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ON THE APPROXIMATION AND SATURATION BY GENERAL SINGULAR INTEGRALS

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Introduction. Let R^n be the n-dimensional Euclidean space whose element will be denoted by $x=(x_1,\cdots,x_n)$. The length or norm of x is denoted by $|x|=[x_1^2+\cdots+x_n^2]^{1/2}$. By E, we denote the space $L^p(R^n)(1\leqq p<\infty)$ or $C_0(R^n)$. For a function $f(x)\in E$, we consider a singular integral of convolution type

$$K(x,\rho;f)=\frac{\rho^n}{(2\pi)^{n/2}}\int_{R^n}f(x-y)k(\rho y)dy \tag{1}$$

where the kernel $k(x)$ is radial and satisfies

$$k(x)\in L^1(R^n) \quad \text{and} \quad \int_{R^n}k(x)dx=(2\pi)^{n/2}. \tag{2}$$

Then, by Jensen's inequality, it is easy to see that

$$\|K(\cdot,\rho;f)-f(\cdot)\|_E\rightarrow 0 \text{ as } \rho\rightarrow\infty. \tag{3}$$

Our object of this paper is to determine the optimal rate of approximation of $f(x)$ by $K(x,\rho;f)$ and the non-trivial class $\mathfrak{K}$ of functions for which this rate is attained. This is called the saturation problem for the singular integral $K(x,\rho;f)$.

For the sake of completeness, we give here the exact definition of saturation.

DEFINITION. Let $f(x)\in E$ and (1) be a given singular integral with kernel $k(x)$ which satisfies (2). Suppose there exist a positive number r and a class $\mathfrak{K}\subset E$ such that

(i) $\|K(\cdot,\rho;f)-f(\cdot)\|_E=o(\rho^{-r})$ as $\rho\rightarrow\infty$ implies $f(x)=0$ a.e.

(ii) $\|K(\cdot,\rho;f)-f(\cdot)\|_E=O(\rho^{-r})$ as $\rho\rightarrow\infty$ implies $f(x)\in\mathfrak{K}$ and vice versa.

Then the singular integral $K(x, \rho; f)$ is called to be saturated with the order ρ^{-r} in the space E and $\mathfrak{K}$ is called its saturation class.

There are many contributions about this problem. P.L.Butzer [3] initiated so-called Fourier transform method and studied the single variable case. P.L. Butzer-R.J.Nessel and R.J.Nessel [4] applied this method to several variables case. However these papers are restricted to the $L^p(1 \leqq p \leqq 2)$ norm. G.Sunouchi [8] applied the generalized Fourier transform of Bochner to the single variable case and studied L^p $(1 \leqq p < \infty)$ approximation. Buchwalter [2] gave a general theorem by distribution theory and E.Görlich [6] applied this to several variables case in $L^p(1 < p < \infty)$ norm. However these papers are somewhat inconvenient in particular, at the application to classical singular integrals.

In this paper, we will give a general theorem by adjoint method. This is essentially equivalent to Fourier transform method in distribution theory, but it has an advantage of doing easy treatment of broader classes of functions. Next we give a simple, nevertheless widely applicable criterion and solve saturation problem for many classical singular integrals of a single variable and several variables in $L^p(1 \leqq p < \infty)$ norm and C_0 norm. Our method will also give an asymptotic approximation theorem and a saturation theorem of higher order by singular integrals.

1. The general theorem for singular integrals in a single variable. Let the Fourier transform $\hat{k}(v)$ of the radial kernel $k(x)$ of a singular integral (1), which is also a radial function $\hat{\kappa}(|v|) = \hat{k}(v)$ say, satisfy the following conditions;

(1. 1) $$\lim_{t\to 0+} \frac{\hat{\kappa}(t)-1}{t^r} = c \neq 0$$

for a positive number r, and

(1. 2) $$\frac{\hat{\kappa}(t)-1}{t^r} \in (S, S),$$

i.e., is a multiplier of class BV into class BV. Then (1.2) is equivalent to the existence of a function $H(x)$ of bounded variation such that

$$\frac{\hat{k}(v)-1}{|v|^r} = \check{H}(v)$$

where $\check{H}(v)$ is the Fourier-Stieltjes transform of $H(x)$. Therefore

$$\frac{\hat{k}(v/\rho)-1}{(|v|/\rho)^r}=\check{H}(v/\rho)=[H(\rho\cdot)]\check{}(v)$$

and so

$$\|H(\rho\cdot)\|_{BV}=O(1)$$

uniformly in ρ, and

$$\int_{-\infty}^{\infty} dH(\rho x)=\int_{-\infty}^{\infty} dH(x)=c'$$

say. We denote by $\mathfrak{D}$ the set of all infinitely differentiable functions with compact support.

LEMMA 1. *If* $\varphi(x)\in\mathfrak{D}$ *and the kernel* $k(x)$ *of a singular integral* (1) *satisfies* (1.1) *and* (1.2), *then*

$$\left\| \frac{K(\cdot,\rho;\varphi)-\varphi(\cdot)}{\rho^{-r}}-\frac{c'}{\sqrt{2\pi}}(-1)^{\left[\frac{r}{2}\right]}\varphi^{\{r\}}(\cdot)\right\|_E \longrightarrow 0 \text{ as } \rho\to\infty$$

where $\varphi^{\{r\}}(x)$ is $\varphi^{(r)}(x)$ *or* $(\tilde{\varphi})^{(r)}(x)$ *according as* r *is even or odd.*

PROOF. Since $\varphi(x)\in\mathfrak{D}$, we have by (1.2)

$$\frac{K(x,\rho;\varphi)-\varphi(x)}{\rho^{-r}}-\frac{c'}{\sqrt{2\pi}}(-1)^{\left[\frac{r}{2}\right]}\varphi^{\{r\}}(x)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\hat{k}(v/\rho)-1}{(|v|/\rho)^r}|v|^r\hat{\varphi}(v)e^{-ivx}dv-\frac{c'}{\sqrt{2\pi}}(-1)^{\left[\frac{r}{2}\right]}\varphi^{\{r\}}(x)$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[H(\rho\cdot)]\check{}(v)(-1)^{\left[\frac{r}{2}\right]}[\varphi^{\{r\}}(\cdot)]\hat{}(v)dv-\frac{c'}{\sqrt{2\pi}}(-1)^{\left[\frac{r}{2}\right]}\varphi^{\{r\}}(x)$$

$$=(-1)^{\left[\frac{r}{2}\right]}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi^{\{r\}}(x-y)dH(\rho y)-\frac{c'}{\sqrt{2\pi}}\varphi^{\{r\}}(x)\right\}.$$

Hence by (3) we get the lemma.

Now we can prove the following general theorem.

THEOREM 1. *Let* $f(x)\in E$ *and the kernel* $k(x)$ *of a singular integral* (1) *satisfy* (1.1) *and* (1.2). *Then this singular integral is saturated with the order* ρ^{-r} *in the space* E *with saturation class* $\mathfrak{K}$ *such as*

$$\mathfrak{K} = \left\{ f(x)\,;\, \begin{matrix} f^{(r-1)}(x) \in BV \ (if\ r\ is\ even) \\ (\tilde{f})^{(r-1)}(x) \in BV (if\ r\ is\ odd) \end{matrix} \right\}^{1)} \quad if\ E = L^1,$$

$$\mathfrak{K} = \{f(x)\,; f^{\{r\}}(x) \in L^p\} \qquad if\ E = L^p (1 < p < \infty),$$

$$\mathfrak{K} = \{f(x)\,; f^{\{r\}}(x) \in L^\infty\} \qquad if\ E = C_0$$

where $f^{\{r\}}(x)$ *is* $f^{(r)}(x)$ *or* $(\tilde{f})^{(r)}(x)$ *according as r is even or odd*[2],

PROOF. (i) Suppose that

$$\|K(\,\cdot\,, \rho\,; f) - f(\,\cdot\,)\|_E = o(\rho^{-r}) \text{ as } \rho \to \infty\,.$$

Then, for any $\varphi(x) \in \mathfrak{D}$, we have

$$\int_{-\infty}^{\infty} \frac{K(x, \rho\,; f) - f(x)}{\rho^{-r}} \varphi(x) dx \to 0 \quad \text{as } \rho \to \infty\,.$$

On the other hand, by the fact that $K(x, \rho; f)$ is of convolution type and by Lemma 1,

$$\int_{-\infty}^{\infty} \frac{K(x, \rho\,; f) - f(x)}{\rho^{-r}} \varphi(x) dx = \int_{-\infty}^{\infty} \frac{K(x, \rho\,; \varphi) - \varphi(x)}{\rho^{-r}} f(x) dx$$

$$\to \frac{c'}{\sqrt{2\pi}} (-1)^{\left[\frac{r}{2}\right]} \int_{-\infty}^{\infty} \varphi^{\{r\}}(x) f(x) dx \quad \text{as } \rho \to \infty\,.$$

Therefore

$$\int_{-\infty}^{\infty} \varphi^{\{r\}}(x) f(x) dx = 0$$

and so, in the sense of distribution, it can be written as

$$f^{\{r\}}(x) = 0\,.$$

Taking Fourier transform in the distribution sense, we get $|v|^r \hat{f}(v) = 0$[3]

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1) $f^{(r-1)}(x)$ (or $(\tilde{f})^{(r-1)}(x)$) $\in BV$ means that $f(x)$ (or $\tilde{f}(x)$) coincide a. e. with a function $G(x)$ with $G^{(r-1)}(x) \in BV$.

2) When $E = C_0$, $\tilde{f}(x)$ means a generalized conjugate function $\tilde{f}_*(x)$. (see remark at the end of this section and N. J. Achieser [1], p. 126–129).

3) In fact, f is a tempered distribution.

and f is a polynomial. (see L. Schwartz [7], p.139). Since $f\in L^p(1\leqq p<\infty)$ or C_0,

$$f(x)=0 \text{ a.e.}$$

(ii) Next we suppose that

$$\|K(\cdot,\rho;f)-f(\cdot)\|_E=O(\rho^{-r}) \quad \text{as } \rho\to\infty.$$

We shall prove only when $E=L^1$, since another cases may be treated by the same idea. At first we discuss the case when r is even. Since the indefinite integral of a function belonging to L^1 is absolutely continuous and so of bounded variation, by the weak*-compactness of the unit ball of space BV, there exist a function $G(x)\in BV$ and a subsequence ρ_ν such that

$$\int_{-\infty}^{\infty}\frac{K(x,\rho_\nu;f)-f(x)}{\rho_\nu^{-r}}\varphi(x)dx\longrightarrow\int_{-\infty}^{\infty}\varphi(x)dG(x) \quad \text{as } \rho_\nu\to\infty$$

for any $\varphi(x)\in\mathfrak{D}$. On the other hand, by the convolution structure of $K(x,\rho;f)$ and by Lemma 1

$$\int_{-\infty}^{\infty}\frac{K(x,\rho;f)-f(x)}{\rho^{-r}}\varphi(x)dx=\int_{-\infty}^{\infty}\frac{K(x,\rho;\varphi)-\varphi(x)}{\rho^{-r}}f(x)dx$$
$$\to\frac{c'}{\sqrt{2\pi}}(-1)^{[\frac{r}{2}]}\int_{-\infty}^{\infty}\varphi^{(r)}(x)f(x)dx \quad \text{as } \rho\to\infty.$$

Therefore if we denote by $G_{r-1}(x)$ the $(r-1)$th integral of $G(x)$, we have

$$(-1)^r\int_{-\infty}^{\infty}\varphi^{(r)}(x)G_{r-1}(x)dx=\frac{c'}{\sqrt{2\pi}}(-1)^{[\frac{r}{2}]}\int_{-\infty}^{\infty}\varphi^{(r)}(x)f(x)dx.$$

By the same argument to the proof of part (i),

$$f(x)=\frac{\sqrt{2\pi}}{c'}(-1)^{r-[\frac{r}{2}]}G_{r-1}(x) \quad \text{a.e.}$$

and so

$$f^{(r-1)}(x)\in BV.$$

Next, we treat the case when r is odd. Similarly as above, we have

$$(-1)^r \int_{-\infty}^{\infty} \varphi^{(r)}(x) G_{r-1}(x) dx = \frac{c'}{\sqrt{2\pi}} (-1)^{\left[\frac{r}{2}\right]} \int_{-\infty}^{\infty} (\tilde{\varphi})^{(r)}(x) f(x) dx$$

$$= \frac{c'}{\sqrt{2\pi}} (-1)^{\left[\frac{r}{2}\right]} \int_{-\infty}^{\infty} \{\varphi^{(r)}\}^{\sim}(x) f(x) dx = \frac{c'}{\sqrt{2\pi}} (-1)^{\left[\frac{r}{2}\right]} (\varphi^{(r)}, \tilde{f})$$

where the last expression is taken in the sense of distribution, because the conjugate function $\tilde{f}(x)$ of $f(x) \in L^1$ does not always belong to L^1. Thus, in the sense of distribution,

$$\left((-1)^r G_{r-1} - \frac{c'}{\sqrt{2\pi}} (-1)^{\left[\frac{r}{2}\right]} \tilde{f},\ \varphi^{(r)}\right) = 0.$$

Hence,

$$\tilde{f}(x) = \frac{\sqrt{2\pi}}{c'} (-1)^{r-\left[\frac{r}{2}\right]} G_{r-1}(x) \text{ a.e.}$$

and so

$$(\tilde{f})^{(r-1)}(x) \in BV.$$

(iii) Conversely, we suppose $f(x) \in L^1$ and $f^{(r-1)}(x)$ or $(\tilde{f})^{(r-1)}(x) \in BV$. Observing that

$$\left[\frac{K(\cdot, \rho; f) - f(\cdot)}{\rho^{-r}}\right]^{\wedge}(v) = \frac{\hat{k}(v/\rho) - 1}{(|v|/\rho)^r} |v|^r \hat{f}(v)$$

$$= (-1)^{\left[\frac{r}{2}\right]} \frac{\hat{k}(v/\rho) - 1}{(|v|/\rho)^r} [f^{(r-1)}]^{\vee}(v) \text{ or } (-1)^{\left[\frac{r}{2}\right]} \frac{\hat{k}(v/\rho) - 1}{(|v|/\rho)^r} [(\tilde{f})^{(r-1)}]^{\vee}(v)$$

and noting that by (1.2) the last term is a Fourier-Stieltjes transform of a function of bounded variation whose norm is uniformly bounded in ρ, we have

$$\int^{x} \frac{K(y, \rho; f) - f(y)}{\rho^{-r}} dy = (-1)^{\left[\frac{r}{2}\right]} \int_{-\infty}^{\infty} f^{(r-1)}(x-y) dH(\rho y)$$

$$\text{or } (-1)^{\left[\frac{r}{2}\right]} \int_{-\infty}^{\infty} (\tilde{f})^{(r-1)}(x-y) dH(\rho y)$$

and

$$\left\| \frac{K(\cdot,\rho;f)-f(\cdot)}{\rho^{-r}} \right\|_{L^1} = O(1) \qquad \text{uniformly in } \rho.$$

REMARK. When $f(x) \in C_0$, $\tilde{f}(x)$ does not always exist in the ordinary sense. So, in this case, we consider as follows. If we put

$$\tilde{f}_*(x) = (x-i)\operatorname*{l.i.m.}_{a\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (i \text{ sign } v)\, \hat{f}_1(v)\, e^{ivx} dv$$

where

$$f_1(x) = \frac{f(x)}{x-i} \in L^2,$$

then, $\tilde{f}'_*(x)$ and $\tilde{f}'(x)$ have the same Fourier transform in the sense of distribution. Therefore in this case we may interprete $\tilde{f}^{(r)}(x)$ as $\tilde{f}_*^{(r)}(x)$.

2. Applications of the general theorem to special singular integrals. When we apply the general theorem to special singular integrals, it is sometimes difficult to verify the condition (1.2). We give here a simple but widely applicable criterion for verifying (1.2).

THEOREM 2. *If, for the Fourier transform $\hat{k}(v)$ of radial kernel $k(x)$, there exists a radial Fourier-Stieltjes transform $\check{m}(v)$ of a function of bounded variation $m(x)$ which satisfies*

$$\hat{\kappa}(t) - 1 = r\int_0^t \check{\mu}(\tau)\tau^{r-1}\, d\tau \tag{2. 1}$$

where $\hat{\kappa}(|v|) = \hat{k}(v)$ and $\check{\mu}(|v|) = \check{m}(v)$ then, the condition (1.2) *is satisfied.*

PROOF. In the formula (2.1), if we change the variable $\tau \to t\tau$, then

$$\frac{\hat{\kappa}(t)-1}{t^r} = r\int_0^1 \hat{\mu}(t\tau)\tau^{r-1} d\tau = \lim_{N\to\infty} \frac{1}{N^r}\, r\sum_{\nu=1}^{N} \check{\mu}\left(t\,\frac{\nu}{N}\right)\nu^{r-1}$$

$$= \lim_{N\to\infty} \check{R}_N(v) \qquad (t=|v|)$$

say, by the definition of integral of Riemann sense. Denote by $\|R_N\|$ and $\|m\|$ the total variation of $R_N(x)$, $m(x)$, then

$$\|R_N\| \leqq \frac{1}{N^r} r \sum_{\nu=1}^{N} \|m\| \nu^{r-1} \leqq \|m\|.$$

Hence the norm $\|R_N\|$ is uniformly bounded and so the uniform limit of $\check{R}_N(v)$ is also representable by a Fourier-Stieltjes transform of a function of bounded variation.

Now we pass to applications.

(1) The singular integral of Gauss-Weierstrass. It is defined by

$$W(x, \rho ; f) = \frac{\rho}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x-y) e^{-|\rho y|^2} dy.$$

Hence, we need only to check the conditions (1.1) and (2.1). Since

$$k(x) = \sqrt{2}\, e^{-|x|^2} \in L^1(-\infty, \infty), \quad \int_{-\infty}^{\infty} k(x) dx = \sqrt{2\pi},$$

$$\hat{k}(v) = e^{-\frac{1}{4}|v|^2} = \hat{\kappa}(|v|)$$

$$\lim_{t \to 0+} \frac{\hat{\kappa}(t) - 1}{t^2} = -\frac{1}{4} \neq 0,$$

$$\hat{\kappa}(t) - 1 = 2 \int_0^t \check{\mu}(\tau) \tau d\tau$$

where

$$\hat{\mu}(|v|) = -\frac{1}{4} e^{-|v|^2/4} = -\frac{1}{4} \hat{k}(v)$$

and $\check{\mu}(|v|)$ is a Fourier-Stieltjes transform, every hypothesis of the general theorem is satisfied. So we get the following saturation theorem.

PROPOSITION (2.1). *The singular integral of Gauss-Weierstass is saturated with the order* ρ^{-2} *and with the classes*

$$\mathfrak{K} = \{f(x) ; f'(x) \in BV\} \qquad if\ E = L^1,$$

$$\mathfrak{K} = \{f(x) ; f''(x) \in L^p\} \qquad if\ E = L^p (1 < p < \infty),$$

$$\mathfrak{K} = \{f(x) ; f''(x) \in L^\infty\} \qquad if\ E = C_0.$$

(2) The singular integral of Cauchy-Poisson. It is defined by

$$P(x, \rho; f) = \frac{\rho}{\pi} \int_{-\infty}^{\infty} f(x-y) \frac{1}{1+|\rho y|^2} dy.$$

Hence,

$$k(x) = \sqrt{\frac{2}{\pi}} \frac{1}{1+|x|^2} \in L^1(-\infty, \infty), \quad \int_{-\infty}^{\infty} k(x)dx = \sqrt{2\pi},$$

$$\hat{k}(v) = e^{-|v|} = \hat{\kappa}(|v|),$$

$$\lim_{t\to 0+} \frac{\hat{k}(t)-1}{t} = -1 \neq 0,$$

$$\hat{\kappa}(t) - 1 = \int_0^t \check{\mu}(\tau)d\tau$$

where

$$\check{\mu}(|v|) = -e^{-|v|} = -\hat{k}(v),$$

and the conditions (1.1) and (2.1) are satisfied and so we get the following saturation theorem.

PROPOSITION (2.2). *The singular integral of Cauchy-Poisson is saturated with the order* ρ^{-1} *and with the classes*

$$\mathfrak{K} = \{f(x)\,;\, \tilde{f}(x) \in BV\} \qquad if\ E = L^1,$$

$$\mathfrak{K} = \{f(x)\,;\, \tilde{f}'(x) \in L^p\} \qquad if\ E = L^p (1 < p < \infty),$$

$$\mathfrak{K} = \{f(x)\,;\, \tilde{f}'_*(x) \in L^\infty\} \qquad if\ E = C_0$$

where $\tilde{f}_*(x)$ *is a generalized conjugate function of* $f(x)$.

(3) The singular integral of Bochner-Riesz of order $\alpha > 1$. This singular integral does not form a semi-group. It is defined by

$$B^\alpha(x, \rho; f) = \frac{\rho}{\sqrt{2\pi}} 2^\alpha \Gamma(\alpha+1) \int_{-\infty}^{\infty} f(x-y) V_{\frac{1}{2}+\alpha}(|\rho y|)\, dy.$$

Hence,

$$k^\alpha(x) = 2^\alpha \Gamma(\alpha+1) V_{\frac{1}{2}+\alpha}(|x|) \in L^1(-\infty, \infty) \qquad \text{for } \alpha > 0,$$

$$\int_{-\infty}^{\infty} k^{\alpha}(x)dx = \sqrt{2\pi},$$

$$\hat{k}^{\alpha}(v)=\begin{cases}(1-|v|^2)^{\alpha} & \text{if } |v| \leqq 1\\ 0 & \text{if } |v| \geqq 1\end{cases} = \hat{\kappa}^{\alpha}(|v|),$$

$$\lim_{t\to 0+} \frac{\hat{\kappa}^{\alpha}(t)-1}{t^2} = -\alpha \neq 0,$$

$$\hat{\kappa}^{\alpha}(t)-1 = 2\int_0^t \check{\mu}(\tau)\tau d\tau$$

where

$$\check{\mu}(|v|) = -\alpha \begin{cases}(1-|v|^2)^{\alpha-1} & \text{if } |v| \leqq 1\\ 0 & \text{if } |v| \geqq 1\end{cases} = -\alpha\hat{k}^{\alpha-1}(v)$$

and

$$k^{\alpha-1}(x) \in L^1(-\infty, \infty) \text{ for } \alpha > 1,$$

so we get the following saturation theorem.

PROPOSITION (2.3). *The saturation strucure for the singular integral of Bochner-Riesz is the same as the singular integral of Gauss-Weierstrass.*

3. The general theorem for singular integral in several variables. Similarly as 1-dimensional case, let the Fourier transform $\hat{k}(v)$ of the radial kernel of a singular integral (1), which is also a radial function $\hat{\kappa}(|v|)=\hat{k}(v)$ say, satisfies the following condition,

$$\lim_{t\to 0+} \frac{\hat{\kappa}(t)-1}{t^{r}} = c \neq 0 \tag{3.1}$$

for a positive number r, and

$$\frac{\hat{\kappa}(t)-1}{t^{r}} \in (S, S), \tag{3.2}$$

i.e., is a multiplier of class S into S[1]. Then (3.2) implies the existence of a bounded measure Λ such that

1) S means the set of bounded measures.

$$\frac{\hat{k}(v)-1}{|v|^r} = \check{\Lambda}(v)$$

where $\check{\Lambda}(v)$ is the Fourier-Stieltjes transform of Λ. Therefore

$$\frac{\hat{k}(v/\rho)-1}{(|v|/\rho)^r} = \check{\Lambda}(v/\rho) = [\Lambda(\rho\,\cdot)]\check{}\,(v)$$

and so

$$\|\Lambda(\rho\,\cdot)\|_S = O(1) \quad \text{uniformly in } \rho,$$

$$\int_{R^n} d\Lambda(\rho x) = \int_{R^n} d\Lambda(x) = c' \text{ say.}$$

Let $\mathfrak{D}$ be the set of all functions with compact support and all of whose derivatives exist and are finite. Then, for any $\varphi(x) \in \mathfrak{D}$, putting

$$\text{(3. 3)} \qquad \varphi^{[r]}(x) = \begin{cases} \triangle^m(\varphi)(x) = \left[\sum_{j=1}^{n} \dfrac{\partial^2}{\partial x_j^2}\right]^m (\varphi)(x) & \text{if } r = 2m \\ \triangle^m(\nabla \cdot \tilde{\Phi})(x) = \triangle^m\left[\sum_{j=1}^{n} \dfrac{\partial \tilde{\varphi}_j}{\partial x_j}\right](x) & \text{if } r = 2m+1 \end{cases}$$

where $\tilde{\varphi}_j(x)$ $(j=1, \cdots, n)$ is j-th conjugate function of $\varphi(x)$ and $\tilde{\Phi}(x) = (\tilde{\varphi}_1(x), \cdots, \tilde{\varphi}_n(x))$, we know that

$$[\varphi^{[r]}]\hat{}(v) = (-1)^{\left[\frac{r}{2}\right]} |v|^r \varphi(v).$$

Through this section we can discuss similarly as 1-dimensional case and so give results analogous to Lemma 1 and Theorem 1 and 2 without proof.

LEMMA 2. *If $\varphi(x) \in \mathfrak{D}$ and the kernel $k(x)$ of a singular integral* (1) *satisfies* (3.1) *and* (3.2), *then*

$$\left\| \frac{K(\cdot, \rho; \varphi) - \varphi(\cdot)}{\rho^{-r}} - \frac{c'}{\sqrt{2\pi}} (-1)^{\left[\frac{r}{2}\right]} \varphi^{[r]}(\cdot) \right\|_E \to 0 \quad as\ \rho \to \infty$$

where $\varphi^{[r]}(x)$ is defined by (3.3).

Hence, we have the following general theorem in several variables.

THEOREM 3. *Let $f(x) \in E$ and the kernel $k(x)$ of a singular integral* (1) *satisfy* (3 1) *and* (3.2). *Then, this singular integral is saturated with the order ρ^{-r} in the space E with saturation class $\mathfrak{K}$ such as*

$$\mathfrak{K} = \left\{ f(x) ; \int_M f^{\{r\}}(x) dx \in S \text{ for every measurable set } M \right\} \quad \text{if } E = L^1,$$

$$\mathfrak{K} = \{ f(x) \; ; \; f^{\{r\}}(x) \in L^p \} \qquad \text{if } E = L^p (1 < p < \infty),$$

$$\mathfrak{K} = \{ f(x) \; ; \; f^{\{r\}}(x) \in L^\infty \} \qquad \text{if } E = C_0$$

where $f^{\{r\}}(x)$ means such as (3.3) *and differential operators $\triangle^m$, $\partial/\partial x_j$ $(j = 1, \cdots, n)$ and, when $E = C_0$, conjugate functions $\tilde{f}_j(x)$ $(j=1, \cdots, n)$, are generally taken in the sense of distribution.*[1)]

Before we pass to applications, we give the criterion for the condition (3.2).

THEOREM 4. *If, for the Fourier transform $\hat{k}(v)$ of radial kernel $k(x)$, there exists a radial Fourier-Stieltjes transform $\check{m}(v)$ of a bounded measure $m(M)$ which satisfies*

$$\hat{\kappa}(t) - 1 = r \int_0^t \check{\mu}(\tau) \tau^{r-1} d\tau \tag{3. 4}$$

where $\hat{\kappa}(|v|) = \hat{k}(v)$ and $\check{\mu}(|v|) = \check{m}(v)$, then, the condition (3.2) *is satisfied.*

(1) The singular integral of Gauss-Weierstrass:

$$W(x, \rho ; f) = \frac{\rho^n}{(2\pi)^{n/2}} 2^{n/2} \int_{R^n} f(x-y) e^{-|\rho y|^2} \, dy.$$

Hence, since

$$k(x) = 2^{n/2} \, e^{-|x|^2} \in L^1(R^n), \quad \int_{R^n} k(x) dx = (2\pi)^{n/2},$$

$$\hat{k}(v) = e^{-\frac{1}{4}|v|^2} = \hat{\kappa}(|v|),$$

$$\lim_{t \to 0+} \frac{\hat{\kappa}(t) - 1}{t^2} = -\frac{1}{4} \neq 0,$$

1) Görlich [6] shows that the differential operation is in ordinary sense if $1 < p < \infty$.

$$\hat{\kappa}(t) - 1 = 2\int_0^t \check{\mu}(\tau)\tau d\tau$$

where

$$\check{\mu}(|v|) = -\frac{1}{4} e^{-\frac{1}{4}|v|^2} = -\frac{1}{4}\hat{k}(v),$$

every hypothesis of Theorem 3 is satisfied. So we get the following theorem.

PROPOSITION (3.1). *The singular integral of Gauss-Weierstrass is saturated with the order ρ^{-2} and with the classes*

$$\mathfrak{K} = \{f(x);\ \int_M \triangle(f)(x)dx \in S \text{ for every measurable set } M\} \text{ if } E = L^1,$$

$$\mathfrak{K} = \{f(x);\ \triangle(f)(x) \in L^p\} \quad \text{if } E = L^p (1 < p < \infty),$$

$$\mathfrak{K} = \{f(x);\ \triangle(f)(x) \in L^\infty\} \quad \text{if } E = C_0$$

where Laplace operation $\triangle$ is taken in the sense of distribution.

(2) The singular integral of Cauchy-Poisson:

$$P(x, \rho;\ f) = \frac{\rho^n}{(2\pi)^{n/2}} \frac{2^{n/2}\Gamma((n+1)/2)}{\sqrt{\pi}} \int_{R^n} f(x-y) \frac{1}{\{1+|\rho y|^2\}^{(n+1)/2}} dy.$$

Hence, since

$$k(x) = \frac{2^{n/2}\Gamma(n+1)}{\sqrt{\pi}} \frac{1}{\{1+|x|^2\}^{(n+1)/2}} \in L^1(R^n),$$

$$\int_{R^n} k(x)dx = (2\pi)^{n/2},$$

$$\hat{k}(v) = e^{-|v|} = \hat{\kappa}(|v|),$$

$$\lim_{t\to 0+} \frac{\hat{\kappa}(t) - 1}{t} = -1 \neq 0,$$

$$\hat{\kappa}(t) - 1 = \int_0^t \check{\mu}(\tau)d\tau$$

where

$$\check{\mu}(|v|) = -e^{-|v|} = -\hat{k}(v),$$

then we get the following theorem by Theorem 3.

PROPOSITION (3.2). *The singular integral of Cauchy-Poisson is saturated with the order* ρ^{-1} *and with the classes*

$$\mathfrak{K} = \{f(x);\ \int_M (\nabla\cdot\tilde{\boldsymbol{f}})(x)dx \in S \text{ for every measurable set } M\} \text{ if } E = L^1,$$

$$\mathfrak{K} = \{f(x);\ (\nabla\cdot\tilde{\boldsymbol{f}})(x) \in L^p\} \text{ if } E = L^p\ (1 < p < \infty),$$

$$\mathfrak{K} = \{f(x);\ (\nabla\cdot\tilde{\boldsymbol{f}})(x) \in L^\infty\} \text{ if } E = C_0$$

where differential operator $\partial/\partial x_j$ *and when* $f(x) \in C_0$, *its conjugate function* $\tilde{f}_j(x)$ $(j = 1, \cdots, n)$, *are taken in the sense of distribution.*

(3) The singular integral of Bochner-Riesz of order $\alpha > (n-1)/2 + 1$:

$$B^\alpha(x, \rho; f) = \frac{\rho^n}{(2\pi)^{n/2}} 2^\alpha\Gamma(\alpha+1)\int_{R^n} f(x-y)V_{\frac{n}{2}+\alpha}(|\rho y|)\,dy.$$

Hence, since

$$k^\alpha(x) = 2^\alpha\Gamma(\alpha+1)\ V_{\frac{n}{2}+\alpha}(|x|) \in L^1(R^n) \text{ for } \alpha > (n-1)/2,$$

$$\int_{R^n} k^\alpha(x)dx = (2\pi)^{n/2},$$

$$\hat{k}^\alpha(v) = \begin{cases}(1-|v|^2)^\alpha & \text{if } |v| \leqq 1\\ 0 & \text{if } |v| \geqq 1\end{cases} = \hat{\kappa}^\alpha(|v|),$$

$$\lim_{t\to 0+}\frac{\hat{\kappa}^\alpha(t)-1}{t^2} = -\alpha \neq 0,$$

$$\hat{\kappa}^\alpha(t) - 1 = 2\int_0^t \check{\mu}(\tau)\tau d\tau$$

where

$$\hat{\mu}(|v|) = -\alpha\begin{cases}(1-|v|^2)^{\alpha-1} & \text{if } |v| \leqq 1\\ 0 & \text{if } |v| \geqq 1\end{cases} = -\alpha\hat{k}^{\alpha-1}(v)$$

and

$$k^{\alpha-1}(x) \in L^1(R^n) \quad \text{for } \alpha > (n-1)/2+1,$$

then we get the following theorem.

PROPOSITION (3.3). *The saturation structure for the singular integral of Bochner-Riesz of order $\alpha > (n-1)/2+1$ is the same as the Gauss-Weierstrass singular integral.*

(3)′ The singular integral of Cesàro-Riesz of order $\alpha > (n-1)/2+1$. It is defined by

$$C_\gamma^\alpha(x, \rho\,; f) = \frac{\rho^n}{(2\pi)^{n/2}} \int_{R^n} f(x-y) k_\gamma^\alpha(\rho y)\, dy$$

where

$$\hat{k}_\gamma^\alpha(v) = \begin{cases} (1-|v|^\gamma)^\alpha & \text{if } |v| \leqq 1 \\ 0 & \text{if } |v| \geqq 1 \end{cases} = \hat{\kappa}_\gamma^\alpha(|v|).$$

It is known that the summability by means of $\hat{k}_\gamma^\alpha(v)$ is equivalent to the summability by means of $\hat{k}^\alpha(v)$ in (3). (see, K.Chandrasekharan and S.Minakshisundaram [5], p.35). So by the uniformly bounded theorem of Banach-Steinhaus and by example (3), we have

$$k_\gamma^\alpha(x) \in L^1(R^n) \quad \text{for } \alpha > (n-1)/2.$$

Moreover

$$\lim_{t\to 0+} \frac{\hat{\kappa}_\gamma^\alpha(t)-1}{t^\gamma} = -\alpha \neq 0,$$

$$\hat{\kappa}_\gamma^\alpha(t) - 1 = \gamma \int_0^t \check{\mu}(\tau)\tau^{\gamma-1}\, d\tau$$

where

$$\check{\mu}(|v|) = -\alpha \begin{cases} (1-|v|^{\gamma})^{\alpha-1} & \text{if } |v| \leqq 1 \\ 0 & \text{if } |v| \geqq 1 \end{cases} = -\alpha \hat{k}_{\gamma}^{\alpha-1}(v)$$

and

$$k_{\gamma}^{\alpha-1}(x) \in L^1(R^n) \quad \text{if } \alpha > (n-1)/2 + 1 .$$

Hence, when γ is a positive integer, the singular integral of Cesàro-Riesz of order $\alpha > (n-1)/2 + 1$ is saturated with the order $\rho^{-\gamma}$.

(4) Spherical average of first order:

$$M_1(x, \rho\,;\, f) = \frac{\rho^n}{(2\pi)^{n/2}} 2^{n/2}\Gamma(n/2+1) \int_{|y| \leqq 1/\rho} f(x-y)\, dy$$

where

$$k_1(x) = \begin{cases} 2^{n/2}\Gamma(n/2+1) & \text{if } |x| \leqq 1 \\ 0 & \text{if } |x| \geqq 1 \end{cases} \in L^1(R^n),$$

$$\int_{R^n} k_1(x)dx = (2\pi)^{n/2}.$$

We note that

$$\hat{k}_1(v) = 2^{n/2}\Gamma(n/2+1)V_{\frac{n}{2}}(|v|) = \hat{\kappa}_1(|v|),$$

$$\lim_{t\to 0+} \frac{\hat{\kappa}_1(t)-1}{t^2} = -\frac{1}{2(n+2)} \neq 0,$$

$$\hat{\kappa}_1(t) - 1 = 2\int_0^t \check{\mu}_1(\tau)\tau d\tau$$

where

$$\check{\mu}_1(|v|) = -2^{n/2-1}\Gamma(n/2+1)V_{\frac{n}{2}+1}(|v|)$$

and

$$-2^{n/2-1}\Gamma(n/2+1)\ [V_{n/2+1}(|\cdot|)]^{\wedge}(x)$$
$$=\begin{cases} -2^{n/2-1}\Gamma(n/2+1)\cdot\dfrac{1}{2}(1-|x|^2) & \text{if } |x|\leqq 1 \\ 0 & \text{if } |x|\geqq 1 \end{cases} \in L^1(R^n).$$

Hence this singular integral is saturated with the order ρ^{-2}.

(5) Spherical average of second order:

$$M_2(x,\rho\,;f)=\frac{\rho^n}{(2\pi)^{n/2}}\int_{R^n} f(x-y)k_2(\rho y)\,dy$$

where

$$k_2(x)=\frac{1}{(2\pi)^{n/2}}\int_{R^n} k_1(x-y)k_1(y)dy\in L^1(R^n),$$

$$\int_{R^n} k_2(x)dx=(2\pi)^{n/2}.$$

We note that

$$\hat{k}_2(v)=[\hat{m}_1(v)]^2=\{2^{n/2}\Gamma(n/2+1)\}^2\{V_{\frac{n}{2}}(|u|)\}^2=\hat{\kappa}_2(|v|),$$

$$\lim_{t\to 0+}\frac{\hat{\kappa}_2(t)-1}{t^2}=-\frac{1}{n+2}\neq 0.$$

$$\hat{\kappa}_2(t)-1=2\int_0^t \check{\mu}_2(\tau)\tau d\tau$$

where

$$\check{\mu}_2(|v|)=-\{2^{n/2}\Gamma(n/2+1)\}^2V_{\frac{n}{2}}(|v|)V_{\frac{n}{2}+1}(|v|)=-\frac{1}{2}\hat{\kappa}_1(|v|)\check{\mu}_1(|v|)\in\hat{L}^1(R^n).$$

(6) Spherical means:

$$S(x,\rho\,;f)=\frac{\rho^n}{(2\pi)^{n/2}}2^{(n-2)/2}\Gamma(n/2)\int_{R^n} f(x-y)\,dk(|\rho y|)$$

where $k(x)$ is a bounded measure with total measure $(2\pi)^{n/2}$ which is distributed on the unit sphere $|x|=1$ uniformly. As we shall mention in Remark 1, our argument is valid for the singular integral with Stieltjes convolution. We note that

$$\check{k}(v) = 2^{(n-2)/2}\Gamma(n/2)V_{(n-2)/2}(|v|)=\check{\kappa}(|v|)$$

say,

$$\lim_{t\to 0+}\frac{\check{\kappa}(t)-1}{t^2} = -\frac{1}{2n} \neq 0,$$

$$\check{\kappa}(t)-1 = 2\int_0^t \check{\mu}(\tau)\tau d\tau$$

where

$$\check{\mu}(|v|) = -2^{(n-2)/2-1}\Gamma(n/2)V_{n/2}(|v|)$$

and

$$-2^{(n-2)/2-1}\Gamma(n/2)V_{n/2}(|x|)=\left.\begin{cases}-2^{(n-2)/2-1}\Gamma(n/2) & \text{if } |x|\leqq 1\\ 0 & \text{if } |x|>1\end{cases}\right\}^{\wedge}(x)\in \hat{L}^1(R^n).$$

Hence Spherical means is saturated with order ρ^{-2}.

(7) The singular integral of Picard:

$$C(x,\rho;f)=\frac{\rho^n}{(2\pi)^{n/2}}\frac{2^{(n-2)2}\Gamma(n/2)}{\Gamma(n)}\int_{R^n} f(x-y)\,e^{-|\rho y|}\,dy$$

where

$$k(x) = \frac{2^{(n-2)/2}\Gamma(n/2)}{\Gamma(n)}e^{-|x|}\in L^1(R^n),$$

$$\int_{R^1} k(x)\,dx = (2\pi)^{n/2}.$$

We note that

$$\hat{k}(v) = \frac{1}{\{1+|x|^2\}^{(n+1)/2}} = \hat{\kappa}(|v|),$$

$$\lim_{t\to 0+}\frac{\kappa(t)-1}{t^2}=-\frac{n+1}{2}\neq 0,$$

$$\hat{\kappa}(t)-1=2\int_0^t \check{\mu}(\tau)\tau d\tau$$

where

$$\check{\mu}(|v|)=-\frac{n+1}{2}\frac{1}{\{1+|v|^2\}^{(n+1)/2+1}}$$

and

$$-\frac{n+1}{2}\left[\frac{1}{\{1+|\cdot|^2\}^{(n+1)/2+1}}\right]\hat{}(x)=-\frac{\sqrt{\pi}}{2^{n/2+1}\Gamma((n+1)/2)}(1+|x|)e^{-|x|}\in L^1(R^n).$$

Hence the singular integral of Picard is saturated with the order ρ^{-2}.

4. Asymptotic approximation and saturation of the higher order for singular integrals. At firtst, we state the asymptotic approximation theorem.

THEOREM 5. *Let* $f(x)\in L^p(R^n)$ $(1<p<\infty)$ *and* $k(x)$ *satisfy* (3.1) *and* (3.2). *If* $f^{\{r\}}(x)$ *exists and belongs to* L^p, *then*

$$\left\|\frac{K(\cdot,\rho;f)-f(\cdot)}{\rho^{-r}}-\frac{c'}{\sqrt{2\pi}}(-1)^{[\frac{r}{2}]}f^{\{r\}}(\cdot)\right\|_{L^p}\to 0 \text{ as } \rho\to\infty.$$

PROOF. For any $\varepsilon>0$, we choose $\varphi(x)\in\mathfrak{D}$ such that

$$\|f(\cdot)-\varphi(\cdot)\|_{L^p}<\varepsilon \text{ and } \|f^{\{r\}}(\cdot)-\varphi^{\{r\}}(\cdot)\|_{L^p}<\varepsilon.$$

Then, by (3.2) and Lemma 2, as $\rho\to\infty$

$$\left\|\frac{K(\cdot,\rho;f)-f(\cdot)}{\rho^{-r}}-\frac{c'}{\sqrt{2\pi}}(-1)^{[\frac{r}{2}]}f^{\{r\}}(\cdot)\right\|_{L^p}$$

$$\leqq\left\|\frac{K(\cdot,\rho;f-\varphi)-(f-\varphi)(\cdot)}{\rho^{-r}}-\frac{c'}{\sqrt{2\pi}}(-1)^{[\frac{r}{2}]}(f-\varphi)^{\{r\}}(\cdot)\right\|_{L^p}$$

$$+\left\|\frac{K(\cdot,\rho;\varphi)-\varphi(\cdot)}{\rho^{-r}}-\frac{c'}{\sqrt{2\pi}}(-1)^{[\frac{r}{2}]}\varphi^{\{r\}}(\cdot)\right\|_{L^p}$$

$$\leqq A_1\|f^{\{r\}}(\cdot)-\varphi^{\{r\}}(\cdot)\|_{L^p}(\|\Lambda\|_S+1)+o(1)\leqq A_2\varepsilon,$$

so we get Theorem 5.

Applying the above theorem to special singular integral given before, we have the following theorem.

PROPOSITION (4.1). *Let* $f(x) \in L^p(R^n)$ $(1 < p < \infty)$.

(i) *If* $\triangle(f)(x) \in L^p$, *then*

$$\left\| \frac{W(\cdot, \rho; f) - f(\cdot)}{\rho^{-2}} - a\triangle(f)(\cdot) \right\|_{L^p} \to 0 \text{ as } \rho \to \infty$$

for some constant a *and this limit operation holds for* $B^\alpha(x, \rho; f)$, $M_1(x, \rho; f)$, $M_2(x, \rho; f)$, $S(x, \rho; f)$ *and* $C(x, \rho; f)$.

(ii) *If* $(\nabla \cdot \tilde{f})(x) \in L^p$, *then*

$$\left\| \frac{P(\cdot, \rho; f) - f(\cdot)}{\rho^{-1}} - a(\nabla \cdot \tilde{f})(\cdot) \right\|_{L^p} \to 0 \text{ as } \rho \to \infty$$

for some constant a.

Next we pass to saturation of the higher order and only give the following saturation theorem, which may be proved by the same methods as before.

THEOREM 6. *Let* $f(x) \in E(R^n)$ *and the radial kernel* $k(x)$ *of singular integral* (1) *satisfy the following conditions*:

$$\text{(4. 1)} \qquad \lim_{t \to 0+} \frac{\hat{\kappa}(t) - \sum_{\nu=0}^{N} a_\nu t^{r\nu}}{t^{r(N+1)}} = c \neq 0 \text{ for some positive number } r,$$

$$\text{(4. 2)} \qquad \hat{\kappa}(t) - \sum_{\nu=0}^{N} a_\nu t^{r\nu} = r \int_0^t (t^r - \tau^r)^N \check{\mu}(\tau) \tau^{r-1} \, d\tau$$

where N *is any given non-negative integer and* $\hat{k}(v) = \hat{\kappa}(|v|)$, $\check{\mu}(|v|)$ *is a radial Fourier-Stieltges transform of a bounded measure. Then*

$$\text{(i)} \qquad \left\| K(\cdot, \rho; f) - \sum_{\nu=0}^{N} a_\nu \rho^{-r\nu} (-1)^{\left[\frac{r\nu}{2}\right]} f^{\{r\nu\}}(\cdot) \right\|_E = o(\rho^{-r(N+1)}) \text{ as } \rho \to \infty$$

if and only if $\quad f^{\{r(N+1)\}}(x) = 0$.

(ii) $$\left\| K(\cdot,\rho;f) - \sum_{\nu=0}^{N} a_\nu \rho^{-r\nu}(-1)^{\left[\frac{r\nu}{2}\right]} f^{\{r\nu\}}(\cdot) \right\|_E = O(\rho^{-r(N+1)}) \text{ as } \rho \to \infty$$

if and only if

$$\int_M f^{\{r(N+1)\}}(x)\,dx \in S \text{ for every measurable set } M, \text{ if } E = L^1,$$
$$f^{\{r(N+1)\}}(x) \in L^p \quad \text{if } E = L^p \ (1 < p < \infty),$$
$$f^{\{r(N+1)\}}(x) \in L^\infty \quad \text{if } E = C_0$$

where $f^{\{r(N+1)\}}(x)$ *is taken in the sense of distribution.*

Moreover, if $f(x) \in L^p$ $(1 < p < \infty)$ *and* $f^{\{r(N+1)\}}(x)$ *exists and belongs to* L^p, *then*

(iii) $$\left\| \frac{K(\cdot,\rho;f) - \sum_{\nu=0}^{N} a_\nu \rho^{-r\nu}(-1)^{\left[\frac{r\nu}{2}\right]} f^{\{r\nu\}}(\cdot)}{\rho^{-r(N+1)}} - b f^{\{r(N+1)\}}(\cdot) \right\|_{L^p} \to 0 \text{ as } \rho \to \infty$$

for some constant b.

For example, if $\hat{k}(v)$ is a function of $|v|^r$ and differentiable and if its derivative is a radial Fourier-Stieltjes transform of a bounded measure, then the conditions (4.1) and (4.2) can be verified by means of Taylor expansion of $\hat{k}(v)$.

$$\hat{\kappa}(t) = \sum_{\nu=0}^{N} a_\nu t^{r\nu} + r\int_0^t (t^r - \tau^r)^N \check{\mu}(\tau)\tau^{r-1}\,d\tau,$$

where $\check{\mu}(|v|)$ is a radial Fourier-Stieltjes transform of a bounded measure $m(M)$.

Change the variable $\tau \longrightarrow t\tau$,

$$r\int_0^t (t^r - \tau^r)^N \check{\mu}(\tau)\tau^{r-1}\,d\tau = rt^{r(N+1)} \int_0^1 (1-\tau^r)^N \check{\mu}(t\tau)\tau^{r-1}\,d\tau.$$

By the definition of Riemann integral,

$$r\int_0^1 (1-\tau^r)^N \check{\mu}(t\tau)\tau^{r-1}d\tau = \lim_{R\to\infty} \frac{r}{R^r} \sum_{\nu=1}^{R} \left\{1 - \left(\frac{\nu}{R}\right)^r\right\}^N \check{\mu}\left(\tau\frac{\nu}{R}\right)\nu^{r-1}$$

and the total variation of the bounded measure which has this formula as its Fourier-Stieltjes transform is less than

$$\frac{r}{R^r}\sum_{\nu=1}^{R}\left\{1-\left(\frac{\nu}{R}\right)^r\right\}^N \|m\|_S\,\nu^{r-1} \leqq \|m\|_S \frac{r}{R^r}\sum_{\nu=1}^{R}\nu^{r-1} \leqq \|m\|_S .$$

So

$$\frac{\hat{\kappa}(t)-\sum_{\nu=0}^{N} a_\nu t^{r\nu}}{t^{r(N+1)}} \in (S, S) .$$

In particular,

(1) The Gauss-Weierstrass kernel:

$$\hat{\kappa}(t)=e^{-\frac{1}{4}t^2}$$

$$=\sum_{\nu=0}^{N}\frac{1}{\nu!}\left(-\frac{1}{4}\right)^{\nu} t^{2\nu}+\frac{2}{(N+1)!}\int_0^t (t^2-\tau^2)^N\left(-\frac{1}{4}\right)^{N+1} e^{-\frac{1}{4}\tau^2}\tau d\tau .$$

Hence we need only to put

$$a_\nu=\frac{1}{\nu!}\left(-\frac{1}{4}\right)^{\nu},$$

$$\check{\mu}(|v|)=\frac{1}{(N+1)!}\left(-\frac{1}{4}\right)^{N+1} e^{-\frac{1}{4}|v|^2} \in \hat{L}^1(R^n).$$

(2) Cauchy-Poisson kernel:

$$\hat{\kappa}(t)=e^{-t}$$

$$=\sum_{\nu=0}^{N}\frac{1}{\nu!}(-1)^{\nu}t^{\nu}+\frac{1}{(N+1)!}\int_0^t (t-\tau)^N(-1)^{N+1}e^{-\tau}d\tau .$$

Hence we need only to put

$$a_\nu=\frac{1}{\nu!}(-1)^{\nu},$$

$$\check{\mu}(|v|)=\frac{1}{(N+1)!}(-1)^{N+1}e^{-|v|} \in \hat{L}^1(R^n).$$

(3) The Bochner-Riesz kernel of fractional order $\alpha>(n-1)/2+1+N$.

For $0 \leqq t \leqq 1$,

$$\hat{\kappa}(t) = (1-t^2)^\alpha$$
$$= \sum_{\nu=0}^{N} \frac{1}{\nu!}(-1)^\nu \alpha(\alpha-1)\cdots(\alpha-\nu+1)t^{2\nu}$$
$$+ \frac{1}{(N+1)!}\int_0^t (t^2-\tau^2)^N(-1)^{N+1}\alpha(\alpha-1)\cdots(\alpha-N)(1-\tau^2)^{\alpha-(N+1)}\,\tau d\tau .$$

Hence we need only to put

$$a_\nu = \frac{1}{\nu!}(-1)^\nu \alpha(\alpha-1)\cdots(\alpha-\nu+1)$$

$$\check{\mu}(|v|) = \begin{cases} \dfrac{1}{(N+1)!}(-1)^{N+1}\alpha(\alpha-1)\cdots(\alpha-N)(1-|v|^2)^{\alpha-(N+1)} & \text{if } |v| \leqq 1 \\ 0 & \text{if } |v| \geqq 1 \end{cases}$$

$\in \hat{L}^1(R^n)$ for $\alpha > (n-1)/2 + 1 + N$.

REMARK 1. We assume that the singular integral of convolution type is defined by the kernel $k(x) \in L^1$. However, as in Theorem 6, it is sufficient to assume that the convolution is defined in distribution sense. In particular, we may consider the kernel of bounded variation and whose Fourier-Stieltjes transform is radial. That is to say,

$$K(x, \rho; f) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-y)dL(\rho y)$$

where the condition (1.1) and (1.2) are assumed in the same type for $\check{L}(v) = \check{\Lambda}(|v|)$. Then all results are valid.

For example, we consider

$$\frac{1}{2}\{f(x+\rho^{-1}) + f(x-\rho^{-1})\} = \int_{-\infty}^{\infty} f(x-y)\,dL(\rho y)$$

where

$$L(x) = \begin{cases} 0 & \text{if } x < -1 \\ \dfrac{1}{2} & \text{if } -1 \leqq x < 1 \\ 1 & \text{if } x \geqq 1. \end{cases}$$

Then

$$\check{L}(v) = \frac{1}{2}(e^{iv} + e^{-iv}),$$

$$\lim_{t\to 0+} \frac{\check{\Lambda}(t)-1}{t^2} = -\frac{1}{2!},$$

$$\check{\Lambda}(t) - 1 = \int_0^t (t-\tau)\frac{e^{i\tau}+e^{-i\tau}}{2}\,d\tau.$$

Hence, in the same way as the proof of Theoıem 6, (1.2) is proved. So $1/2\ \{f(x+\rho^{-1})+f(x-\rho^{-1})\}$ is saturated with order ρ^{-2}. This is generalized to the higher order difference formula or the mixed formula with differences and derivatives.

REMARK 2. P.Malliavin showed to the one of the authors that (1.1) does not always imply (1.2). That is, there exists a function of bounded variation such that $\lim_{t\to 0} \frac{\check{\Lambda}(t)-1}{|t|} = c \neq 0$ and $\frac{\check{\Lambda}(t)-1}{|t|}$ is not Fourier-Stieltjes transform of any function of bounded variation.

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