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TOPOLOGICAL NON-DEGENERATE FUNCTIONS

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1. **Introduction**. The theory of C^∞ non-degenerate functions has been useful in the study of differentiable manifolds. The theory of topological non-degenerate functions is much less developed. Morse [5] has established the Morse inequalities. Kuiper [3] has shown that any compact n-manifold which admits a topological non-degenerate function with two critical points is homeomorphic to S^n. Eells and Kuiper [2] have studied compact manifolds which admit a topological non-degenerate function with three critical points. In this paper we prove:

THEOREM 1.1. *Suppose f is a topological non-degenerate function defined on a compact n-manifold. If $[a,b]$ is an interval of regular values of f then $f^{-1}(a)\times(0,1)$ is homeomorphic to $f^{-1}(b)\times(0,1)$.*

THEOREM 1.2. *Suppose a compact n-manifold M admits a topological non-degenerate function f such that all the critical points of f of index λ lie at the level λ. Then M admits a cell decomposition with exactly as many cells of dimension λ as f has critical points of index λ.*

Theorem 1.1 illustrates some of the difficulties in the theory of topological non-degenerate functions. If $[a,b]$ is an interval of regular values of a C^∞ non-degenerate function, it is easy to show that $f^{-1}(a)$ is homeomorphic to $f^{-1}(b)$ (Milnor [4], p. 12). Whether or not this is true in the topological case is an open question. Theorem 1.2 is a partial solution of a problem of Eells and Kuiper [2], p. 195. We intend to give a more complete solution to this problem in a future paper.

2. **Notation and Terminology**. We refer to Morse [5] as a general reference for §2 and §3. We denote cartesian n-space by R^n. Let $N^i_\varepsilon=\{(z_1,\cdots,z_n)\in R^n\,|\,(z_1^2+\cdots+z_i^2)^{\frac{1}{2}}<\varepsilon$ and $(z_{i+1}^2+\cdots+z_n^2)^{\frac{1}{2}}<\varepsilon\}$, $i=0,\cdots,n;\varepsilon>0$. An *n-manifold* is a separable metric space each point of which has a neighbour-

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hood homeomorphic to R^n.

Suppose f is a real-valued function on M. A point $x \in M$ is a *topological regular point* if there exists an $\varepsilon > 0$ and a homeomorphism $h: N_\varepsilon^{n-1} \longrightarrow M$ such that the image of h is an open neighborhood of x and $f \circ h(z_1, \cdots, z_n) = f(x) + z_n$ if $(z_1, \cdots, z_n) \in N_\varepsilon^{n-1}$. A point $x \in M$ is a *topological critical point* if it is not a topological regular point. A point $x \in M$ is a *topological critical point of index* λ if there exists $\varepsilon > 0$ and a homeomorphism h: $N_\varepsilon^\lambda \longrightarrow M$ such that the image of h is an open neighborhood of x and $f \circ h(z_1, \cdots, z_n) = f(x) - z_1^2 - \cdots - z_\lambda^2 + z_{\lambda+1}^2 + \cdots + z_n^2$ if $(z_1, \cdots, z_n) \in N_\varepsilon^\lambda$. It is easy to see that every topological critical point of index λ is indeed a topological critical point. In both of the above cases we refer to the function h as an *f-coordinate function* and to the open set $h(N_\varepsilon^i)$ as an *f-neighborhood.* The function f is a *topological non-degenerate function* if every critical point of f is a topological critical point of index λ for some λ $(0 \leqq \lambda \leqq n)$. If f is a topological non-degenerate function it is clear that the critical points of f are isolated. If, further, M is compact, they will be finite in number.

We speak of the set $f^{-1}(a)$ as the *f-level a*. We say $x \in M$ is *below the f-level a* if $f(x) \leqq a$. Corresponding definitions can be given of x is *above, strictly above,* and *strictly below* the f-level a.

If $M = X_n \supset \cdots \supset X_1 \supset X_0$ is a sequence of closed subsets of a compact n-manifold M such that X_0 is a finite set of points and $X_i - X_{i-1}$ is the disjoint union of a finite number of open i-cells, $i = 1, \cdots, n$, then $\{X_i\}$ is called a *cell decomposition* of M.

3. Level Lowering Homeomorphisms. Suppose $x_0 \in M$ and $h: N_\varepsilon^i \longrightarrow M$ is an f-coordinate function such that $h(0) = x_0$. If $z = (z_1, \cdots, z_n) \in N_\varepsilon^i$, $i = 0, \cdots, n$ we agree to write $z = (\zeta_1, \zeta_2)$ where $\zeta_1 = (z_1, \cdots, z_i)$, $\zeta_2 = (z_{i+1}, \cdots, z_n)$. Note that if $i = n$, $z = \zeta_1$ and if $i = 0$, $z = \zeta_2$. We have two cases to consider.

Case 1. x_0 is a topological regular point and $i = n - 1$. Define a homeomorphism $\rho: N_\varepsilon^{n-1} \longrightarrow N_\varepsilon^{n-1}$ by

$$\rho(\zeta_1, z_n) = (\zeta_1, z_n + (1 - |\zeta_1|/\varepsilon)(z_n^2/2\varepsilon - \varepsilon/2)).$$

Define $H: M \longrightarrow M$ by:

$$H(x) = h \circ \rho \circ h^{-1}(x) \text{ if } x \in h(N_\varepsilon^{n-1})$$

$$= x \qquad \text{if } x \notin h(N_\varepsilon^{n-1}).$$

It is clear that H is a homeomorphism and that $f(H(x)) < f(x)$ if $x \in h(N_\varepsilon^{n-1})$.

Case 2. x_0 is a topological critical point of index λ and $i = \lambda$. Define homeomorphisms $\rho_1, \rho_2 : N_\varepsilon^\lambda \longrightarrow N_\varepsilon^\lambda$ by

$$\rho_1(\zeta_1, \zeta_2) = ((2 - |\zeta_2|/\varepsilon)\zeta_1, \zeta_2) \quad \text{if } 0 \leqq |\zeta_1| \leqq \varepsilon/3$$
$$= \left(\frac{1}{2}(1 + |\zeta_2|/\varepsilon + \varepsilon/|\zeta_1| - |\zeta_2|/|\zeta_1|)\zeta_1, \zeta_2\right)$$
$$\text{if } \varepsilon/3 \leqq |\zeta_1| \leqq \varepsilon.$$
$$\rho_2(\zeta_1, \zeta_2) = (\zeta_1, (|\zeta_2|^2/\varepsilon^2 - |\zeta_1||\zeta_2|^2/\varepsilon^3 + |\zeta_1|/\varepsilon)\zeta_2).$$

Define $H : M \longrightarrow M$ by:

$$H(x) = h \circ \rho_2 \circ \rho_1 \circ h^{-1}(x) \quad \text{if } x \in h(N_\varepsilon^\lambda)$$
$$= x \quad \text{if } x \notin h(N_\varepsilon^\lambda).$$

It is clear that H is a homeomorphism, that $f(H(x)) < f(x)$ if $x \in h(N_\varepsilon^\lambda)$, $x \neq x_0$, and that H maps $\{h(\zeta_1, \zeta_2) | \, |\zeta_1| \leqq \varepsilon/3, \zeta_2 = 0\}$ homeomorphically onto $\{h(\zeta_1, \zeta_2) | \, |\zeta_1| \leqq 2\varepsilon/3, \zeta_2 = 0\}$.

In both cases we refer to H as the *f-homeomorphism* corresponding to h.

We need two lemmas. Hereafter we assume M is a compact n-manifold and f is a topological non-degenerate function on M.

LEMMA 3.1. *If $a < c < d < b$ and $[a,b]$ is an interval of regular values, then there exists a homeomorphism $D : M \longrightarrow M$ such that $D(x) = x$ if $f(x) \geqq b$ or $f(x) \leqq a$ and $f(D(x)) \leqq c$ if $f(x) \leqq d$.*

PROOF. Let $h_1, \cdots, h_q$ be a set of f-coordinate functions such that the f-neighborhoods $\{h_i(N_{\varepsilon_i}^{n-1})\}$ form an open cover of $f^{-1}([c,d])$ and do not meet $f^{-1}((-\infty, a] \cup [b, \infty))$. Let H_i be the f-homeomorphism corresponding to h_i, $i = 1, \cdots, q$. Define a homeomorphism $E : M \longrightarrow M$ by $E = H_q \circ \cdots \circ H_2 \circ H_1$. Clearly if $x \in f^{-1}([c,d])$ then $x \in h_i(N_{\varepsilon_i}^{n-1})$ for some i so that $f(E(x)) < f(x)$. Furthermore if $f(x) \geqq b$ or $f(x) \leqq a$ it is clear that $H_i(x) = x, i = 1, \cdots, q$ so that $E(x) = x$. By compactness there exists a positive integer m such that $f(E(x)) \leqq f(x) - (d-c)/m$ for every $x \in f^{-1}([c,d])$. Let $D = E^m$. Clearly D is a homeomorphism and $D(x) = x$ if $f(x) \geqq b$ or $f(x) \leqq a$. It is also clear that $f(D(x)) \leqq d - m(d-c)/m = c$ for all $x \in f^{-1}([c,d])$. We have established Lemma 3.1.

LEMMA 3.2. *There exists a homeomorphism $D : M \longrightarrow M$ such that:*
i) *$f(D(x)) < f(x)$ if x is a topological regular point;*
ii) *for each critical point $\xi \in M$ of index λ $(0 \leqq \lambda \leqq n)$ there exists an*

$\varepsilon > 0$ and an f-coordinate function h: $N_\varepsilon^\lambda \longrightarrow M$ such that $D(h(\zeta_1, 0)) = h(2\zeta_1, 0)$ if $|\zeta_1| \leqq \varepsilon/3$.

PROOF. Let $h_1, \cdots, h_q$ be a set of f-coordinate functions such that the f-neighborhoods $h_i(N_{\varepsilon_i}^{j(i)})$ cover M. Suppose that $h_1(0), \cdots, h_p(0)$ are regular points and that $h_{p+1}(0), \cdots, h_q(0)$ are critical points. Suppose further that the h_i have been chosen so that $h_i(N_{\varepsilon_i}^{n-1}) \cap \{h_k(\zeta_1, 0) \mid |\zeta_1| \leqq 2\varepsilon_k/3\} = \emptyset$, $i = 1, \cdots, p$; $k = p+1, \cdots, q$. Let H_i be the f-homeomorphism corresponding to h_i, $i=1, \cdots, q$. Define $D = H_q \circ \cdots \circ H_2 \circ H_1$. Clearly $f(D(x)) < f(x)$ if x is a topological regular point. If ξ is a topological critical point there exists k $(p+1 \leqq k \leqq q)$ such that $h_k(0) = \xi$. Then $D(h_k(\zeta_1, 0)) = h_k(2\zeta_1, 0)$ if $|\zeta_1| \leqq \varepsilon_k/3$. Lemma 3.2 is proved.

4. **Proof of Theorem 1.1.** We first prove:

THEOREM 4.1. *If $[a, b]$ is an interval of regular values, then there exists a homeomorphism K_∞ from $f^{-1}(b) \times (0, 1]$ onto $f^{-1}((a, b])$ such that $K_\infty(x, 1) = x$ for all $x \in f^{-1}(b)$.*

PROOF. Since a and b are both regular values, $f^{-1}(a)$ and $f^{-1}(b)$ are both locally flat $(n-1)$-manifolds in M. Therefore by a result of Brown [1], $f^{-1}(a)$ and $f^{-1}(b)$ are both collared. Thus there exist homeomorphisms $H: f^{-1}(a) \times [0, 1] \longrightarrow f^{-1}([a, b))$ and $K: f^{-1}(b) \times [0, 1] \longrightarrow f^{-1}((a, b])$ such that $H(x, 0) = x$ for all $x \in f^{-1}(a)$, $K(x, 1) = x$ for all $x \in f^{-1}(b)$, and $H(f^{-1}(a) \times [0, 1]) \supset f^{-1}([a, a+\varepsilon])$, $K(f^{-1}(b) \times [0, 1]) \supset f^{-1}([b-\varepsilon, b])$ for some $\varepsilon > 0$. By Lemma 3.1 there exists a homeomorphism $D: M \longrightarrow M$ such that D is the identity outside $f^{-1}((a, b))$ and D moves the set $f^{-1}(b-\varepsilon)$ strictly below the f-level $(a+\varepsilon)$.

Let $K_0 = D \circ K$ and $H_0 = H$. Then $H_0: f^{-1}(a) \times [0, 1] \rightarrow M$ and $K_0: f^{-1}(b) \times [0, 1] \longrightarrow M$ are homeomorphisms, $H_0(x, 0) = x$ for all $x \in f^{-1}(a)$, $K_0(x, 1) = x$ for all $x \in f^{-1}(b)$, $H_0(f^{-1}(a) \times [0, 1)) \cup K_0(f^{-1}(b) \times (0, 1]) = f^{-1}([a, b])$, and there exists c_0 $(0 < c_0 < 1)$ such that $H_0(f^{-1}(a) \times [0, 1]) \cap K_0(f^{-1}(b) \times [c_0, 1]) = \emptyset$.

If $0 < c < d < 1$ and $k: f^{-1}(a) \times [0, 1] \rightarrow M$ or $k: f^{-1}(b) \times [0, 1] \longrightarrow M$ is a homeomorphism into, define a homeomorphism $E(c, d, k): M \longrightarrow M$ by:

$$\begin{aligned} E(c, d, k)(k(x, t)) &= k(x, c + (1-c)(t-d)/(1-d)) && \text{if } d \leqq t \leqq 1 \\ &= k(x, ct/d) && \text{if } 0 \leqq t \leqq d, \\ E(c, d, k)(y) &= y && \text{if } y \notin \text{image } k. \end{aligned}$$

We proceed to define a sequence of real numbers c_m and a double sequence H_m and K_m of homeomorphisms using induction. Our inductive hypothesis is:

P_m : There exists a real number c_m $(0 < c_m \leqq (1/2)^m$ and $c_m < c_{m-1})$ and homeomorphisms $H_m : f^{-1}(a) \times [0, 1] \longrightarrow M$ and $K_m : f^{-1}(b) \times [0, 1] \longrightarrow M$ such that $H_m(x, 0) = x$ for all $x \in f^{-1}(a)$ and $K_m(x, 1) = x$ for all $x \in f^{-1}(b)$ and:

i) $K_m(f^{-1}(b) \times [0, c_m])$ lies strictly below the f-level $a + (1/2)^m$;

ii) $K_m(f^{-1}(b) \times 0) \subset H_m(f^{-1}(a) \times [0, 1))$;

iii) $H_m(f^{-1}(a) \times [0, 1]) \cap K_m(f^{-1}(b) \times [c_m, 1]) = \emptyset$.

We first establish P_1. Choose $0 < d_1 < e_1 < 1$ such that $H_0(f^{-1}(a) \times [0, d_1])$ lies strictly below the f-level $a + 1/2$ and $K_0(f^{-1}(b) \times 0) \subset H_0(f^{-1}(a) \times [0, e_1])$. Define $K_1 = E(d_1, e_1, H_0) \circ K_0$. Clearly $K_1 : f^{-1}(b) \times [0, 1] \longrightarrow M$ is a homeomorphism such that $K_1(x, 1) = x$ for all $x \in f^{-1}(b)$, $K_1(f^{-1}(b) \times 0)$ lies strictly below the f-level $a + 1/2$, $K_1(f^{-1}(b) \times 0) \subset H_0(f^{-1}(a) \times [0, 1))$, and $H_0(f^{-1}(a) \times [0, 1]) \cap K_1(f^{-1}(b) \times [c_0, 1]) = \emptyset$. Choose $c_1 (0 < c_1 \leqq 1/2$ and $c_1 < c_0)$ such that $K_1(f^{-1}(b) \times [0, c_1])$ lies strictly below the f-level $a + 1/2$. Define $H_1 = E(c_1, c_0, K_1) \circ H_0$. Clearly $H_1 : f^{-1}(a) \times [0, 1] \longrightarrow M$ is a homeomorphism such that $H_1(x, 0) = x$ for all $x \in f^{-1}(a)$, $K_1(f^{-1}(b) \times 0) \subset H_1(f^{-1}(a) \times [0, 1))$, and $H_1(f^{-1}(a) \times [0, 1]) \cap K_1(f^{-1}(b) \times [c_1, 1]) = \emptyset$. P_1 is established.

Now suppose P_m is true. Choose $0 < d_{m+1} < e_{m+1} < 1$ such that $H_m(f^{-1}(a) \times [0, d_{m+1}])$ lies strictly below the f-level $a + (1/2)^{m+1}$ and $K_m(f^{-1}(b) \times 0) \subset H_m(f^{-1}(a) \times [0, e_{m+1}])$. Define $K_{m+1} = E(d_{m+1}, e_{m+1}, H_m) \circ K_m$. $K_{m+1} : f^{-1}(b) \times [0, 1] \rightarrow M$ is a homeomorphism such that $K_{m+1}(x, 1) = x$ for all $x \in f^{-1}(b)$, $K_{m+1}(f^{-1}(b) \times 0)$ lies strictly below the f-level $a + (1/2)^{m+1}$, $K_{m+1}(f^{-1}(b) \times 0) \subset H_m(f^{-1}(a) \times [0, 1))$, and $H_m(f^{-1}(a) \times [0, 1]) \cap K_{m+1}(f^{-1}(b) \times [c_m, 1]) = \emptyset$. Choose $c_{m+1} (0 < c_{m+1} \leqq (1/2)^{m+1}$ and $c_{m+1} < c_m)$ such that $K_{m+1}(f^{-1}(b) \times [0, c_{m+1}])$ lies strictly below the f-level $a + (1/2)^{m+1}$. Define $H_{m+1} = E(c_{m+1}, c_m, K_{m+1}) \circ H_m$. Clearly $K_{m+1}(f^{-1}(b) \times 0) \subset H_{m+1}(f^{-1}(a) \times [0, 1))$ and $H_{m+1}(f^{-1}(a) \times [0, 1]) \cap K_{m+1}(f^{-1}(b) \times [c_{m+1}, 1] = \emptyset$. We have proven P_{m+1}. Note that K_{m+1} is defined so that $K_{m+1}(x, t) = K_m(x, t)$ if $c_m \leqq t \leqq 1$.

We define $K_\infty(x, t) = \lim_{m \to \infty} K_m(x, t)$. For $t \geqq c_m$, $K_m(x, t) = K_{m+1}(x, t) = \cdots$. Therefore this limit exists for all $(x, t) \in f^{-1}(b) \times (0. 1]$ and $K_\infty(x, t) = K_m(x, t)$ if $t \geqq c_m$. Thus $K_\infty : f^{-1}(b) \times (0, 1] \longrightarrow M$ is a one-one continuous map. Clearly $K_\infty(x, 1) = x$ for $x \in f^{-1}(b)$. Further, if $y_0 \in f^{-1}((a, b])$ then there exists an m such that y_0 is strictly above the f-level $a + (1/2)^m$. Thus there exists $(x_0, t_0) \in f^{-1}(b) \times (c_m, 1]$ such that $y_0 = K_m(x_0, t_0) = K_\infty(x_0, t_0)$. Therefore K_∞ maps onto $f^{-1}((a, b])$. Further, if V is a neighborhood of (x_0, t_0) we may assume $t > c_m$ if $(x, t) \in V$. Therefore $K_\infty(V) = K_m(V)$ is a neighborhood of y_0. Therefore K_∞^{-1} is continuous at y_0 so K_∞ is a homeomorphism. Theorem 4.1 is proved.

We can now prove Theorem 1.1. By Theorem 4.1 there exists a homeomorphism K_∞ from $f^{-1}(b) \times (0, 1]$ onto $f^{-1}((a, b])$ such that $K_\infty(x, 1) = x$ for all $x \in f^{-1}(b)$. Similarly there exists a homeomorphism K_∞^* from $f^{-1}(a)\times[0,1)$ onto $f^{-1}([a,b))$ such that $K_\infty^*(x,0)=x$ for all $x\in f^{-1}(a)$. Then F defined by $F(x,t)=(K_\infty^*)^{-1}(K_\infty(x,t))$ for $x\in f^{-1}(b)$ and $0<t<1$ is a homeomorphism from $f^{-1}(b)\times(0,1)$ onto $f^{-1}(a)\times(0,1)$.

5. Proof of Theorem 1.2. Let $D: M \longrightarrow M$ be the homeomorphism whose existence was established in Lemma 3.2. Let $\xi_1,\cdots,\xi_p$ be the critical points of f of index $\geqq 1$. Let $\lambda(i)$ be the index of ξ_i, $i=1,\cdots,p$. Then there exist $\varepsilon>0$ and homeomorphisms $h_i: N_\varepsilon^{\lambda(i)} \longrightarrow M, i=1,\cdots,p$, such that $h_i(0)=\xi_i$, $h_i(z_1,\cdots,z_n)=f(\xi_i)-z_1^2-\cdots-z_{\lambda(i)}^2+z_{\lambda(i)+1}^2+\cdots+z_n^2$, and $h_i(\zeta_1,0)=h_i(2\zeta_1,0)$ if $0\leqq|\zeta_1|\leqq\varepsilon/3$. For $i=1,\cdots,p$ define $H_i: R^{\lambda(i)}\longrightarrow M$ by:

$$\begin{aligned} H_i(x) &= h_i(x,0) && \text{if } 0\leqq|x|\leqq 2\varepsilon/3 \\ &= D\circ h_i(x(1-\varepsilon/3|x|),0) && \text{if } 2\varepsilon/3\leqq|x|\leqq\varepsilon \\ &\ \vdots \\ &= D^m\circ h_i(x(1-m\varepsilon/3|x|),0) && \text{if } (m+1)\varepsilon/3\leqq|x|\leqq(m+2)\varepsilon/3 \\ &\ \vdots \end{aligned}$$

Let X_0 be the set of critical points of f of index 0. Let $X_j=X_0\cup(\cup\{H_i(R^{\lambda(i)})\mid\lambda(i)\leqq j\})$. If $y\in M$, by a compactness argument, there exist $N\geqq 0$ and a critical point ξ_k such that $D^{-m}(y)$ lies within ε of ξ_k for all $m\geqq N$. Thus $y\in H_k(R^{\lambda(k)})$. Therefore $X_n=M$. Clearly H_i is a homeomorphism. If y is a limit point of $H_i(R^{\lambda(i)})$, then so also is $D^{-m}(y)$, $m\geqq N$ (N as above). Thus if $y\notin H_i(R^{\lambda(i)})$ then $y\in H_k(R^{\lambda(k)})$ with $\lambda(k)<\lambda(i)$. Therefore X_j is closed and X_j-X_{j-1} is the disjoint union of open j-cells. Theorem 1.2 is proved.

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