\mathfrak{O}$=the structure-sheaf of the complex n-space $\mathbf{C}^n$. For $r>0$, B_r^n or B_r denotes the ball in $\mathbf{C}^n$ with radius r and centered at the origin. The boundary of a set E in $\mathbf{C}^n$ is denoted by ∂E. Suppose $g=(g_1, \cdots, g_p)$ is a p-tuple of complex-valued functions defined on a set K. Then $\|g\|_K$ denotes $\sup\{|g_i(x)| \mid 1 \leqq i \leqq p,\ x \in K\}$. If $\mathfrak{U}$ is an open covering of a topological space, then $N(\mathfrak{U})$ denotes the nerve of $\mathfrak{U}$.

DEFINITION 1. Suppose $\gamma_i < \delta_i$ and $\delta_i > 0$, $1 \leqq i \leqq n$. The domain $\{z=(z_1, \cdots, z_n) \in \mathbf{C}^n \mid \gamma_i < |z_i| < \delta_i,\ 1 \leqq i \leqq n\}$ is called a *polyannulus*.

In this definition γ_i can be negative. Hence a polydisc is a polyannulus.

DEFINITION 2. Suppose p_j, $1 \leqq j \leqq n+r$, are polynomials on $\mathbf{C}^n$ such that $p_i = z_i$ for $1 \leqq i \leqq n$. Suppose $\alpha_j < \beta_j$ and $\beta_j > 0$, $1 \leqq j \leqq n+r$. The domain $D=\{z \in \mathbf{C}^n \mid \alpha_j < |p_j(z)| < \beta_j,\ 1 \leqq j \leqq n+r\}$ is called a *polynomial polyannulus*. Suppose $(k_1, \cdots, k_{n+r})$ is a permutation of $(1, \cdots, n+r)$ such that $\alpha_{k_j} \geqq 0$ for $1 \leqq j \leqq m$ and $\alpha_{k_j} < 0$ for $m < j \leqq n+r$. The polynomials p_{k_j}, $1 \leqq j \leqq m$, are called *essential defining polynomials* for D.

Trivial modifications of the proofs of I.D.1, 2, 3 in [4] give us:

(1) Suppose P is a polyannulus in $\mathbf{C}^n$. If $q>0$ and ω is a C^∞ $\bar{\partial}$-closed

$(0, q)$-form on a neighborhood of P^-, then there is a C^∞ $(0, q-1)$-form η on P such that $\bar{\partial}\eta=\omega$.

By using (1) instead of I.D.3, [4], we can easily modify the proof of I.F.5, [4] to obtain:

(2) Suppose D is a polynomial polyannulus in $\mathbf{C}^n$. If $q>0$ and ω is a C^∞ $\bar{\partial}$-closed $(0, q)$-form on a neighborhood of D^-, then there is a C^∞ $(0, q-1)$-form η on D such that $\bar{\partial}\eta=\omega$.

By using (2) instead of I.F.5,[4], we can easily modify the proof of I.F.8, [4] to obtain:

(3) Suppose D is a polynomial polyannulus and p_k, $1\leqq k\leqq m$, are essential defining polynomials for D. Let $G=\{z\in\mathbf{C}^n \mid p_k(z)\neq 0, 1\leqq k\leqq m\}$. Then any holomorphic function on D can be approximated uniformly on compact subsets of D by holomorphic functions on G.

DEFINITION 3. Suppose $\mathfrak{F}$ is a coherent analytic sheaf on a σ-compact complex space $(X, \mathfrak{O})$ and K is a compact subset of X. Suppose $\varphi: \mathfrak{O}^p \to \mathfrak{F}$ is a sheaf-epimorphism such that φ induces an epimorphism $\tilde{\varphi}: \Gamma(X, \mathfrak{O}^p) \to \Gamma(X, \mathfrak{F})$. For $f\in\Gamma(X, \mathfrak{F})$, $\|f\|_K^\varphi$ is defined as $\inf\{\|g\|_K \mid g\in\Gamma(X, \mathfrak{O}^p), \tilde{\varphi}(g)=f\}$.

LEMMA 1. *Under the assumptions of Def. 3, the norms $\{\|\cdot\|_K^\varphi | K$ is a compact subset of $X\}$ define a Fréchet space topology in $\Gamma(X, \mathfrak{F})$.*

PROOF. Let $\mathfrak{R}=\mathrm{Ker}\ \varphi$. $\Gamma(X, \mathfrak{R})$ is a closed subspace of the Fréchet space $\Gamma(X, \mathfrak{O}^p)$ with the topology of uniform convergence on compact subsets (cf. VIII.A. 2, [4]). The surjectivity of $\tilde{\varphi}$ implies that the topology defined by the norms $\|\cdot\|_K^\varphi$ in $\Gamma(X, \mathfrak{F})$ is identical with the quotient topology induced by $\tilde{\varphi}$ and that the quotient topology is a Fréchet space topology. q.e.d.

This Fréchet space topology of $\Gamma(X, \mathfrak{F})$ is independent of the choice of φ and hence is canonical.

PROPOSITION 1. *Suppose $\varphi^{(1)}, \cdots, \varphi^{(m)}$ are real-valued C^∞ functions on $\mathbf{C}^n$ satisfying:*

$$(*) \qquad |\varphi_{ij}^{(k)}(z)| < (6n^2)^{-1} \text{ and } |\varphi_{i\bar{j}}^{(k)}(z)-\delta_{ij}| < (3n^2)^{-1}$$

for $z\in\mathbf{C}^n$, $1\leqq i, j\leqq n$, and $1\leqq k\leqq m$, where δ_{ij} is the Kronecker delta, $\varphi_{ij}^{(k)}=\dfrac{\partial^2\varphi^{(k)}}{\partial z_i\partial z_j}$, and $\varphi_{i\bar{j}}^{(k)}=\dfrac{\partial^2\varphi^{(k)}}{\partial z_i\partial\bar{z}_j}$. Suppose $D=\{z\in\mathbf{C}^n \mid \varphi^{(k)}(z)<0, 1\leqq k\leqq m\}$

is a bounded domain. Then $H^p(D, {}_n\mathfrak{O})=0$ *for* $p \geqq 1$.

PROOF. First we prove that

(4) for $z^0=(z_1^0, \cdots, z_n^0) \in \partial D$, there exists a polynomial f such that $f(z^0)=0$ and f is nowhere zero on D.

Fix $z^0 \in \partial D$. Then $\varphi^{(k)}(z^0)=0$ for some k. Define a polynomial $f(z) = \sum_{i=1}^{n} \frac{\partial \varphi^{(k)}}{\partial z_i}(z^0)(z_i - z_i^0)$. Then $\varphi^{(k)}(z)=2\mathrm{Re}\,(f(z)+ \sum_{1 \leqq i,j \leqq n} \varphi_{ij}^{(k)}(z^*)(z_i-z_i^0)(z_j-z_j^0)) + \sum_{1 \leqq i,j \leqq n} \varphi_{i\bar{j}}^{(k)}(z^*)(z_i-z_i^0)\overline{(z_j-z_j^0)}$ for some z^* depending on z. (*) implies that

$\varphi^{(k)}(z) \geqq 2\mathrm{Re} f(z)+ \frac{1}{3}\left(\sum_{i=1}^{n} |z_i-z_i^0|^2\right)$. Hence f is nowhere zero on D.

Construct open subsets P_k of D, $1 \leqq k < \infty$, such that (i) P_k is a union of topological components of a polynomial polyannulus whose essential defining polynomials are nowhere zero on D, (ii) $P_k \subset\subset P_{k+1}$, and (iii) $\bigcup_{k=1}^{\infty} P_k = D$. This is possible by (4).

Now by using (2) and (3) we can complete the proof in almost the same way as the proof of I.D.5, [4]. q.e.d.

COROLLARY. *Suppose* $\mathfrak{F}$ *is a coherent analytic sheaf on* D *admitting a finite free resolution. Then* $H^p(D, \mathfrak{F})=0$ *for* $p \geqq 1$.

PROPOSITION 2. *Suppose* $\mathfrak{F}$ *is a coherent analytic sheaf defined on an open neighborhood* G *of* $\bar{B}_r$ *in* $\mathbf{C}^n$. *Then* $\dim_{\mathbf{C}} H^p(B_r, \mathfrak{F}) < \infty$ *for* $p \geqq 1$.

PROOF. Choose in $\mathbf{C}^n$ balls $U_k \subset\subset V_k \subset\subset G$, $1 \leqq k \leqq m$, such that (i) $\partial B_r \subset \bigcup_{k=1}^{m} U_k$, and (ii) $\mathfrak{F}$ admits a finite free resolution on V_k. Let ψ_k be a C^∞ non-negative function on $\mathbf{C}^n$ such that $\psi_k \equiv 0$ outside V_k and $\psi_k > 0$ on U_k, $1 \leqq k \leqq m$. Let $\varphi^{(0)} = \sum_{i=1}^{n} |z_i|^2 - r^2$. Choose positive numbers λ_k, $1 \leqq k \leqq m$, so small that $\varphi^{(k)} = \varphi^{(0)} - \sum_{i=1}^{k} \lambda_i \psi_i$ satisfies (*) for $z \in \mathbf{C}^n$, $1 \leqq i,j \leqq n$ and $1 \leqq k \leqq m$. Let $D_k = \{z \in \mathbf{C}^n \mid \varphi^{(k)}(z) < 0\}$, $0 \leqq k \leqq m$. Then $D_0 = B_r \subset\subset D_m$, $D_k = D_{k-1} \cup (D_k \cap V_k)$, and $D_{k-1} \cap V_k = D_{k-1} \cap (D_k \cap V_k)$. By Cor. to Prop. 1, $H^p(D_{k-1} \cap V_k, \mathfrak{F})=0$ for $p \geqq 1$ and $1 \leqq k \leqq m$. From the exactness of the Mayor-Vietoris sequence $H^p(D_k, \mathfrak{F}) \to H^p(D_{k-1}, \mathfrak{F}) \oplus H^p(D_k \cap V_k, \mathfrak{F}) \to H^p(D_{k-1} \cap V_k, \mathfrak{F})$, we conclude that $H^p(D_k, \mathfrak{F}) \to H^p(D_{k-1}, \mathfrak{F})$ is surjective for $p \geqq 1$ and $1 \leqq k \leqq m$. Hence

(5) the restriction map $H^p(D_m, \mathfrak{F}) \to H^p(B_r, \mathfrak{F})$ is surjective, $p \geqq 1$.

Choose two finite collections of balls in G, $\{\widetilde{U}_j^i\}_{j=1}^l$, $i=1,2$, such that (i) $\widetilde{U}_j^1 \subset\subset \widetilde{U}_j^2$, (ii) $B_r \subset \bigcup_{j=1}^l \widetilde{U}_j^1$, (iii) $D_m \subset \bigcup_{j=1}^l \widetilde{U}_j^2$, and (iv) on $\widetilde{U}_j^2$ we have a sheaf-epimorphism $\xi_j : {}_n\mathfrak{O}^{p_j} \to \mathfrak{F}$ which is part of a finite free resolution. Let $U_j^1 = \widetilde{U}_j^1 \cap B_r$, $U_j^2 = \widetilde{U}_j^2 \cap D_m$, and $\mathfrak{U}_i = \{U_j^i\}_{j=1}^l$, $i=1,2$. Fix $p \geqq 1$. Since $H^1(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \operatorname{Ker} \xi_{j_0}) = 0$ by Cor. to Prop. 1, the map $\Gamma(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, {}_n\mathfrak{O}^{p_{j_0}}) \to \Gamma(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \mathfrak{F})$ induced by ξ_{j_0} is surjective for $l \geqq j_0, \cdots, j_q \geqq 1$ and $i=1,2$. By Lemma 1 $\Gamma(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \mathfrak{F})$ has a canonical Fréchet space topology. $Z^p(N(\mathfrak{U}_i), \mathfrak{F})$, $i=1,2$, and $C^{p-1}(N(\mathfrak{U}_1), \mathfrak{F})$ can be given Fréchet space structures canonically. Let $\rho: Z^p(N(\mathfrak{U}_2), \mathfrak{F}) \to Z^p(N(\mathfrak{U}_1), \mathfrak{F})$ be the restriction map and δ: $C^{p-1}(N(\mathfrak{U}_1), \mathfrak{F}) \to Z^p(N(\mathfrak{U}_1), \mathfrak{F})$ be the coboundary map. Since $H^s(U_{j_0}^i \cap \cdots \cap U_{j_q}^i, \mathfrak{F}) = 0$ for $s \geqq 1$, $i=1,2$, $l \geqq j_0, \cdots, j_q \geqq 1$ by Cor. to Prop.1, $H^p(N(\mathfrak{U}_1), \mathfrak{F}) \approx H^p(B_r, \mathfrak{F})$ and $H^p(N(\mathfrak{U}_2), \mathfrak{F}) \approx H^p(D_m, \mathfrak{F})$. By (5) $\rho \oplus \delta$: $Z^p(N(\mathfrak{U}_2), \mathfrak{F}) \oplus C^{p-1}(N(\mathfrak{U}_1), \mathfrak{F}) \to Z^p(N(\mathfrak{U}_1), \mathfrak{F})$ defined by $(\rho \oplus \delta)(a \oplus b) = \rho(a) + \delta(b)$ is surjective. Since $U_j^1 \subset\subset U_j^2$, the map $\rho \oplus 0$: $Z^p(N(\mathfrak{U}_2), \mathfrak{F}) \oplus C^{p-1}(N(\mathfrak{U}_1), \mathfrak{F}) \to Z^p(N(\mathfrak{U}_1), \mathfrak{F})$ defined by $(\rho \oplus 0)(a \oplus b) = \rho(a)$ is compact. By Schwartz Theorem (App.B, 12, [4]), $0 \oplus \delta = \rho \oplus \delta - \rho \oplus 0$ has finite-dimensional cokernel. Hence δ has finite-dimensional cokernel. $\dim_{\mathbf{C}} H^p(B_r, \mathfrak{F}) < \infty$. q.e.d.

PROPOSITION 3. *Under the assumption of Prop. 2, $H^p(B_r, \mathfrak{F}) = 0$ for $p \geqq 1$.*

PROOF. By shrinking G, w.l.o.g. we can assume $\dim \operatorname{Supp} \mathfrak{F} < \infty$. Fix $p \geqq 1$. Use induction on $\dim \operatorname{Supp} \mathfrak{F}$. The case $\dim \operatorname{Supp} \mathfrak{F} = 0$ is trivial. Suppose the proposition is true for $\dim \operatorname{Supp} \mathfrak{F} < d$. Now assume $\dim \operatorname{Supp} \mathfrak{F} = d > 0$. Let $\operatorname{Supp} \mathfrak{F} = (\cup_{i \in I} X_i') \cup (\cup_{j \in J} X_j)$ be the decomposition into irreducible branches, where $\dim X_i' < d$ and $\dim X_j = d$. Let $\pi : \mathbf{C}^n \to \mathbf{C}$ be the projection $\pi(z_1, \cdots, z_n) = z_1$. After a linear coordinates transformation in $\mathbf{C}^n$ we can assume that no X_j is contained in $\pi^{-1}(a)$ for any $a \in \mathbf{C}$. Let M be the set of entire functions on $\mathbf{C}$. Take $f \in M - \{0\}$. Let $\varphi_f : \mathfrak{F} \to \mathfrak{F}$ be the sheaf-homomorphism defined by multiplication by $f \circ \pi$ and let $\mathfrak{K}_f = \operatorname{Ker} \varphi_f$ and $\mathfrak{L}_f = \operatorname{Coker} \varphi_f$. Then $\dim \operatorname{Supp} \mathfrak{K}_f < d$ and $\dim \operatorname{Supp} \mathfrak{L}_f < d$. By induction hypothesis

$$H^q(B_r, \mathfrak{K}_f) = H^q(B_r, \mathfrak{L}_f) = 0 \quad \text{for} \quad q \geqq 1. \tag{6}$$

The exact sequence $0 \longrightarrow \mathfrak{K}_f \xrightarrow{\alpha} \mathfrak{F} \longrightarrow \mathfrak{F}/\mathfrak{K}_f \longrightarrow 0$ (where α is the inclusion) implies that $H^p(B_r, \mathfrak{F}) \xrightarrow{\approx} H^p(B_r, \mathfrak{F}/\mathfrak{K}_f)$ by (6). The exact sequence $0 \longrightarrow \mathfrak{F}/\mathfrak{K}_f$

$\xrightarrow{\beta} \mathfrak{F} \longrightarrow \mathfrak{L}_f \longrightarrow 0$ (where β is induced by φ_f) implies that $H^p(B_r, \mathfrak{F}/\mathfrak{R}_f) \xrightarrow{\approx} H^p(B_r, \mathfrak{F})$ by (6). Hence φ_f induces an isomorphism

$$\varphi_f^* : H^p(B_r, \mathfrak{F}) \xrightarrow{\approx} H^p(B_r, \mathfrak{F}). \tag{7}$$

Suppose $0 \neq \omega \in H^p(B_r, \mathfrak{F})$. Define $\Phi: M \to H^p(B_r, \mathfrak{F})$ by $\Phi(f) = \varphi_f^*(\omega)$ for $f \in M - \{0\}$ and $\Phi(0) = 0$. Then Φ is a linear injection by (7). $\dim_c H^p(B_r, \mathfrak{F}) \geqq \dim_c M = \infty$, contradicting Prop.2. q.e.d.

A proof similar to [6] gives us

COROLLARY 1. *Under the assumption of Prop.2, $\mathfrak{F}$ is generated on B_r by* $\Gamma(B_r, \mathfrak{F})$.

COROLLARY 2. *Suppose $\mathfrak{F}$ is a coherent analytic sheaf on a Stein space X and G is a relatively compact open subset of X. Then $\mathfrak{F}$ is generated on G by* $\Gamma(G, \mathfrak{F})$.

PROOF. Follows from the fact that some open neighborhood of G^- in X is biholomorphic to a subvariety of a ball in a complex number space. q.e.d.

COROLLARY 3. *Suppose D is an open subset of a Stein space $(X, \mathfrak{O})$ and $\varphi: X \to \mathbf{C}^n$ is holomorphic such that* (i) *for some open neighborhood G of D^- φ maps G biholomorphically onto a subvariety of some open subset H of $\mathbf{C}^n$ and* (ii) *$\varphi(D)$ is a subvariety in a ball B_r in H. Then $\Gamma(X, \mathfrak{O})$ is dense in $\Gamma(D, \mathfrak{O})$ with the topology of uniform convergence on compact subsets.*

PROOF. Let $\mathfrak{J}$ be the ideal-sheaf of $\varphi(G)$ on H. Since $H^1(B_r, \mathfrak{J}) = 0$, the natural map: $\Gamma(B_r, {}_n\mathfrak{O}) \longrightarrow \Gamma(B_r, {}_n\mathfrak{O}/\mathfrak{J}) (\approx \Gamma(D, \mathfrak{O}))$ is surjective. This means that the map α: $\Gamma(B_r, {}_n\mathfrak{O}) \longrightarrow \Gamma(D, \mathfrak{O})$ induced by $\varphi | D$ is surjective. Let $\beta: \Gamma(\mathbf{C}^n, {}_n\mathfrak{O}) \longrightarrow \Gamma(X, \mathfrak{O})$ be induced by φ.

$$\begin{array}{ccc} \Gamma(\mathbf{C}^n, {}_n\mathfrak{O}) & \xrightarrow{\beta} & \Gamma(X, \mathfrak{O}) \\ \rho\downarrow & & \downarrow\sigma \\ \Gamma(B_r, {}_n\mathfrak{O}) & \xrightarrow{\alpha} & \Gamma(D, \mathfrak{O}) \end{array}$$

is commutative, where ρ and σ are restriction maps. Since $\Gamma(\mathbf{C}^n, {}_n\mathfrak{O})$ is dense in $\Gamma(B_r, {}_n\mathfrak{O})$ (I.F.9, [4]), $\Gamma(X, \mathfrak{O})$ is dense in $\Gamma(D, \mathfrak{O})$. q.e.d.

PROOF OF CARTAN'S THEOREM B. Suppose $\mathfrak{F}$ is a coherent analytic sheaf on a Stein space $(X, \mathfrak{O})$. We construct open subsets X_k and holomorphic maps $\varphi^{(k)}: X \to \mathbf{C}^{n_k}$, $1 \leqq k < \infty$, such that (i) $X = \bigcup_{k=1}^{\infty} X_k$, (ii) $X_k \subset\subset X_{k+1}$, (iii) $\varphi^{(k)}$ maps X_{k+1} biholomorphically onto a subvariety of an open subset of $\mathbf{C}^{n_k}$, and (iv) $\varphi^{(k)}(X_k)$ is a subvariety in a ball of $\mathbf{C}^{n_k}$. By Cor.2 to Prop.3, there exist sheaf-epimorphisms $\psi^{(k)}: \mathfrak{O}^{r_k} \to \mathfrak{F}$ on X_k for $k \geqq 1$. By Prop. 3, $H^1(X_k, \operatorname{Ker} \psi^{(k+s)}) = 0$ for $k \geqq 1$ and $s \geqq 1$. Hence

(8) $\tilde{\psi}_{k,s}: \Gamma(X_k, \mathfrak{O}^{r_{k+s}}) \to \Gamma(X_k, \mathfrak{F})$ induced by $\psi^{(k+s)}$ is surjective for $k \geqq 1$ and $s \geqq 1$.

By Lemma 1 $\Gamma(X_k, \mathfrak{F})$ has a canonical Fréchet space structure for $k \geqq 1$. For $k \geqq 1$ and $s \geqq 1$,

$$\begin{array}{ccc}
\Gamma(X_{k+s}, \mathfrak{O}^{r_{k+s+1}}) & \longrightarrow & \Gamma(X_k, \mathfrak{O}^{r_{k+s+1}}) \\
\tilde{\psi}_{k+s,1} \Big\downarrow & & \tilde{\psi}_{k,s+1} \Big\downarrow \\
\Gamma(X_{k+s}, \mathfrak{F}) & \longrightarrow & \Gamma(X_k, \mathfrak{F})
\end{array}$$

is commutative, where the horizontal maps are restriction maps. By (8) and Cor. 3 to Prop. 3,

(9) $\Gamma(X_{k+s}, \mathfrak{F})$ is dense in $\Gamma(X_k, \mathfrak{F})$ for $k \geqq 1$ and $s \geqq 1$.

By Prop. 3 $H^p(X_k, \mathfrak{F}) = 0$ for $p \geqq 1$ and $k \geqq 1$. Let $\mathfrak{X}^{(k)} = \{X_m\}_{m=1}^{k}$ for $k \geqq 1$, and $\mathfrak{X} = \{X_m\}_{m=1}^{\infty}$. Then $H^p(N(\mathfrak{X}^{(k)}), \mathfrak{F}) = 0$ for $k \geqq 1$ and $p \geqq 1$, and $H^p(X, \mathfrak{F}) \approx H^p(N(\mathfrak{X}), \mathfrak{F})$ for $p \geqq 1$.

Fix $q \geqq 1$ and $\sigma \in Z^q(N(\mathfrak{X}), \mathfrak{F})$. Let $\sigma^{(k)} = \sigma | N(\mathfrak{X}^{(k)})$. Then $\sigma^{(k)} = \delta\alpha^{(k)}$ for some $\alpha^{(k)} \in C^{q-1}(N(\mathfrak{X}^{(k)}), \mathfrak{F})$. $\alpha^{(k)} - \alpha^{(k-1)} \in Z^{q-1}(N(\mathfrak{X}^{(k-1)}), \mathfrak{F})$.

Case $q=1$. Construct by induction on k $\beta^{(k)} \in C^0(N(\mathfrak{X}^{(k)}), \mathfrak{F})$ such that $\delta\beta^{(k)} = \sigma^{(k)}$ and $\sup_{3 \leqq j \leqq k} \| \beta^{(k)} - \beta^{(k-1)} | X_{j-1} \|_{X_{j-2}}^{\psi^{(j)}} < 2^{-k}$: Choose $\beta^{(1)} = \alpha^{(1)}$. Suppose we have chosen $\beta^{(1)}, \cdots, \beta^{(k-1)}$. Then $\alpha^{(k)} - \beta^{(k-1)}$ is a section of $\mathfrak{F}$ on X_{k-1}. By (8) and (9) there exists $\tau \in \Gamma(X_k, \mathfrak{F})$ such that $\sup_{3 \leqq j \leqq k} \| \tau - (\alpha^{(k)} - \beta^{(k-1)}) | X_{j-1} \|_{X_{j-2}}^{\psi^{(j)}} < 2^{-k}$. Set $\beta^{(k)} = \alpha^{(k)} - \tau$. The construction is complete. Define $\beta \in C^0(N(\mathfrak{X}), \mathfrak{F})$ by $\beta(X_k) = \lim_{m \geqq k} \beta^{(m)}(X_k)$. It is easily verified that β is well-defined and $\delta\beta = \sigma$.

Case $q > 1$. Since $\alpha^{(k)} - \alpha^{(k-1)} \in Z^{q-1}(N(\mathfrak{X}^{(k-1)}), \mathfrak{F})$, there exists $\beta^{(k-1)} \in C^{q-1}(N(\mathfrak{X}^{(k-1)}), \mathfrak{F})$ such that $\delta\beta^{(k-1)} = \alpha^{(k)} - \alpha^{(k-1)}$ on $N(\mathfrak{X}^{(k)})$. Define $\gamma \in C^{q-1}(N(\mathfrak{X}), \mathfrak{F})$ by $\gamma = \alpha^{(k)} - \delta\left(\sum_{m<k} \beta^{(m)}\right)$ on $N(\mathfrak{X}^{(k)})$. γ is well-defined and $\delta\gamma = \sigma$. q.e.d.

PROOF OF CARTAN'S THEOREM A. Follows from Theorem B by [6]. q.e.d.

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MATHEMATICS DEPARTMENT
UNIVERSITY OF NOTRE DAME
NOTRE DAME, INDIANA, U. S. A.