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AN OSCILLATION THEOREM OF TAUBERIAN TYPE

HARI SHANKAR

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1. Introduction and notations. The purpose of this paper is to review the inequalities established earlier by the author. The aim was to study the rates of growth of the Ahlfors-Shimizu characteristic function of a meromorphic function $f(z)$ and the area function of the image of the disc $|z|<r$ on the Riemann sphere under the mapping $f(z)$ by using a general kind of comparison function which on specialization gives all previously known results. That study resulted in a set of inequalities (Cf. sec. 2). In the present paper we try to sharpen those and establish some new ones. This attempt culminates in the establishment of two basic independent inequalities, and a Tauberian-type theorem as their by-product which is the main result of this paper.

Let $f(z)$ be meromorphic and non-constant in the open complex plane. Following Hayman [2, p.11], we write

$$m^*(r, a) = (1/2\pi) \int_0^{2\pi} \log\{(k(f(z), a))^{-1}\} d\theta \, , z = re^{i\theta},$$

where $k(w, a)$ denotes the chordal distance on the Riemann sphere of the two points which project to w and a, and

$$A(r, f) = A(r) = (1/\pi) \int_0^r \int_0^{2\pi} \{|f'(z)|^2/(1 + |f(z)|^2)^2\} u du d\theta \quad , \ z = ue^{i\theta}.$$

Clearly enough $\pi A(r)$ gives, with due regard to the multiplicity, the area of the image of $|z| < r$ onto the Riemann sphere under the mapping $f(z)$. Evidently, $A(r) \equiv 0$ if and only if $f(z)$ is constant, otherwise $A(r, f)$ is non-negative, non-decreasing and continuous function of r [8, p. 125]. This being so, $A(r, f)$ represents the behaviour of the growth of $f(z)$ on $|z| = r$. In fact, if $n(r, a)$ denotes the number of a-points of $f(z)$ in $|z| \leqq r$, then $A(r, f)$ is the average value of $n(r, a)$ as a moves over the Riemann sphere.

Below is the statement only of the first fundamental theorem of Nevanlinna in the form of Ahlfors [1] and Shimizu [8], which is frequently used hereafter.

THEOREM: [2, p. 12; 5, p. 166] *Suppose that $f(z)$ is meromorphic in the open disc $|z| < R$, $0 < R < \infty$. Then for every a, finite or infinite, and $0 < r < R$ we hav*

$$\int_0^r (A(t)/t)dt = N(r, a) + m^*(r, a) - m^*(o, a), \tag{1.1}$$

provided $f(0) \neq a$, and $N(r, a)$ has the usual meaning assigned in the Nevanlinna theory of meromorphic functions. (If $f(0) = a$, a slight modification is necessary).

The integral in the left side of (1.1) will be called the Ahlfors-Shimizu characteristic function of $f(z)$ and subsequently will be denoted as $T^*(r, f) = T^*(r)$. Nevanlinna [5, p. 167] calls $T^*(r, f)$ "spherical normal" form of his characteriatic $T(r, f)$. For non-constant meromorphic functions, it is clear that $T^*(r)$ is a positive, strictly increasing and convex function of $\log r$. Also it can easily be shown [2, p. 13] that $T(r, f)$ and $T^*(r, f)$ differ by a bounded function of r. However, $T^*(r)$ gives a very elegant geometrical interpretation of the Nevanlinna's characteristic.

We get for all $r > 1$, since $A(r)$ is increasing, $T^*(r) > A(1)\log r$; and hence $\liminf_{r\to\infty} T^*(r)/\log r > 0$, unless $f(z)$ is constant. If $f(z)$ is a rational function as the quotient $P_m(z)/P_n(z)$, of two polynomials of degree m and n respectively, then $A(r, f) = O(1)$ and conversely. In fact [8, p. 131] $A(r, f) = \operatorname{Max}(m, n) + O(1)$. By fairly elementary methods it can be shown [7, 8] that for a meromorphic function of order ρ and lower order λ we have, as $r \to \infty$ lim sup [inf] $\log T^*(r)/\log r = \rho[\lambda] =$ lim sup [inf] $\log A(r)/\log r$.

For our use we set the following notations:

$$\limsup\,[\inf]_{r\to\infty} T^*(r, f)/r^\rho = T[t],$$

$$\limsup\,[\inf]_{r\to\infty} A(r, f)/r^\rho = A[a].$$

Wherever it appears, $\exp(x) = e^x$.

2. **Comments and statements of results**. The following inequalities were established [6] among these numbers A, a, T and t.

$$a \leqq A \exp((a/A) - 1) \leqq \rho T \leqq A \leqq e\rho T,\ t < \infty, \tag{2.1}$$

$$a \leqq \rho t \leqq a \cdot \log(eA/a) \leqq A,\ a \neq 0,\ t < \infty, \tag{2.2}$$

$$A + a \leqq e\rho T, \tag{2.3}$$

(2. 4) $A = a$ if and only if $a = \rho T$.

Consequently equality can not hold simultaneously in (2. 3) and (2. 4), and hence 'a fortiori' it can never hold simultaneously in the inequalities $A \leqq e\rho T$ and $a \leqq \rho T$ provided $T > 0$. We also observe that if $A = 0$ then $T = 0$ and conversely. If $a = \infty$ then $t = \infty$; and if $t = \infty$ then $A = \infty$. Thus A is finite and non-zero if and only if T is finite and non-zero. Furthermore, if $t = 0$ then $a = 0$; and if $a = 0$, $A < \infty$ then $t = 0$. If $t < \infty$ then either $A = T = \infty$ or $A = T = 0$ or $0 < A < \infty$, $0 < T < \infty$. Thus $0 \leqq a \leqq A \leqq \infty$ if and only if $0 \leqq t \leqq T \leqq \infty$.

The inequalities (2.1) – (2.4) were established by comparing the functions $T^*(r)$ and $A(r)$ with a more general function $r^\rho L(r)$ where $L(r)$ is a 'slowly changing' function in the sense of Karamata; that is $L(r) > 0$ and continuous for $r > r_0$ and $L(kr) \sim L(r)$ as $r \to \infty$, for every $k > 0$. It is remarked that these results hold if we use $r^{\rho(r)}$ as a comparison function, where $\rho(r)$ is the Lindelöf's proximate order of $f(z)$. In fact, $r^{\rho(r)-\rho}$ is a slowly changing function [3]. In this paper, for the sake of simplicity, we confine ourselves to the function r^ρ but the results and the arguments hold also with reference to the function $r^\rho L(r)$.

It is clear that if $a \neq 0$ and $t < \infty$, then the in-equalities (2.1) and (2.2) give us the following set of inequalities

(2. 5) $a \leqq \rho t \leqq \rho T \leqq A$,

(2. 6) $A/\rho T \leqq \exp \{1 - (a/A)\}$,

(2. 7) $a/\rho t \geqq \{1 + \log(A/a)\}^{-1}$

Therefore the existence of the limit of the ratio $A(r)/r^\rho$, as $r \to \infty$ implies that the limit of the ratio $T^*(r)/r^\rho$ exists and is equal to A/ρ. We now raise the question, which might well be raised for any Tauberian theorem, that does the existence of the limit of the ratio $T^*(r)/r^\rho$ as $r \to \infty$ imply that of the ratio $A(r)/r^\rho$, and will it be equal to the number ρT? The problem would rather be better appreciated if we reduce it to the form; Do there exist unique functions (say) g and h of the parameters T and t such that $g(T, t) \geqq A/\rho T \geqq 1 \geqq a/\rho t \geqq h(T, t)$; and as $t \to T$, both the functions g and h approach to unity? In spite of the simplicity of the problem, it does not seem possible to solve it, in any manner, from the set of the inequalities proved earlier. Nevertheless, the answer to the problem is in the affirmatve and the Tauberian theorem we propose to prove may well be stated as follows.

THEOREM 1. *If* $0 < t = T < \infty$, *then* $0 < a = A = \rho T$.

Among several other examples to support the theorem we pick the following [4, p. 15] doubly-periodic Weierstrass $\wp$-function.

$$\wp(z) = (z)^{-2} + \Sigma_{m,\,n}\{(z + m + in)^{-2} - (m + in)^{-2}\},$$
$$(m, n = \pm 1, \pm 2, \pm 3, \cdots)$$
$$T^*(r, \wp(z)) \sim \pi r^2 + B(r), \text{ where } B(r) = O(1),$$
$$A(r, \wp(z)) \sim 2\pi r^2.$$

The results cited below in Theorem 2 are basic for the subsequent and contribute to the proof of Theorem 1.

THEOREM 2. *For any non-constant meromorphic function $f(z)$ of finite non-zero order ρ,*

$$\rho T \geqq A \exp((\rho t/A) - 1), \tag{2.8}$$

$$\rho T \geqq a \exp((\rho t/a) - 1). \tag{2.9}$$

It should be observed that neither (2.8) nor (2.9) could be deduced from the previous results (2.1), (2.2) and (2.3). On the contrary, by a few easy calculations it can be seen that the inequalities (2.8) and (2.9) respectively sharpen the middle part of the inequalities (2.1) and (2.2). Moreover, from (2.8) one easily gets that $e\rho T \geqq A\exp(\rho t/A)$, and since for any real x, $\exp(x) \geqq 1+x$, we finally get, $e\rho T \geqq A+\rho t$ which is a strengthening of (2.3). Furthermore, from (2.9) we obtain $\rho t \leqq a(1+\log(\rho T/a))$ and now, since for all $x \geqq 1$, $\log x \leqq x/e$, we have $0 \leqq (\rho t - a) \leqq (\rho T)/e$. We collect all these deductions, and the changes produced by (2.8) and (2.9) in the previous results, in the form of the following corollaries.

COROLLARY 1. *For $t < \infty, a \neq 0$, $A \neq 0$, we therefore have*

$$a \leqq A\exp((a/A) - 1) \leqq A\exp((\rho t/A) - 1) \leqq \rho T \leqq A \leqq e\rho T, \tag{2.10}$$

$$a \leqq \rho t \leqq a(1 + \log(\rho T/a)) \leqq a(1 + \log(A/a)) \leqq A. \tag{2.11}$$

COROLLARY 2. *Under the restrictions of corollary* 1, *we have*

$$(a + A) \leqq A + \rho t \leqq e\rho T, \tag{2.12}$$

$$0 \leqq (\rho t - a) \leqq (\rho T)/e. \tag{2.13}$$

From corollary 2 without any efforts we obtain

COROLLARY 3.

$$(A - ea) \leqq \rho(1 + e)(T - t). \tag{2.14}$$

From Theorem 2 we also deduce the theorem cited below, which with the aid of (2.4) proves the assertion claimed in Theorem 1.

THEOREM 3. *For any non-constant meromorphic function $f(z)$ of finite non-zero order ρ,*

$$(a/\rho t) \geqq h(T/t)\,, \tag{2.15}$$

$$(A/\rho t) \leqq (A/\rho t) \leqq g(T/t)\,, \tag{2.16}$$

where both the functions h and g are unique and continuous; $h(T/t)$ approaches to unity as $(T/t) \to 1$, and it decreases from unity to zero as the ratio T/t increases from unity to infinity. The funtion $g(T/t)$ approaches also to unity as $(T/t) \to 1$, but it increases from unity to infinity as the ratio T/t increases from unity to infinity. In fact, when T/t is large $g(T/t) \sim e(T/t)$; $h(T/t) = o\,(eT/t)$. While, when T/t is close to unity we have $g(T/t) \sim 1 + \sqrt{2((T/t) - 1)} \sim h(T/t)$.

In the light of Theorem 3, we further state the following

COROLLARY 4.

$$\rho T \leqq A \leqq \rho t g(T/t) \leqq \rho t g(A/\rho t), \tag{2.17}$$

$$\rho t h(T/t) \leqq a \leqq \rho t. \tag{2.18}$$

We observe that the two pairs of inequalities {(2.9), (2.15)} and {(2.8), (2.16)} yield easily that $a \geqq \text{Max}\,\{\rho t h(T/t), \rho t(1 + \log(\rho T/a)^{-1}\}$ and $A \leqq \text{Min}\,\{\rho t g\,(T/t),\ \rho T \exp(1 - (\rho t/A))\}$. But we present the following precise order relations between these numbers.

THEOREM 4. *For any non-constant meromorphic function $f(z)$ of finite non-zero order ρ,*

$$a \geqq \rho t(1 + \log(\rho T/a))^{-1} \geqq \rho t h(T/t), \tag{2.19}$$

$$A \leqq \rho T \exp(1 - (\rho t/A)) \leqq \rho t g(T/t). \tag{2.20}$$

3. Subsequently we shall need the following simple lemma.

LEMMA. *Given the function*

$$ey = x\exp(1/x),\ 0 < x < \infty, \tag{3.1}$$

there do exist two unique real-valued inverse functions $x=g(y)$ and $x=h(y)$ defined for $1 \leqq y < \infty$, which are positive, continuous, $g(1)=1=h(1)$ and both of them satisfying the relation (3.1). *Furthermore,*

(i) *$g(y)$ increases steadily from unity to infinity, and $h(y)$ decreases steadily from unity to zero,*

(ii) *both the functions $g(y)$ and $h(y)$ are differentiable in a finite open interval $(1, y_0)$ and*

$$g'(y) = g(y)\{g(y) - 1\}^{-1}.\ \exp\ (1 - (1/g(y)))\,,$$

same for $h'(y)$ with $g(y)$ replaced by $h(y)$ in the above expression,

(iii) *when y is large, $g(y) \sim ey$; $h(y) = o(ey)$,*

(iv) *when y is close to unity, $g(y) \sim 1+\sqrt{2(y-1)} \sim h(y)$,*

PROOF OF LEMMA. The proof is based upon the elementary techniques of calculas. First, consider (3.1) for all $x \geqq 1$. We notice that for $x \geqq 1$, $y(x)$ is positive, continuous and $y(1)=1$. Its derivative $y'(x)=\{1-(1/x)\}\ \exp\ ((1/x)-1)$ remains positive for all $x \geqq 1$. Hence $y(x)$ increases steadily from unity to infinity for $x \geqq 1$. Therefore there exists a unique inverse function, say, $g(y)$ defined for $y \geqq 1$, continuous, positive, $g(1)=1$ and differentiable for $(1, y_0), y_0 < \infty$. Clearly $g(y)$ satisfies the equation $ey=g(y)\ \exp(1/g(y))$. Moreover, since the derivative

$$g'(y) = g(y)\{g(y) - 1\}^{-1}\exp(1 - (1/g(y)))$$

stays positive for all $y > 1$, hence $g(y)$ increases from unity to infinity. Obviously, for large y, $g(y) \sim ey$. Thes proves (i), (ii) and (iii). To prove (iv), we observe that for $x=1+\varepsilon$, $\varepsilon > 0$ and arbitrarily small

$$\begin{aligned} y &= (1 + \varepsilon)\exp\{(1 + \varepsilon)^{-1} - 1\} = (1 + \varepsilon)\exp\{- \varepsilon + \varepsilon^2 - \varepsilon^3 + \cdots\} \\ &= (1 + \varepsilon)\{1 - \varepsilon + (3/2)\varepsilon^2 + O(\varepsilon^3)\} = 1 + (1/2)\varepsilon^2 + O(\varepsilon^3) \\ &= 1 + (1/2)(x - 1)^2 + O(x - 1)^3. \end{aligned}$$

Hence $g(y) \sim 1+\sqrt{2(y-1)}$. This proves all the assertions made for $g(y)$. The assertions connected with $h(y)$ can similarly be proved by considering the relatien (3.1) on the interval $0 < x \leqq 1$. This completes the proof of the Lemma.

4. **Proof of Theorems**. We begin by proving Theorem 2. The proof of Theorem 1 has already been furnished in the comments right after the corollary 3.

PROOF OF THEOREM 2. The proof is straightforward. Pick a number $A' > 0$ such that $A - A' = \varepsilon > 0$. Suppose $A(r_1) > A' r_1^\rho$, where $r_1 = r_1(A')$. Then by Ahlfors-Shimizu theorem, since $A(r)$ increasing, we have for all $r > r_1$

$$T(r) \geqq T(r_1) + A' r_1^\rho \int_{r_1}^{r} \left(\frac{du}{u}\right) + O(1),$$

$$= T(r_1) + A' r_1^\rho \{\log(r/r_1)\} + O(1)\,. \tag{4.1}$$

Also, it is possible to choose r_1 such that $T(r_1) > t' r_1^\rho$ with $t - t' = \varepsilon$. Therefore from (4.1) for all $r > r_1$ we obtain

$$T(r)/r^\rho > (r_1/r)^\rho \{t' + A' \log(r/r_1)\} + o(r^\rho)\,. \tag{4.2}$$

Now, by usual mathods of calculus we maximize the first term of the right hand side of (4.2). We find that its maxima is attained for that value of r which satisfies the relation

$$r/r_1 = \exp\{(A' - \rho t')/\rho A'\} \tag{4.3}$$

and that maximum value is $(A'/\rho)\exp\{(\rho t' - A')/A'\}$ Therefore from (4.2) we finally get,

$$T(r)/r^\rho \geqq (A'/\rho)\exp\{(\rho t' - A')/A'\} + o(r^\rho) \tag{4.4}$$

for r satisfying (4.3). We may take r such that $r_1 \to \infty$. In that case, r satisfying (4.3) tends to infinity and we see that $\rho T \geqq A' \exp\{(\rho t'/A) - 1)\}$.
Now, since A' can be picked arbitrarily close to A and t' arbitrarily close to t, we immediately deduce from (4.4) the required result, namely $\rho T \geqq A \exp\{(\rho t/A) - 1\}$. This proves (2.8). The inequality (2.9) may be proved in a similar manner. This completes the proof of Theorem 2.

PROOF OF THEOREM 3. With the aid of the lemma, the proof is simple. The inequality (2.8) yields

$$T/t \geqq (A/\rho t)\exp\{(\rho t/A) - 1\}\,. \tag{4.5}$$

Let us consider the function (3.1)

$$y = x \exp\{(1/x) - 1\}$$

with $x = A/\rho t \geqq 1$. Then, in view of the conclusions made in the lemma, there exists a unique, continuous and steadily increasing function $g(y)$ such that $g(y) = A/\rho t$. But from (4.5) $T/t \geqq y$ and hence, since g is increasing, we finally get

$$g(T/t) \geqq g(y) = A/\rho t.$$

This proves (2.16). The proof for (2.15) also depends on the lemma and is similar to that of (2.16). This proves Theorem 3.

PROOF OF THEOREM 4. The proof of (2.19) is based upon the fact that (2.15) of Theorem 3 holds. In fact, we try to show that (2.15) is a necessary and sufficient condition for the validity of (2.19). So we begin with (2.15). We have

$$a \geqq \rho t\, h(T/t)\,, \tag{4.6}$$

and hence

$$(T/t)\,/h(T/t) \geqq \rho T/a\,.$$

Since the function h satisfies the relation (3.1), we obtain

$$(T/t)/h(T/t) \equiv \exp\{(1/h(T/t)) - 1\} \geqq \rho T/a\,.$$

Therefore, we finally get

$$\rho t \cdot h(T/t) \leqq \rho t\{1 + \log(\rho T/a)\}^{-1}\,. \tag{4.7}$$

On the other hand, by retracing the steps, (4.7) leads us back to (4.6) from which we now deduce that

$$h^{-1}(a/\rho t)/(a/\rho t) \leqq \rho T/a\,. \tag{4.8}$$

Since $h^{-1}(x) = y = x \exp\{(1/x) - 1\}$, (4.8) yields

$$(\rho t/a)(h^{-1}(a/\rho t)) = \exp\{(\rho t/a) - 1\} \leqq \rho T/a\,,$$

that is,

$$a \geqq \rho t\{1 + \log(\rho T/a)\}^{-1}. \tag{4.9}$$

Again by retracing the steps we observe that (4.9) leads us back to (4.6). In fact, from (4.9) we can also deduce the relation

$$T/t \geqq h^{-1}(a/\rho t),$$

and since h is decreasing, therefore

$$a \geqq \rho t \cdot h(T/t),$$

which, in turn, as shown above gives us (4.9). This completes the proof of the inequality (2.19). The arguments to prove (2.20) run parallel to that of (2.19) and details may be omitted. In fact, now one tries to show that (2.16), namely $A \leqq \rho t g(T/t)$ is a necessary and sufficient condition for the validity of the inequality (2.20). Thus Theorem 4 is completely proved.

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DEPARTMENT OF MATHEMATICS
OHIO UNIVERSITY, ATHENS, OHIO
U. S. A.