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PROPERTIES OF KERNELS FOR A CLASS OF CONVOLUTION-TRANSFORMS

Z. DITZIAN AND A. JAKIMOVSKI

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1. Introduction. In this paper the kernels for the class of convolution transforms that was introduced by I.I.Hirschman and D.V.Widder (see[2, p 696]) and was treated in many papers by Y.Tanno (see [4], [5] and [6]) will be investigated.

A meromorphic function $F(s)$ will be of class F if:

$$F(s) = \prod_{k=1}^{\infty} [(1-s/a_k)\exp(s/a_k)/(1-s/c_k)\exp(s/c_k)] \tag{1. 1}$$

where $\operatorname{Re} a_k = a_k$, $\operatorname{Re} c_k = c_k$, $0 \leqq a_k/c_k < 1$, $\sum_{k=1}^{\infty} a_k^{-2} < \infty$ and c_k may be equal to $\pm\infty$ in which case $(1-s/c_k)\exp(s/c_k) \equiv 1$.

The kernels of our transforms will be functions $H(t)$ satisfying for some $F(s) \in F$

$$F(iy)^{-1} = \int_{-\infty}^{\infty} e^{-iyt} dH(t). \tag{1. 2}$$

Asymptotic properties of $H(t)$ and its derivatives will be found on basis of zeros and poles, of $F(s)$ which are analogous to those achieved by I.I.Hirschman and D.V.Widder for the case where all $c_k = \pm\infty$. Also conditions for $H(t)$ to have derivatives are set and in Section 6, the strict positive character of $H'(t)$ on at least half the real axis is established.

In the literature one can find treatments of $F_1(s) = e^{bs}F(s)$ where $F(s) \in F$ instead of $F(s)$, this will represent only a shift of b in $H(t)$ and we shall avoid it.

2. $H(t)$ as a distribution function. In this section we shall relate to $F(s)$ a function $H(t)$ satisfying: $H(t)$ is non-decreasing, $H(-\infty) \equiv \lim_{t\to-\infty} H(t) = 0$ and $H(\infty) \equiv \lim_{t\to\infty} H(t) = 1$.

THEOREM 2.1. *Suppose we have a function $F_n(s)$ such that*

$$F_n(s) = \prod_{k=1}^{n} (1-s/a_k)e^{s/a_k} \Big/ \prod_{k=1}^{n} (1-s/c_k)e^{s/c_k}$$

$0 \leqq a_k/c_k < 1$, $\operatorname{Re} a_k = a_k$ *and* $\operatorname{Re} c_k = c_k$.

Then there exist $H_n(t)$ *satisfying*;

(1) $H_n(t)$ *is a non-decreasing normalized function*, $H_n(-\infty)=0$ *and* $H_n(\infty) = 1$,

(2) $H_n(t)$ *is continuous except at* $t_n = \sum_{k=1}^{n} (a_k^{-1} - c_k^{-1})$ *where it has a jump of* $\prod_{k=1}^{n} a_k c_k^{-1}$,

(3) $$F_n(s)^{-1} = \int_{-\infty}^{\infty} e^{-st} dH_n(t)$$

converges for $\alpha_1 < \operatorname{Re} s < \alpha_2$

and

(4) $$H_n(t) \equiv \frac{1}{2\pi i} \lim_{T\to\infty} \int_{d-iT}^{d+iT} \frac{e^{st}}{sF_n(s)} ds$$

for $0 < d < \alpha_2$,

where

(2. 1) $$\alpha_1 = \max_{a_k<0} \{a_k, -\infty\} \text{ and } \alpha_2 = \min_{a_k>0} \{a_k, \infty\}.$$

To prove this theorem we shall need the following lemma.

LEMMA 2.2. *Let* ${}_iF(s)$ *be defined by*

(2. 2) $$ {}_iF(s) = \left(1 - \frac{s}{a_i}\right) e^{s/a_i} \Big/ \left(1 - \frac{s}{c_i}\right) e^{s/c_i}$$

for $0 < a_i/c_i < 1$ *and let* $h_i(t)$ *be defined by*

(2. 3) $$h_i(t) = \begin{cases} \exp(-1 + a_i c_i^{-1} + a_i t)(c_i - a_i)/c_i & t < a_i^{-1} - c_i^{-1} \\ (2c_i - a_i)/2c_i & t = a_i^{-1} - c_i^{-1} \\ 1 & t > a_i^{-1} - c_i^{-1} \end{cases}$$

when $a_i > 0$ *and by*

(2. 4)
$$h_i(t) = \begin{cases} 0 & t < a_i^{-1} - c_i^{-1} \\ a_i/2c_i & t = a_i^{-1} - c_1^{-i} \\ 1 - \exp(-1 + a_i c_i^{-1} + a_i t)(c_i - a_i)/c_i & t > a_i^{-1} - c_i^{-1} \end{cases}$$

when $a_i < 0$. *Then*:

(1)
$$_iF(s)^{-1} = \int_{-\infty}^{\infty} e^{-st} dh_i(t)$$

converges for Re $s < a_i$ *in case* $a_i > 0$ *and for* Re $s > a_i$ *in case* $a_i < 0$.

(2)
$$h_i(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{d-iT}^{d+iT} \frac{e^{st}}{s_i F(s)} ds$$

for $0 < d < |a_i|$.

PROOF. Substitution of (2.3) and (2.4) yields (1) for $a > 0$ and $a < 0$ respectively. Equation (2) follows from Theorem 5.6 [7, p. 242].

REMARK. When $c = \infty$ this reduces to the classical result (see [3, p. 24]).

We shall need also the following lemma, the proof of which we omit being simple and straightforward.

LEMMA 2.3. *Let* $h_i(t)$ *be a distribution function continuous in* $-\infty < t < \infty$ *except at* $t = a_i$ *where it has a jump of* α_i, *then the function* $H(t)$ *defined by*

$$H(t) = \int_{-\infty}^{\infty} h_1(t-u) dh_2(u)$$

is a distribution function which has its only jump of $\alpha_1 \cdot \alpha_2$ *at* $a_1 + a_2$.

PROOF OF THEOREM 2.1. We define $H_n(t)$ by induction as follows

(2. 5)
$$H_n(t) = \int_{-\infty}^{\infty} H_{n-1}(t-u) dh_n(u).$$

By induction one sees easily that Lemmas 2.2 and 2.3 yield (1) and (2) (we have to recall that $h_i(t)$ has by Lemma 2.2 its only jump of a_i/c_i at $a_i^{-1}-c_i^{-1}$). Now Theorem 16a of [7, p. 257] yields (3) and therefore Theorem 5.6 of [7, p. 242] yields (4). Q.E.D.

COROLLARY 2.4. *$H_n(t)$ defined by Theorem* 3.1 *satisfies*

$$H_n(t)-H_n(0)=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{-iT}^{iT}\frac{e^{st}-1}{sF_n(s)}ds. \tag{2. 6}$$

THEOREM 2.5. *Let $F(s)$ be defined by* (1.1), *then there exists a function $H(t)$ satisfying*;

(1) *$H(t)$ is a normalized non-decreasing function, $H(-\infty)=0$ and $H(\infty)=1$,*

(2) *for $-\infty<y<\infty$*

$$F(iy)^{-1}=\int_{-\infty}^{\infty}e^{-iyt}dH(t),$$

(3)
$$H(t)-H(0)=\frac{1}{2\pi i}\lim_{T\to\infty}\int_{-iT}^{iT}\frac{e^{st}-1}{sF(s)}ds.$$

PROOF. For every $A>0$

$$\lim_{n\to\infty}\mathrm{F}_n(iy)^{-1}=F(iy)^{-1}$$

uniformly in $-A\leqq y\leqq A$ where $F_n(s)$ are defined in Theorem 2.1. Therefore by Theorem 2.1 here and Corollary 2.3 of [3, p. 41] there exists a function $H_*(t)$ satisfying: (a) $H_*(t)$ is non-decreasing, (b) $H_*(-\infty)=0$ and $H_*(\infty)=1$,(c) $\lim_{n\to\infty}H_n(t)=H_*(t)$ in all points of continuity of $H_*(t)$, (d) $\frac{1}{F(iy)}=\int_{-\infty}^{\infty}e^{-iyt}dH_*(t)$.

Using Theorem 4.5 of [8, vol. 2, pp. 259-260] we obtain that $H(t)$ derived from $H_*(t)$ by normalization satifies assumption (1), (2) and (3) of our theorem. Q.E.D.

For the following theorems let us recall the definition of $N=N(\{a_k\},\{c_k\})$ introduced in [1]

$N=\liminf_{x\to\infty}[N(\{a_k\},x)-N(\{c_k\},x)]+\liminf_{x\to-\infty}[N(\{a_k\},x)-N(\{c_k\},x)]\equiv N_+ + N_-$, where $N(\{a_k\},x)$ is the number of a_k's between zero and x.

We shall also need the following definition.

DEFINITION. The meromorphic function $F(s)$ satisfying (1.1) will satisfy condition $A(n)$ if there exists a function $\chi(\tau)>0$ such that $\int_0^{\infty}\chi(\tau)\cdot\tau^{-1}d\tau<\infty$ and

(2.7) $$|F(\sigma+i\tau)|^{-1}=O(|\tau|^{-n}\chi(\tau)) \qquad |\tau|\to\infty$$

uniformly in $-R<\sigma<R$ for any R.

THEOREM 2.6. *If in addition to the assumption of Theorem* 2.5 *we have that* $F(s)$ *satisfies* $A(0)$ (*as a special case we have* $N>0$) *then we have also*:

(4) $$F(s)^{-1}=\int_{-\infty}^{\infty} e^{-st}dH(t) \qquad \alpha_1<\text{Re } s<\alpha_2,$$

(5) $$H(t)=\int_{d-i\infty}^{d+i\infty}\frac{e^{st}ds}{sF(s)} \qquad \textit{for all } d,\ 0<d<\alpha_2.$$

PROOF. Since $\dfrac{e^{st}-1}{sF(s)}$ is regular in $\alpha_1<\text{Re}\, s<\alpha_2$ the residue theorem yields for $0<d<\alpha_2$

$$0=\left\{-\int_{-iT}^{iT}+\int_{d-iT}^{d+iT}+\int_{-iT}^{d-iT}-\int_{iT}^{d+iT}\right\}\frac{e^{st}-1}{sF(s)}ds\equiv I_1+I_2+I_3+I_4.$$

Using Theorem 2.1 of [1] or (2.7) with $n=0$ we have

$$\lim_{T\to\infty} I_3=\lim_{T\to\infty} I_4=0.$$

The above calculation with condition $A(0)$ imply

$$H(t)-H(0)=\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{e^{st}ds}{sF(s)}-\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{ds}{sF(s)}.$$

Letting $t\to-\infty$ we have $H(0)=\dfrac{1}{2\pi i}\displaystyle\int_{d-i\infty}^{d+i\infty}\frac{ds}{sF(s)}$ and therefore conclusion (5) is valid. A similar method will yield for $\alpha_1<d_1<0$

(2.8) $$H(t)-1=\frac{1}{2\pi i}\int_{d_1-i\infty}^{d_1+i\infty}\frac{e^{st}ds}{sF(s)}.$$

Combining (5), (2.8) and the condition $A(0)$ we obtain for some positive K

(2.9) $$H(t)\leqq Ke^{dt} \quad 0<d<\alpha_2 \quad \text{and} \quad 1-H(t)\leqq Ke^{d_1t} \quad \alpha_1<d_1<0$$

which implies (4). Q.E.D.

REMARK. The condition $A(0)$ is not fully required here, it is enough for (2.7) to hold uniformly in $\alpha_1 < \mathrm{Re}\, s < \alpha_2$.

3. Continuity and differentiality of $H(t)$.

THEOREM 3.1. *If $F(s)$ satisfies condition $A(n)$, $H(t) \in C^n(-\infty, \infty)$.*

PROOF. One can easily see that for $k \leqq n$

$$H^{(k)}(t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} \frac{s^k ds}{sF(s)} \qquad 0 < d < \alpha_2 \tag{3.1}$$

which immediately implies $H(t) \in C^n(-\infty, \infty)$. Q.E.D.

COROLLARY 3.2. *If $N_+ + N_- > n$ then $H(t) \in C^n(-\infty, \infty)$ and if $N_+ + N_- = \infty$ then $H(t) \in C^\infty(-\infty, \infty)$.*

DEFINITION. $H'(t) = G(t)$ if it exists.

THEOREM 3.3. *If $N_+ + N_- \geqq n$ then there exists a function $h(t)$ of bounded variation such that $h(t) = H^{(n)}(t)$ at all but a denumerable subset of $(-\infty, \infty)$ at most.*

PROOF. Rearranging the sequences $\{a_k\}$ and $\{c_k\}$ by a method similar to that used in the proof of Theorem 2.2 in [9] we obtain for $N_+ + N_- \geqq n$

$$F(s) \equiv \prod_{i=1}^{n} (1 - s/a_{k(i)}) \exp(s/a_{k(i)}) \cdot \prod_{j=1}^{\infty} [(1 - s/a_{k_j})/(1 - s/c^*_{k_j})] \exp(-s a_{k_j}^{-1} + s c^{*-1}_{k_j})$$

$$\equiv F_1(s) \cdot F_2(s),$$

where $0 \leqq a_{k_j}/c^*_{k_j} < 1$. Define $H_l(t)$ for $l = 1, 2$ by

$$H_l(t) = H_l(0) + \frac{1}{2\pi i} \lim_{T\to\infty} \int_{-iT}^{iT} \frac{e^{st} - 1}{sF_l(s)} ds$$

$H_2(t)$ is a distribution function by Theorem 2.5. $H_1(t)$ is well known by results on $G_1(t)$ in [3, Ch. II], $G_1(t)$ defined by

$$H_1(t) = \int_{-\infty}^{t} G_1(t) dt$$

where

$$\frac{1}{F_1(s)}=\int_{-\infty}^{\infty} e^{-st}G_1(t)dt.$$

By Theorems 6.3 and 8.2 of [3, p. 25 and p. 31] $H_1(t)\in C^{n-1}$ and moreover $H_1^{(n)}(t)$ is continuous except at

$$t=t_n\equiv\sum_{i=1}^{n} a_{k(i)}^{-1}$$

where it is not defined. In fact by using Theorem 8.2 of [3, p.31] and a simple calculation $H_1^{(n)}(t)$ is of bounded variation with a single jump at t_n.

$$H(t)=\int_{-\infty}^{\infty} H_1(t-u)dH_2(u).$$

The integral

$$h(t)=\int_{-\infty}^{\infty} H_1^{(n)}(t-u)dH_2(u)$$

is defined everywhere except at $\{t_n+P_{H_2}\}$ where P_{H_2} is the set of discontinuities of $H_2(t)$ and by Theorem 12 of [7, p. 250] $h(t)$ is of bounded variation in $(-\infty,\infty)$. Straightforward computation shows that at points different from $\{t_n+P_{H_2}\}$ $h(t)=H^{(n)}(t)$ and $H^{(n)}(t)$ exist there. Q.E.D.

COROLLARY 3.4. *If* $N_+ + N_- \geqq n \geqq 1$ *then there exists a normalized function* $G(t)$, $G(t)=H'(t)$ *(at all but a denumerable set at most).*

THEOREM 3.5. *If for* $F(s)$ $N_+ + N_- \geqq 2$ *then a density function* $G(t)$ *exists satisfying*:

(1) $$\frac{1}{F(s)}=\int_{-\infty}^{\infty} e^{-st}G(t)dt \qquad \alpha_1<\operatorname{Re} s<\alpha_2.$$

(2) $$G(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{st}}{F(s)}ds.$$

(3) $$G^{(n)}(t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{s^n e^{st}}{F(s)}ds \qquad n\leqq N_+ + N_- -2.$$

PROOF. Theorem 3.1 yields $G(t)\in C$ and since $H'(t)=G(t)$, $G(t)\geqq 0$.

Theorem 2.6 implies (1). Formulae (2) and (3) are immediate corollaries of (1) and Theorem 2.2 of [1]. Q.E.D.

THEOREM 3.6. *Suppose*: (1) $F_r(s) \in F$. (2) *An integer* p *exists such that for all* $F_r(s)$ $N_+ + N_- \geqq p+2 \geqq 2$. (3) $H_r(t) \equiv \dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} \frac{e^{st}}{sF_r(s)}\,ds$ *satisfies* $\lim_{r\to\infty} H_r(t) = H_0(t)$ (*at points of continuity of* $H_0(t)$). (4) *There exists a* $B > 0$ *independent of* r *such that*

$$|F_r(iy)|^{-1} \leqq (1 + By^{2N_+ + 2N_-})^{-1/2}. \qquad n=0,1,2,\cdots.$$

Then

$$\lim_{r\to\infty} \|G_r^{(l)}(t) - G_0^{(l)}(t)\|_\infty = 0 \qquad l=0,1,\cdots,p$$

(*where* $\|f(x)\|_\infty = \sup_x |f(x)|$).

PROOF. It would be enough to prove the theorem for $l=p$. By Theorem 2.2 of [3, p.14] and Theorem 2.1 of [1] we have

$$\lim_{r\to\infty} F_r(iy)^{-1} = F_0(iy) \tag{3.2}$$

uniformly for $|y| \leqq A$ for any positive A.
By (3) of Theorem 3.5 we may write

$$\|G_r^{(p)}(t) - G_0^{(p)}(t)\|_\infty \leqq \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \frac{y^p}{F_r(iy)} - \frac{y^p}{F_0(iy)} \right| dy.$$

One can easily conclude the proof splitting the integral into the following three parts

$$\int_{-\infty}^{\infty} \cdots = \left\{ \int_{-\infty}^{-A} + \int_{-A}^{A} + \int_{A}^{\infty} \right\} \cdots.$$

Estimating the first and the third by (4) and the second by (3.2). Q.E.D.

REMARK. r may be both a continuous parameter or a sequence of integers.

REMARK. In Theorem 3.5 $A(1)$ can replace the condition $N_+ + N_- \geqq 2$ and also the conditions of Theorem 3.6 can be reduced but we can use the condition related to the zeros and the poles more readily.

4. Asymptotic behaviour. Denote by μ_1+1 the number of a_k equal to α_1 and by μ_2+1 the number of a_k equal to α_2.

THEOREM 4.1. *If $F(s)\in F$ has infinitely many zeros and $N_+ + N_- \geqq 2$, then for all n satisfying $0 \leqq n \lneqq N_+ + N_- - 2$ or if condition $A(n+1)$ is satisfied, one has*

A. $\alpha_1 > -\infty$ *implies*

$$G^{(n)}(t)=[e^{\alpha_1 t}p(t)]^{(n)}+O(e^{kt}) \quad t\to\infty$$

for any k satisfying $\max\{a_r, -\infty \mid a_r<0, a_r \neq \alpha_1\} < k < \alpha_1$ *and $p(t)$ is a real polynomial of degree μ_1.*

B. $\alpha_1 = -\infty$ *implies*

$$G^{(n)}(t)=O(e^{kt}) \qquad t\to\infty$$

for every negative k.

C. $\alpha_2 < \infty$ *implies*

$$G^{(n)}(t)=[e^{\alpha_2 t}q(t)]^{(n)}+O(e^{kt}) \qquad t\to-\infty$$

for any k satisfying $\alpha_2 < k <$ $\min\{a_k, -\infty \mid a_k>0, a_k \neq \alpha_2\}$ *and $q(t)$ is a real polynomial of degree μ_2.*

D. $\alpha_2 = \infty$ *implies*

$$G^{(n)}(t)=O(e^{kt}) \qquad t\to-\infty$$

for every positive k.

The proof is standard following that of I.I. Hirschman and D.V.Widder (see [3, p. 108]) and was used on many occasions. The estimations of $F(s)$ used are mainly those of [1].

REMARK. In case there are some (or infinitely many) different negative zeros of $F(s)$ we can define A_k by $A_1=\alpha_1$,

$$A_k=\max\{a_r,-\infty \mid a_r<0,\ a_r\neq A_p\ 1\leqq p<k\}$$

and if there are at least m finite A_k's

$$G(t)=\sum_{i=1}^{m} P_i(t)e^{A_i t}+O(e^{At}) \qquad t\to\infty$$

where A satisfies $\max\{a_r, -\infty \mid a_r<0,\ a_r\neq A_p,\ 1\leqq p\leqq m\} < A < A_m$, and a similar result is achieved in case $F(s)$ has a number of different positive zeros.

5. Asymptotic estimates in case $F(s)$ has only positive zeros. The restriction is really that either all a_k are positive or all are negative and in the second case treat $G(-t)$ (which has positive a_k's).

For $F(s) \in F$ we shall define;

$$\lambda(r)=\sum_{k=1}^{\infty}\frac{r}{a_k(a_k+r)}-\sum_{k=1}^{\infty}\frac{r}{c_k(c_k+r)}, \tag{5.1}$$

$$\sigma(r)=\left[\sum_{k=1}^{\infty}\left(\frac{1}{(a_k+r)^2}-\frac{1}{(c_k+r)^2}\right)\right]^{1/2} \tag{5.2}$$

and

$$\Lambda(r)=e^{-r\lambda(r)}[\sigma(r)F(-r)]^{-1}. \tag{5.3}$$

These definitions are analogous to those of I.I. Hirschman and D.V. Widder see [3, p.111].

DEFINITION. Suppose $F(s) \in F$, $a_k > 0$ and $N_+ = \infty$ then the corresponding $G(t)$ will belong to class B if there exists a subsequence $\{a_{k_i}\}$ of $\{a_k\}$ satisfying:

(1) $N_*(x)=N(\{a_k\},x)-N(\{a_{k_i}\},x)-N(\{c_{k_i}\},x)\geqq 0 \qquad$ for $x>0$,

(2) $$\sum_{i=1}^{\infty}(r+a_{k_i})^{-2}>\alpha\sigma(r)^2$$

for all r and for some fixed $\alpha, 0<\alpha<1$,

(3) $$\sum_{i=1}^{\infty}(r+a_{k_i})^{-3}=o(\sigma(r)^3) \qquad r\to\infty.$$

To show that not all those $G(t)$ for which $a_k>0$ and $N_+=\infty$ are of class B we will show that there exists a function $F(s)$ for which $a_k>0$, $0\leqq a_k/c_k<1$, $N_+=\infty$ but no subsequence of $\{a_k\}$ satisfies both (1) and (2) (each assumption alone is easily satisfied).

EXAMPLE 5.1. Let $F(s)$ be defined by a

$$a_k=2^n \qquad \text{for} \qquad \sum_{r=0}^{n-1}2^r+(n-1)\leqq k<\sum_{r=0}^{n}2^r+n$$

and

$$c_k = 2^{n+1} - 2^{-2n} \qquad \text{for} \qquad \sum_{r=0}^{n-1} 2^r \leqq k < \sum_{r=0}^{n} 2^r$$

for $n \geqq 1$.

It is easily seen that in order that (1) be satisfied and $\sum_{i=1}^{\infty} (r+a_{k_i})^{-2}$ be the greatest possible $k_i = \sum_{r=0}^{i-1} 2^r + i - 1$ and $a_{k_i} = 2^i$.

$$\begin{aligned} \sigma(r)^2 &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^n}{(r+2^n)^2} + \sum_{n=1}^{\infty} 2^n \left\{ \frac{1}{(r+2^{n+1})^2} - \frac{1}{(r+2^{n+1}-2^{-2n})^2} \right\} \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{(r+2^n)^2} + \frac{1}{(r+2)^2} \\ &\geqq \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(r+2^n)^2} - \sum_{n=1}^{\infty} \frac{2^{-n}}{(r+2^{n+1})(r+2^{n+1}-2^{-2n})} \\ &\geqq \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^n}{(r+2^n)^2} . \end{aligned}$$

It is obvious that

$$\sum_{i=1}^{\infty} (r+a_{k_i})^{-2} = \sum_{n=1}^{\infty} (r+2^n)^{-2} = o\Big(\sum_{n=1}^{\infty} \frac{2^n}{(r+2^n)^2} \Big) = o(\sigma(r)^2) \qquad r \to \infty .$$

We also can see that (3) is satisfied. One can also mention that in this example $\sum (a_k^{-1} - c_k^{-1}) = \infty$.

REMARK 5.2. If $F(s) \in F$, $a_k > 0$ and $N_+ = \infty$ then the existence of $\{a_{k_i}\}$ satisfying (1) of the definition of class B and

$$\sum_{i=1}^{\infty} (r+a_{k_i})^{-2} > \beta \sum_{k=1}^{\infty} (r+a_k)^{-2} \tag{4}$$

for all $r > 0$ and for some $\beta > 0$ implies conditions (1) and (2) of class B. Condition (4) obviously implies condition (2) since by (1)

$$\sigma(r)^2 < \sum_{k=1}^{\infty} (r+a_k)^{-2}.$$

$N_+ = \infty$ implies $r\sigma(r) \to \infty$ and since $\sigma(r)^2 > \sum_{i=1}^{\infty} (r+a_{k_i})^{-2}$ we have

$$\frac{1}{\sigma(r)^3}\sum_{k=1}^{\infty}(r+a_k)^{-3}\leqq\frac{1}{r\sigma(r)}\left\{\frac{1}{\sigma(r)^2}\sum_{k=1}^{\infty}(r+a_k)^{-2}\right\}$$

$$\leqq\frac{1}{r\sigma(r)}B=o(1)\qquad r\to\infty.$$

REMARK. It is not hard to see that if $a_n>0$, $F(s)\in F$, $a_n\leqq Kn^{\gamma}$ $\gamma>1/2$ and $K_1n^{\gamma_1}\leqq c_n$ for $n\geqq n_0$ where $\gamma_1\geqq\gamma$ and when $\gamma_1=\gamma$, $K_1>K$ then both $N_+=\infty$ and $G(t)$ belongs to B. In fact all the above mentioned transforms satisfy (4) of Remark 5.2 not only (2) and (3) of the definition of B. The same is true if we mix two or more sequences of a_k's and c_k's respectively of the type mentioned above. It seems to us that the transforms that were dealt with as a special case of the class F of convolution transforms are of the above mentioned type (as a matter of fact $\gamma=\gamma_1=1$).

THEOREM 5.3. *If $G(t)$ belongs to class B then for $n\geqq 0$*

$$G^{(n)}[\lambda(r)]\sim(2\pi)^{-1/2}(-r)^n\Lambda(r)\qquad r\to\infty.$$

PROOF. Since $N_+=\infty$ we obtain by substitution for all $r>0$

$$G^{(n)}(u)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{s^ne^{su}}{F(s)}ds \tag{5.4}$$

$$=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}(s-r)^ne^{(s-r)u}[F(s-r)]^{-1}ds.$$

Using the Residue theorem on the rectangle $\pm iR$, $r\pm iR$, and since $F(s-r)^{-1}$ is regular in this rectangle (for all R) we have

$$G^{(n)}(u)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(s-r)^ne^{-(s-r)u}[F(s-r)]^{-1}ds$$

$$=\frac{e^{ru}(-r)^n}{\sigma(r)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(1-\frac{s}{r\sigma(r)}\right)^ne^{su/\sigma(r)}\left[F\left(\frac{s}{\sigma(r)}-r\right)\right]^{-1}ds.$$

Define now

$$A_k(r)=\sigma(r)(a_k+r)\quad\text{and}\quad C_k(r)=\sigma(r)(c_k+r). \tag{5.5}$$

Therefore by a similar method to [3, p. 112]

$$F\left(\frac{s}{\sigma(r)}-r\right)=F(-r)\prod_{k=1}^{\infty}(1-s/A_k(r))\exp(s/a_k\cdot\sigma(r))\Big/\prod_{k=1}^{\infty}(1-s/C_k(r))\exp(s/c_k\cdot\sigma(r))$$

$$= \exp(\lambda(r)s/\sigma(r))F(-r) \prod_{k=1}^{\infty} (1-s/A_k(r))\exp(s/A_k(r)) \Big/ \prod_{k=1}^{\infty} (1-s/C_k(r))\exp(s/C_k(r))$$
$$\equiv \exp(\lambda(r)s/\sigma(r))F(-r)F_r(s).$$

From which follows

$$(5.6)\quad G^{(n)}(u) = \frac{e^{-ru}(-r)^n}{F(-r)\sigma(r)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(1 - \frac{s}{r\sigma(r)}\right)^n [F_r(s)]^{-1} \cdot \exp\left(\frac{s}{\sigma(r)}(u-\lambda(r))\right) ds.$$

Substituting $u=\lambda(r)$

$$G^{(n)}(\lambda(r)) = \Lambda(r)(-r)^n I_r \equiv \Lambda(r)(-r)^n \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [F_r(s)]^{-1} ds.$$

By (5.2) and (5.5) one can see that

$$(5.7)\quad \sum_{k=1}^{\infty} A_k(r)^{-2} - \sum_{k=1}^{\infty} C_k(r)^{-2} = 1.$$

Choose the sequence a_{k_i} as in the definition of class B and define

$$\sigma_1(t)^2 = \sum_{i=1}^{\infty} (a_{k_i}+r)^{-2}.$$

Obviously $\lim_{r\to\infty} r^2\sigma_1(r)^2 = \infty$ and since $\sigma_1(r)^2 < \sigma(r)^2$ we have $\lim_{r\to\infty} r^2\sigma(r)^2 = \infty$. Denote

$$(5.8)\quad A(r) \equiv \sigma(r)(r+\alpha_2).$$

Recalling the inequality

$$\left| \log\left\{(1-s)\exp\left(s+\frac{1}{2}s^2\right)\right\} \right| \leqq 2|s|^3 \qquad \text{for } |s| < \frac{1}{2}$$

(see [3, p. 113]) we obtain for $|s| \leqq \frac{1}{2} A(r)$

(5.9) $$|\log\{F_r(s)e^{s^2/2}\}|\leqq 2|s|^3\Big(\sum_{k=1}^{\infty}A_k(r)^{-3}+\sum_{k=1}^{\infty}C_k(r)^{-3}\Big)$$
$$\leqq 4|s|^3\sum_{k=1}^{\infty}A_k(r)^{-3}.$$

Writing (3) of the definition of class B we have

(5.10) $$\sum_{k=1}^{\infty}A_k(r)^{-3}=o(1) \qquad r\to\infty.$$

This yields

(5.11) $$\lim_{r\to\infty}F_r(s)=e^{-s^2/2}$$

uniformly in every disc $|s|\leqq K$ $(K<\infty)$.

For every $N>0$ and all real y

(5.12) $$\exp\Big(-\frac{1}{2}(iy)^2\Big)^{-2}\leqq\Big(1+\frac{1}{N!}y^{2N}\Big)^{-1}.$$

Denoting $\sigma_1(r)^2=\sum_{i=1}^{\infty}(a_{k_i}+r)^{-2}$ we shall prove that since $\sum_{i=1}^{\infty}[\sigma_1(r)(a_{k_i}+r)]^{-2}=1<\infty$ there exists for every integer N, a constant $B(N)$, $B(N)>0$ independent of r such that

(5.13) $$|F_r(iy)|^{-2}\leqq[1+B(N)y^{2N}]^{-1}$$
$$|F_r(iy)|^{-2}\leqq\Big|\prod_{i=1}^{\infty}\Big(1-\frac{iy}{\sigma(r)(a_{k_i}+r)}\Big)\Big|^{-2}\cdot\Big|\prod_{k=1}^{\infty}\frac{(1-iy/\sigma(r)(a_k^*+r))}{(1-iy/\sigma(r)(c_k^*+r))}\Big|^{-2}$$

where $\{a_k^*\}$ and $\{c_k^*\}$ are subsequences of $\{a_k\}$ and $\{c_k\}$ such that $a_k^*\leqq a_{k+1}^*$, $c_k^*\leqq c_{k+1}^*$; $\{a_k^*\}$ is the sequence $\{a_k\}$ from which $\{a_{k_i}\}$ were omitted and $\{c_k^*\}$ is the rearranged sequence $\{c_k\}$ from which at most infinite $+\infty$ terms were omitted. Obviously $0\leqq a_k^*/c_k^*<1$ and $0\leqq(a_k^*+r)/(c_k^*+r)<1$ and by Theorem 2.1 of [1]

$$|F_r(iy)|^{-2}\leqq\Big|\prod_{i=1}^{\infty}\Big(1-\frac{i\Big(y\frac{\sigma_1(r)}{\sigma(r)}\Big)}{\sigma_1(r)(a_{k_i}+r)}\Big)\Big|^{-2}=\prod_{i=1}^{\infty}\Big(1+\frac{\Big(y\frac{\sigma_1(r)}{\sigma(r)}\Big)^2}{\sigma_1(r)^2(a_{k_i}+r)^2}\Big)^{-1}.$$

Now we have by the argument used in [3, pp. 64-65, pp.111-113]

$$\prod_{i=1}^{\infty}\left(1+\frac{\tau^2}{[\sigma_1(r)(a_{k_i}+r)]^2}\right)\geqq 1+B_1(N)\tau^{2N}$$

(in fact $B_1(N)$ is as near to $\frac{1}{N!}$ from below as we wish). Choosing $B(N) = B_1(N)\alpha^{2N}$ we complete the proof of (5.13).

Using Theorem 3.6 where $G_0(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ and $F_0(s)=e^{s^2/2}$ we get

$$\lim_{r\to\infty}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{s^m e^{st}}{F_r(s)}ds=\left(\frac{d}{du}\right)^m((2\pi)^{-1/2}e^{-u^2/2})_{u=0}.$$

We have

$$I_r=\frac{1}{2\pi i}\sum_{m=0}^{n}(-r\sigma(r))^{-m}\binom{n}{m}\int_{-i\infty}^{i\infty}s^m F_r(s)^{-1}e^{st}\,ds$$

but since $r\sigma(r)\to\infty$

(5.14) $$I_r=\frac{1}{\sqrt{2\pi}}(1+o(1))\qquad r\to\infty\,.$$

Combining (5.6) and (5.14) we complete the proof. Q.E.D.

6. The positive character of $G(t)$. It is known that $G(t)$ as described in the former sections satisfies $G(t)\geqq 0$. It is interesting to know if $G(t)>0$ (at least on a ray) which will generalize results by Hirschman and Widder and permit us to treat more asymptotic properties in the next section.

We define three classes of kernels:

$F(s)\in F$ belongs to class I if there exist k and j such that $a_k\cdot a_j<0$.

$F(s)\in F$ belongs to class II if $a_k>0$ for all k and

$$\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})=\infty\,.$$

$F(s)\in F$ belongs to class III if $a_k>0$ for all k and

$$\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})<\infty\,.$$

Either $F(s)$ or $F(-s)$ is in one of these classes. One can assume $a_k\leqq a_{k+1}$

and $c_k \leqq c_{k+1}$ for convergence or divergence of $\sum (a_k^{-1}-c_k^{-1})$ since $F(s)$ as a meromorphic function for which the order of zeros and poles is not important, one can also show that changes in order of a_k's and c_k's that preserve $0 \leqq a_k/c_k < 1$ preserve the sum.

THEOREM 6.1. *If $F(s) \in F$ and $F(s)$ belong to class I then $H(t)$ is strictly monotonic.*

LEMMA 6.2. *Let $-\infty < \gamma_1 < \alpha_1 < 0$ and $0 < \alpha_2 < \gamma_2 < \infty$ and*

$$F_1(s) = \frac{(1-s/\alpha_1)(1-s/\alpha_2)}{(1-s/\gamma_1)(1-s/\gamma_2)}$$

then

$$h(t) = \begin{cases} \dfrac{\gamma_2-\alpha_2}{\gamma_2}\,\dfrac{\alpha_1}{\gamma_1}\left(\dfrac{\gamma_1-\alpha_1}{\alpha_1-\alpha_2}+1\right)e^{\alpha_2 t} & t<0 \\[2ex] \dfrac{\alpha_1}{\gamma_1\gamma_2}\left(\dfrac{\gamma_1-\alpha_1}{\alpha_1-\alpha_2}(\gamma_2-\alpha_2)+\dfrac{2\gamma_2-\alpha_2}{2}\right) & t=0 \\[2ex] 1-\dfrac{\gamma_1-\alpha_1}{\gamma_1}\left(1-\dfrac{\alpha_1}{\alpha_1-\alpha_2}\,\dfrac{\gamma_2-\alpha_2}{\gamma_2}\right)e^{\alpha_1 t} & t>0 \end{cases} \tag{6.1}$$

satisfies

$$F_1(s)^{-1} = \int_{-\infty}^{\infty} e^{-st}dh(t).$$

PROOF. Since

$$h(t) = \int_{-\infty}^{8} h_1(t-u)dh_2(u)$$

where

$$h_1(u) = \begin{cases} 0 & u<0 \\ \alpha_1/2\gamma_1 & u=0 \\ 1-\dfrac{\gamma_1-\alpha_1}{\gamma_1}\,e^{\alpha_1 u} & u>0 \end{cases}$$

and

$$h_2(u)=\begin{cases}\frac{\gamma_2-\alpha_2}{\gamma_2}e^{\alpha_2 u} & u>0\\ 1-\alpha_2/2\gamma_2 & u=0\\ 0 & u>0\end{cases}$$

the proof is just straight forward calculation. Q.E.D.

LEMMA 6.3. *Let* $\alpha_1<0,\ 0<\alpha_2<\gamma_2<\infty$ *and*

$$F_1(s)=\frac{(1-s/\alpha_1)(1-s/\alpha_2)}{(1-s/\gamma_2)}$$

then

$$h(t)=\begin{cases}\frac{\gamma_2-\alpha_2}{\gamma_2}\frac{\alpha_1}{\alpha_1-\alpha_2}e^{\alpha_2 t} & t\leqq 0\\ 1-\left(1-\frac{\gamma_2-\alpha_2}{\gamma_2}\frac{\alpha_1}{\alpha_1-\alpha_2}\right)e^{\alpha_1 t} & t>0\end{cases}\tag{6.2}$$

satisfies

$$F_1(s)^{-1}=\int_{-\infty}^{\infty}e^{-st}\,dh(t).$$

PROOF. Simple calculation. Q.E.D.

PROOF OF THEOREM 6.1. Define α_1 and α_2 as in (2.1) and γ_i, $i=1,2$ as follows

$$\gamma_1=\max\{c_k,-\infty\,|\,c_k<0\},\ \gamma_2=\min\{c_k,\infty\,|\,c_k>0\}.\tag{6.3}$$

If $\gamma_1=-\infty$ and $\gamma_2=\infty$ our theorem is the classical result of Hirschman and Widder. Define $h(t)$ as in Lemma 6.2 in case $\gamma_1>-\infty$, $\gamma_2<\infty$ and in Lemma 6.3 when $\gamma_1=-\infty$, $\gamma_2<\infty$. In these cases

$$H(t)=\int_{-\infty}^{\infty}H_2(t-u)dh(u)=\int_{-\infty}^{\infty}h(t-u)dH_2(u)$$

where $H_2(u)$ satisfies

$$F_2(iy)^{-1} = \int_{-\infty}^{\infty} e^{iyt} dH_2(t)$$

and

$$F(s) = F_1(s) \cdot F_2(s).$$

There exists a constant A such that

$$\int_{-A}^{A} dH_2(u) \geqq \frac{1}{2},$$

$$H(t+h) - H(t) = \int_{-\infty}^{\infty} [h(t-u+h) - h(t-u)] dH_2(u)$$

$$\geqq \int_{-A}^{A} [h(t-u+h) - h(t-u)] dH_2(u).$$

One can see from (6.1) and (6.2) that $h(t)$ is strictly monotonic and therefore exists a constant $m(t, A, h) > 0$ such that

$$h(t-u+h) - h(t-u) \geqq m(t, A, h) > 0 \qquad \text{for } -A < u < A$$

and therefore

$$H(t+h) - H(t) \geqq \frac{1}{2} m(t, A, h) > 0.$$

If $\gamma_1 > -\infty$ and $\gamma_2 = \infty$ we shall treat $H(-t)$ instead of $H(t)$. Q.E.D.

COROLLARY 6.4. *If $F(s) \in F$ and class I; and $N_+ \geqq 1$ then $\frac{1}{2}(G(t+) + G(t-)) > 0$ (where $G(t \pm h) - G(t\pm) = o(1)$ $h \downarrow 0$).*

For the next positivity theorem we need the following lemmas.

LEMMA 6.5. *Let $c_k > a_k > 0$, $F(s) = \prod_{k=1}^{n} [(1 - s/a_k)/(1 - s/c_k)]$, then the corresponding $H_n(t)$ is strictly increasing in $t < 0$ and 1 for $t > 0$.*

PROOF. For $n=1$ this is a simple corollary of 2.1. Assume it for $n = l-1$.

$$H_l(t) = \int_{-\infty}^{\infty} H_{l-1}(t-u)\, dh_l(u)$$

where

$$[(1-s/c_l)/(1-s/a_l)] = \int_{-\infty}^{\infty} e^{-st} dh_l(u).$$

For $t<0$ choose $0<h<-t/4$

$$H_l(t+h)-H_l(t) = \int_{-\infty}^{\infty} [H_{l-1}(t+h-u) - H_{l-1}(t-u)] dh_l(u)$$

$$\geqq \int_t^0 [H_{l-1}(t+h-u) - H_{l-1}(t-u)] dh_l(u) \geqq \int_t^{t/2} [H_{l-1}(t-u+h) - H_{l-1}(t-u)]\, dh_l(u)$$

$$\geqq m[h_l(t/2) - h_l(t)]$$

where

$$m = \inf_{t \leqq u \leqq t/2} [H_{l-1}(t-u+h) - H_{l-1}(t-u)].$$

One can see $m>0$; assume $m=0$ then a sequence u_n, $u_n \to u_0 \leqq t/2$ exists such that $[H_{l-1}(t-u_n+h) - H_{l-1}(t-u_n)] < \frac{1}{n}$ from which one can see

$$H_{l-1}\left(t-u_n+\frac{3}{4}h\right) - H_{l-1}\left(t-u_n+\frac{1}{4}h\right) < \frac{1}{n}$$

and therefore

$$A = H_{l-1}\left(t-u_0+\frac{2}{3}h\right) - H_{l-1}\left(t-u_0+\frac{1}{3}h\right) < \frac{1}{n} \quad \text{for all } n \geqq n_0$$

and this yields $A \leqq 0$ but on the other hand strict monotonicity of $H_{l-1}(t)$ in $t<0$ contradicts $A \leqq 0$. Now we have

$$H_l(t+h) - H_l(t) \geqq m(h_l(t/2) - h_l(t)) > 0.$$

For $t>0$

$$H_l(t) = \int_{-\infty}^{\infty} H_{l-1}(t-u) dh_l(u) = \int_{-\infty}^{0} H_{l-1}(t-u) dh_l(u) = \int_{-\infty}^{0} dh_l(u) = 1.$$

DEFINITION. The n-th moment of $H(t)$:

$$M_1 = \int_{-\infty}^{\infty} t\,dH(t) \quad \text{and} \quad M_n = \int_{-\infty}^{\infty} (t-M_1)^n dH_1(t). \tag{6.4}$$

LEMMA 6.6. *Let $F(s)\in F$, $N_+ + N_- \geqq 1$ then for the $H(t)$ corresponding to $F(s)$, $M_1=0$ and*

$$M_2=\sum_{k=1}^{\infty}(a_k^{-2}-c_k^{-2}).$$

PROOF. The standard proof used by Hirschman-Widder, Tanno and others applies here where $N_+ + N_- \geqq 1$ implies convergence via (2.9). Q.E.D.

LEMMA 6.7. *Let $F(s)\in F$ and $N_+ + N_- \geqq 1$ then*

$$H(t)\leqq \frac{1}{t^2}\sum_{k=1}^{\infty}(a_k^{-2}-c_k^{-2}) \qquad \text{for } t<0$$

$$1-H(t)\leqq \frac{1}{t^2}\sum_{k=1}^{\infty}(a_k^{-2}-c_k^{-2}) \qquad \text{for } t>0.$$

PROOF. For $t<0$

$$H(t)=\int_{-\infty}^{t} dH(u)\leqq \int_{|u|>|t|} dH(u)\leqq \frac{1}{t^2}\int_{-\infty}^{\infty} u^2 dH(t)$$
$$\leqq \frac{1}{t^2}\sum_{k=1}^{\infty}(a_k^{-2}-c_k^{-2}).$$

For $t>0$ the proof is similar. Q.E.D.

THEOREM 6.8. *If $F(s)\in F$ $N_+ + N_- \geqq 1$ and $a_k>0$ then $(1/2)(G(t+)+G(t-))>0$ in $-\infty<t<\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})$ when $F(s)$ is of class* III *and* $(1/2)$ $(G(t+)+G(t-))>0$ *always if $F(s)$ is of class* II.

PROOF. It is enough to show that $H(t)$ is strictly increasing for $t<\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})$ or always if $F(s)$ belong to classes III or II respectively. For every $t<\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})$ there exists a δ such that $t+2\delta<\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})$. Choose n so that

$$S_n=\sum_{k=n+1}^{\infty}(a_k^{-2}-c_k^{-2})<\frac{\delta^2}{3} \quad\text{and}\quad t+2\delta<\sum_{k=1}^{n}(a_k^{-1}-c_k^{-1}).$$

Let $F_1(s)$ be defined by

$$F_1(s)=\prod_{k=1}^{n}[(1-s/a_k)\exp(s/a_k)/(1-s/c_k)\exp(s/c_k)].$$

Define $F_2(s)$ by $F_2(s)=F(s)/F_1(s)$. Define $H_i(s)$ by

$$F_i(s)^{-1}=\int_{-\infty}^{\infty}e^{-st}dH_i(t) \qquad i=1,2.$$

One can see easily by Lemma 6.5 that $H_1(t)$ is strictly increasing for

$$t<\sum_{k=1}^{n}(a_k^{-1}-c_k^{-1})\equiv t_n.$$

Choose h so that $t+3h<t_n$.

$$\begin{aligned}H(t+h)-H(t)&=\int_{-\infty}^{\infty}[H_1(t-u+h)-H_1(t-u)]dH_2(u)\\&\geqq\int_{t-\delta}^{t+\delta}[H_1(u+h)-H_1(u)]dH_2(t-h)\\&\geqq\inf_{t-\delta\leqq u\leqq t+\delta}[[H_1(u+h)-H_1(u)]\cdot\int_{-\delta}^{\delta}dH_2(t)\\&\geqq m\cdot\left(1-\frac{2S_n}{\delta^2}\right)\geqq\frac{m}{3}.\end{aligned}$$

By considerations similar to those of Lemma 6.5 $m>0$. Q.E.D.

7. More Asymptotic estimates. Section 6 permits us to write at least in case $N_+ + N_- \geqq 2$ $G(t)=e^{-\chi(t)}$ where $F(s)$ is of class I or II and $G(t)=e^{-\chi(t)}$ for $t<\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})$ where $F(s)$ is of class III.

Theorem. 5.1 with the above notation yields

(7.1) $$\chi'(\lambda(r))\sim r,\quad r\to\infty \quad \text{where} \quad F(s)\in \text{class } B.$$

Define the function $M(t)$ when $F(s)\in$ class III by

(7.2) $$t=\sum_{k=1}^{\infty}[(M(t)+a_k)^{-1}-(M(t)+c_k)^{-1}], \qquad t>0.$$

Define the function $L(t)$ when $F(s)\in$ class II by

(7.3) $$t=\sum_{k=1}^{\infty} L(t)[(a_k(a_k+L(t)))^{-1}-(c_k(c_k+L(t)))^{-1}], \qquad t>0.$$

THEOREM 7.1. *Let $F(s)$ belong to class B then:*
(a) *$F(s)$ belongs to class III implies*

(7.4) $$\chi'\Big(\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})-t\Big)\sim M(t) \qquad t\downarrow 0,$$

(b) *$F(s)$ belongs to class II implies*

(7.5) $$\chi'(t)\sim L(t) \qquad t\to\infty.$$

PROOF. We shall prove (a) ((b) is similar)

$$\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})-t=\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})-\sum_{k=1}^{\infty}[(M+a_k)^{-1}-(M+c_k)^{-1}]=\lambda(M(t)).$$

Since $\chi'(\lambda(r))\sim r$ $(r\to\infty)$ and since $M(t)\to\infty$ when $t\downarrow 0$ we obtain

$$\chi'\Big(\sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})-t\Big)=\chi'(\lambda(M(t)))\sim M(t),\ t\downarrow 0. \qquad \text{Q.E.D.}$$

THEOREM 7.2. *If $F(s)$ belongs to class II and to class B, then*

$$\chi'(t)=L(t+o(1)) \qquad t\to\infty.$$

PROOF. The proof is analogous to that of Theorem 3.4 of [3, p.116]. Define

$$H_r'(u)\equiv\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}[F_r(s)]^{-1}e^{su}\,ds$$

where $A_k(r)=(a_k+r)\sigma(r)$, $C_k(r)=(c_k+r)\sigma(r)$ and

$$F_r(s)=\prod_{k=1}^{\infty}[(1-s/A_k(r))\exp(s/A_k(r))/(1-s/C_k(r))\exp(s/C_k(r))].$$

Via the proof of Theorem 5.1 we know

$$e^{ru-\chi(u)}=H_r'\Big(\frac{u-\lambda(r)}{\sigma(r)}\Big)\Big/\sigma(r)F(-r).$$

$F(s) \in$ class B implies

$$\lim_{r\to\infty} \|H_0^{(n)}(u) - H_r^{(n)}(u)\|_\infty = 0 \tag{7.6}$$

where

$$H_0'(u) = (2\pi)^{-1/2} e^{-u^2/2}. \tag{7.7}$$

It is clear that $H_0''(0)=0$, $H_0''(-\eta)=-H_0''(\eta)=\frac{\eta}{\sqrt{2\pi}} e^{-\eta^2/2}$, $H_0'''(u)=\frac{1}{\sqrt{2\pi}}(u^2-1)e^{-u^2/2}$ and therefore for $-\frac{1}{2}<u<\frac{1}{2}$

$$H_0'''(u) \leqq \frac{1}{2\sqrt{2\pi}} e^{-1/2}.$$

We have

$$H_0''\left(\frac{1}{n}\right) = \frac{-1}{n\sqrt{2\pi}} e^{-1/2n^2} \text{ and } H_0''\left(-\frac{1}{n}\right) = \frac{1}{n\sqrt{2\pi}} e^{-1/2n^2}$$

and therefore for each n we can choose $r \geqq r_n > r_1$ so that by (7.6)

$$H_r''\left(-\frac{1}{n}\right) \geqq \frac{1}{2n\sqrt{2\pi}} e^{-1/2},\ H_r''\left(\frac{1}{n}\right) \leqq \frac{-1}{2n\sqrt{2\pi}} e^{-1/2}$$

and also for $-1/n \leqq u \leqq 1/n$

$$H_r'''(u) \leqq \frac{-1}{2\sqrt{2\pi}} e^{-1/8} < 0.$$

From these inequalities the existence of one and only one $z(r)$ in $[-1, 1]$ such that for $r \geqq r_n$, $-1/n < z(r) < 1/n$ $H_r''(z(r))=0$ follows. Since

$$e^{ru-\chi(u)} = [F(-r)\sigma(r)]^{-1} H_r'\left[\frac{u-\lambda(r)}{\sigma(r)}\right]$$

$$(r-\chi'(u))e^{ru-\chi(u)} = \frac{1}{F(-r)\sigma(r)^2} H_r''\left(\frac{u-\lambda(r)}{\sigma(r)}\right)$$

$r=\chi'(u)$ for $z(r)=\frac{u-\lambda(r)}{\sigma(r)}$ and

$$\chi'(\lambda(r)+\sigma(r)z(r)) = r.$$

Since $z(r)$ is continuous for $r \geqq r_1$ and defining $r(t)$ by $t=\lambda(r)+\sigma(r)z(r)$

$$\chi'(t)=r(t)$$

$r(t)\to\infty$ whenever $t\to\infty$

$$t=\lambda(r(t))+\sigma(r(t))z(r(t))$$

and therefore

$$r(t)=L(t-\sigma(r(t))z(r(t)))$$

and hence

$$\chi'(t)=L(t+o(1)) \qquad r\to\infty. \qquad \text{Q.E.D.}$$

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DEPERTMENT OF MATHEMATICS
THE UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA, CANADA

AND

TEL-AVIV UNIVERSITY
TEL-AVIV, ISRAEL