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A NOTE ON CONVERGENCE AND SUMMABILITY FACTORS

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This paper deals with summability factors of the type which transform summable series into absolutely summable series, i.e., factors of type $(A, |B|)$. All matrices under consideration will be normal ($a_{nk}=0$ for $k > n$; $a_{nn} \neq 0$). Theorem 1 gives the summability factors of type $(A, |B|)$ for a class of matrices where A satisfies a certain mean value theorem and $|A| \subseteq |B|$. In Theorem 2 the inclusion requirement is dropped and A is taken to be a weighted arithmetical mean. Theorem 3 gives $(A, |I|)$ factors, A having a mean value theorem. Theorem 4 uses an induction argument to obtain factors of type $(A^*, |B|)$, where $A^*=AP^k$, P is a weighted mean, and the factors are known for $(A, |B|)$. Finally, applications are given for specific methods.

1. Definitions and theorems.

DEFINITION 1. If a method A has the property that given integers n, m with $n \geqq m$ there exists an integer n' and a constant K depending only on A such that

$$\left| \sum_{k=0}^{m} a_{nk} s_k \right| \leqq K \left| \sum_{k=0}^{n'} a_{n'k} s_k \right| \qquad (0 \leqq n' \leqq m \leqq n)$$

then A is said to have a mean value theorem.

DEFINITION 2. If $\sum a_n \varepsilon_n \in |B|$ whenever $\sum a_n \in A$ we say that $\varepsilon_n \in (A, |B|)$.

DEFINITION 3. The matrices $\overline{A}$ and $\widehat{A}$ are defined by the equations

$$\sigma_n = \sum_{k=0}^{n} a_{nk} s_k = \sum_{k=0}^{n} \overline{a}_{nk} a_k ; \quad \sigma_n - \sigma_{n-1} = \sum_{k=0}^{n} \widehat{a}_{nk} a_k$$

where as usual $s_n = a_0 + \cdots + a_n$.

LEMMA 1. *If A has a mean value theorem and $s_n \in A$, then* $s_n = O\left(\frac{1}{a_{nn}}\right)$.

PROOF. See [4], Lemma 1.

THEOREM 1. *Suppose A has a mean value theorem,* $\sum_{k=0}^{n} a_{nk} = 1$, $a_{nk} \searrow 0$ *as* $n \nearrow \infty$, $a_{nk} > 0\ (k \leqq n)$, $\frac{a_{mk}}{a_{nk}} \widehat{a}_{nk} \nearrow$ *as* $k \nearrow (k \leqq n,\ m > n)$. *Also assume B is regular,* $b_{nk} \geqq b_{n+1,k}\ (k \leqq n)$, $|A| \subseteq |B|$ *and* $\frac{b_{kk}}{a_{kk}} = O\left(\frac{b_{k+1,k+1}}{a_{k+1,k+1}}\right)$.

Then $\varepsilon_n \in (A, |B|)$ *if and only if*

(i) $$\varepsilon_n = \sum_{k=n}^{\infty} \alpha_k \bar{a}_{kn}\,; \quad \sum_{k=0}^{\infty} |\alpha_k| < \infty$$

(ii) $$\sum_{n=0}^{\infty} \frac{b_{nn}}{a_{nn}} |\varepsilon_n| < \infty .$$

PROOF. A partial summation gives the formula

$$\sum_{k=0}^{n} \widehat{b}_{nk} a_k \varepsilon_k = \sum_{k=0}^{n} \widehat{b}_{nk} s_k \Delta\varepsilon_k + \sum_{k=0}^{n} (\Delta_k \widehat{b}_{nk}) s_k \varepsilon_{k+1}$$
$$= A_n + B_n .$$

The author has shown ([4], The proof of Theorem 1) that condition (i) is sufficient for $\sum s_k \Delta\varepsilon_k \in |A|$, hence by the assumed inclusion we have $\sum_{n=0}^{\infty} |A_n| < \infty$. For the remaining term we have the estimation

$$\sum_{n=0}^{\infty} |B_n| \leqq O(1) \sum_{k=0}^{\infty} \frac{|\varepsilon_{k+1}|}{a_{kk}} \sum_{n=k}^{\infty} |\widehat{b}_{nk} - \widehat{b}_{n,k+1}|$$
$$\leqq O(1) \sum_{k=0}^{\infty} \frac{b_{k+1,k+1}}{a_{k+1,k+1}} |\varepsilon_{k+1}| < \infty .$$

Thus the conditions are seen to be sufficient. To show the necessity of (ii) we introduce the inverse matrix.

$$\sum_{k=0}^{n} \widehat{b}_{nk} a_k \varepsilon_k = \sum_{j=0}^{n} \sigma_j \sum_{k=j}^{n} \widehat{b}_{nk} \bar{a}'_{kj} \varepsilon_k = \sum_{j=0}^{n} A_{nj} \sigma_j .$$

Since the matrix (A_{nj}) transforms every convergent sequence into an absolutely convergent series it is necessary [7] that

$$\sum_{n=0}^{\infty} |A_{nn}| = \sum_{n=0}^{\infty} \frac{b_{nn}}{a_{nn}} |\varepsilon_n| < \infty .$$

The necessity of (i) follows from a theorem of Peyerimhoff [6] since $\varepsilon_n \in (A, |B|)$ implies $\varepsilon_n \in (A, B)$.

THEOREM 2. *Let P be a weighted mean with $p_n > 0$, $P_n \to \infty$ and $\frac{P_n}{p_n} = O\left(\frac{P_{n+1}}{p_{n+1}}\right)$. Suppose in addition that B sasisfies $\sum_{k=0}^{n} b_{nk} = 1$, $b_{nk} \searrow 0$ as $n \nearrow \infty$, $b_{nn} = O(b_{n+1,n+1})$. Then $\varepsilon_n \in (P, |B|)$ if and only if*

(i) $$\varepsilon_n = \sum_{k=n}^{\infty} \alpha_k \left(1 - \frac{P_{n-1}}{P_k}\right); \sum_{k=0}^{\infty} |\alpha_k| < \infty$$

(ii) $$\sum_{n=0}^{\infty} \frac{P_n b_{nn}}{p_n} |\varepsilon_n| < \infty .$$

PROOF. The necessity follows from Theorem 1. Computing the $\widehat{B}$ transform of $\sum a_k \varepsilon_k$ we obtain the formula

$$\sum_{k=0}^{n} \widehat{b}_{nk} a_k \varepsilon_k = \sum_{j=0}^{n} \sigma_j \sum_{k=j}^{n} \widehat{b}_{nk} \overline{p}'_{kj} \varepsilon_k .$$

For the sufficiency it clearly suffices to show that

$$\sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \left| \sum_{k=j}^{n} \widehat{b}_{nk} \overline{p}'_{kj} \varepsilon_k \right| < \infty .$$

A straightforward computation gives the formula

$$\sum_{k=j}^{n} \widehat{b}_{nk} \overline{p}'_{kj} \varepsilon_k = \frac{P_j}{p_j} \varepsilon_j \Delta_j \widehat{b}_{nj} + P_j \widehat{b}_{nj} \Delta \left(\frac{\Delta \epsilon_j}{p_j}\right) - \frac{P_j}{p_{j+1}} \varepsilon_{j+2} \Delta_j \widehat{b}_{n,j+1}$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III} .$$

For I we calculate

$$\sum_{j=0}^{\infty} \frac{P_j}{p_j} |\varepsilon_j| \sum_{n=j}^{\infty} |\Delta_j \widehat{b}_{nj}| = 2 \sum_{j=0}^{\infty} \frac{P_j b_{jj}}{p_j} |\varepsilon_j| < \infty .$$

In case II we use the representation (i) to obtain

$$\sum_{j=0}^{\infty} P_j \left| \Delta\left(\frac{\Delta\varepsilon_j}{p_j}\right) \right| \sum_{n=j+1}^{\infty} \hat{b}_{n,j+1} = \sum_{j=0}^{\infty} |\alpha_j| < \infty.$$ (Note that our assumptions imply $\hat{B}$ is positive and absolutely regular [4].) In case III we have

$$\sum_{j=0}^{\infty} \frac{P_j}{p_{j+1}} |\varepsilon_{j+2}| \sum_{n=j+1}^{\infty} |\Delta_j \hat{b}_{n,j+1}| = 2 \sum_{j=0}^{\infty} \frac{P_j}{p_{j+1}} |\varepsilon_{j+2}| b_{j+1,j+1}$$

$$= O(1) \sum_{j=0}^{\infty} \frac{P_{j+2}}{p_{j+2}} b_{j+2,j+2} |\varepsilon_{j+2}| < \infty.$$

THEOREM 3. *If A has a mean value theorem and $a_{n+1,n+1} = O(a_{nn})$, then $\varepsilon_n \in (A, |I|)$ if and only if $\sum_{n=0}^{\infty} \frac{|\varepsilon_n|}{a_{nn}} < \infty$.*

PROOF. The necessity follows again from Theorem 1. On the other hand,

$$\sum_{k=0}^{\infty} |a_k \varepsilon_k| = \sum_{n=0}^{\infty} |\varepsilon_k| |s_k - s_{k-1}| \leqq O(1) \sum_{k=0}^{\infty} \left| \frac{\varepsilon_k}{a_{kk}} \right| < \infty,$$

using Lemma 1 and the condition on the diagonal elements of A.

DEFINITION 4. If $\varepsilon_n = \sum_{k=n}^{\infty} \alpha_k \bar{a}_{kn}$ with $\sum_{k=0}^{\infty} |\alpha_k| < \infty$ we say that $\varepsilon_n = \varepsilon_n(\bar{A}, \alpha)$.

DEFINITION 5. If $s_{n-1} \in A$ whenever $s_n \in A$ we use the notation $A \dot{\subseteqq} A$. The notation $|A| \dot{\subseteqq} |A|$ is defined analogously.

THEOREM 4. *Let P be a weighted mean and suppose A, B and P satisfy the following condions.*

(1) $p_n > 0$; $P_n \to \infty$; $\frac{p_{n+1}}{P_{n+1}} = O\left(\frac{p_n}{P_n}\right)$

(2) *A has a mean value theorem;* $a_{n+1,n+1} = O(a_{nn})$ *and* $\sum_{k=0}^{n} |a_{nk}| < M$

(3) $\sum_{k=0}^{n} b_{nk} = 1$; $b_{nk} \searrow 0$ *as* $n \nearrow \infty$

(4) $\frac{p_n}{P_n} = O(b_{nn})$; $\frac{b_{nn}}{a_{nn}} = O\left(\frac{b_{n+1,n+1}}{a_{n+1,n+1}}\right)$

(5) $A \subseteqq AP$; $|B| \dot{\subseteqq} |B|$.

If $\sum_{n=0}^{\infty} \left| \frac{b_{nn}}{a_{nn}} \varepsilon_n(\overline{A}, \alpha) \right| < \infty$ *implies* $\varepsilon_n \in (A, |B|)$, *then also* $\sum_{n=0}^{\infty} \left| \left(\frac{P_n}{p_n}\right)^k \frac{b_{nn}}{a_{nn}} \times \varepsilon_n(\overline{AP^k}, \alpha) \right| < \infty$ *implies* $\varepsilon_n \in (AP^k, |B|)$ *for* $k = 1, 2, \cdots$.

LEMMA 2. *Let* $D = CP$. *If* $\overline{D}$ *and* $\overline{C}$ *have bounded columns, and if* $C \subseteqq D$, *then for every* $\alpha = \{\alpha_n\}$; $\sum_{n=0}^{\infty} |\alpha_n| < \infty$

(a) *there exists* $\beta = \{\beta_n\}$; $\sum_{n=0}^{\infty} |\beta_n| < \infty$ *such that* $\varepsilon_n(\overline{D}, \alpha) = \varepsilon_n(\overline{C}, \beta)$

(b) *there exists a* $\gamma = \{\gamma_n\}$; $\sum_{n=0}^{\infty} |\gamma_n| < \infty$ *such that* $\Delta\varepsilon_n(\overline{D}, \alpha) = \frac{p_n}{P_{n-1}} \varepsilon_n(C, \gamma)$.

PROOF. See [3].

LEMMA 3. *If* $\sum_{k=0}^{n} b_{nk} = 1$, $b_{nk} \searrow 0$ *as* $n \nearrow \infty$, *and* $\frac{p_n}{P_n} = O(b_{nn})$, *then* $\sum_{n=k}^{\infty} \left| \hat{b}_{nk} - \frac{P_{k-1}}{P_k} \hat{b}_{n,k+1} \right| = O(b_{kk})$.

PROOF.

$$\sum_{n=k}^{\infty} \left| \hat{b}_{nk} - \frac{P_{k-1}}{P_k} \hat{b}_{n,k+1} \right| \leqq \sum_{n=k}^{\infty} |\hat{b}_{nk} - \hat{b}_{n,k+1}| + \frac{p_k}{P_k} \sum_{n=k+1}^{\infty} \hat{b}_{n,k+1} = O(b_{kk}).$$

LEMMA 4. *If* A *has a mean value theorem,* $a_{n+1,n+1} = O(a_{nn})$, *and* $\frac{p_{n+1}}{P_{n+1}} = O\left(\frac{p_n}{P_n}\right)$, *then* $s_n \in AP^k$ *implies* $s_n = O\left(\frac{1}{a_{nn}}\left(\frac{P_n}{p_n}\right)^k\right)$.

PROOF. See [5], Theorem 1.14.

PROOF OF THEOREM 4. Consider first the case $k=1$, and let $\delta_n = \frac{P_{n-1}}{p_n} \Delta\varepsilon_n(\overline{AP}, \alpha)$. By Lemma 2, $\delta_n = \delta_n(\overline{A}, \gamma)$ and

$$\sum_{n=0}^{\infty}\left|\frac{b_{nn}}{a_{nn}}\delta_n\right| \leqq O(1)\left\{\sum_{n=0}^{\infty}\left|\frac{b_{nn}P_n}{a_{nn}p_n}\varepsilon_n\right| + \sum_{n=0}^{\infty}\left|\frac{b_{n+1,n+1}P_{n+1}}{a_{n+1,n+1}p_{n+1}}\varepsilon_{n+1}\right|\right\} < \infty,$$

hence $\delta_n(\overline{A}, \gamma) \in (A, |B|)$. Setting $B = BP'P$ and performing a partial summation yields the formula

$$\begin{aligned}\sum_{k=0}^{n} \widehat{b}_{nk}a_k\varepsilon_k(\overline{AP}, \alpha) &= \sum_{k=0}^{n} \widehat{(BP')}_{nk}\varepsilon_k(\overline{AP}, \alpha)\widehat{P}_k(a_i) + \sum_{k=0}^{n} \widehat{b}_{nk}\frac{P_{k-2}}{p_{k-1}}\Delta\varepsilon_{k-1}(\overline{AP}, \alpha)\widehat{P}_{k-1}(a_i) \\ &= \sum_{k=0}^{n} \widehat{(BP')}_{nk}\varepsilon_k(\overline{AP}, \alpha)\widehat{P}_k(a_i) + \sum_{k=0}^{n} \widehat{b}_{nk}\delta_{k-1}(\overline{A}, \gamma)\widehat{P}_{k-1}(a_i) \\ &= \mathrm{I} + \mathrm{II}\end{aligned}$$

where $\widehat{P}_k(a_i) = \sum_{i=0}^{k} \widehat{p}_{ki}a_i = \frac{p_k}{P_kP_{k-1}}\sum_{i=1}^{k} P_{i-1}a_i$. Now if $\sum a_k \in AP$ then $\sum \widehat{P}_k(a_i) \in A$,

hence $\sum \delta_k\widehat{P}_k(a_i) \in |B|$, but then also $\sum \delta_{k-1}\widehat{P}_{k-1}(a_i) \in |B|$.

For I we have the estimation

$$\begin{aligned}\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} \widehat{(BP')}_{nk}\varepsilon_k\widehat{P}_k(a_i)\right| &\leqq O(1)\sum_{k=0}^{\infty}\left|\frac{\varepsilon_k}{a_{kk}}\right|\sum_{n=k}^{\infty}\left|(\widehat{\widehat{BP}}')_{nk}\right| \\ &\leqq O(1)\sum_{k=0}^{\infty}\left|\frac{P_k\varepsilon_k}{p_ka_{kk}}\right|\sum_{n=k}^{\infty}\left|\widehat{b}_{nk} - \frac{P_{k-1}}{P_k}\widehat{b}_{n,k+1}\right| \\ &\leqq O(1)\sum_{k=0}^{\infty}\left|\frac{P_kb_{kk}}{p_ka_{kk}}\varepsilon_k\right| < \infty,\end{aligned}$$

making use of Lemma 1 and Lemma 3. This completes the proof for $k = 1$. The induction step is completely analogous to the case just proved. We simply write $AP^{k+1} = (AP^k)P$ and use Lemma 4 instead of Lemma 1.

COROLLARY. *Let $P^{(i)}$ be a weighted mean which satisfies the conditions of Theorem 4 for each $i = 1, 2, \cdots$, and set $A^* = A\prod_{i=1}^{k} P^{(i)}$. Suppose also that $A \subseteqq AP^{(1)} \subseteqq \cdots \subseteqq A^*$.*

If $\sum_{n=0}^{\infty}\left|\frac{b_{nn}}{a_{nn}}\varepsilon_n(\overline{A}, \alpha)\right| < \infty$ implies $\varepsilon_n \in (A, |B|)$, then also $\sum_{n=0}^{\infty}\left|\frac{b_{nn}}{a^*_{nn}}\varepsilon_n(\overline{A}^*, \alpha)\right| < \infty$ implies $\varepsilon_n \in (A^*, |B|)$.

2. Applications. In Theorem 1, take $A=(C,\alpha)$ and $B=(C,\beta)$, where $0<\alpha\leqq\beta\leqq 1$.

A typical application of Theorem 2 is the case $P=(R,\log(n+1),1)$, the discontinuous Riesz mean of order 1, and $B=(C,\beta)$, with $0\leqq\beta\leqq 1$. It is easily seen that P is equivalent to the weighted mean Q, with $q_n=1/(n+1)$. Using this fact together with Lemma 2, part (a), we invert the representation (i) of Theorem 2 and arrive at the equivalent statement that $\varepsilon_n\in(P,|B|)$ if and only if

$$\text{(i)}'\qquad \sum_{n=1}^{\infty} n\log n|\Delta^2\varepsilon_n|<\infty$$

$$\text{(ii)}'\qquad \sum_{n=1}^{\infty} n^{1-\beta}\log n|\varepsilon_n|<\infty.$$

If $\beta=0$, then (ii)′ alone is necessary and sufficient, which one may also obtain immediately from Theorem 3.

Finally, we let $A=(C,\alpha)$ $(0<\alpha\leqq 1)$ and $B=P=(C,1)$. Theorem 2 and Theorem 4 together give the summability factors $(C_\alpha,|C_1|)$ for $\alpha>0$.

The theorems proved here clearly cover a much wider class of methods than just the Cesàro means, but on the other hand it would be desirable to be able to offer a complete generalization of the Cesàro results. One would at least expect Theorem 1 to be true when $|B|\subseteqq|A|$, perhaps with other minor restrictions. This question remains open.

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