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SAMPLE PROPERTIES OF MARTINGALES AND THEIR ARITHMETIC MEANS

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1. Introduction. Let $X=\{X_n, \mathfrak{F}_n, n \geqq 1\}$ be a martingale on a probability space $(\Omega, \mathfrak{F}, \mathrm{P})$, where $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ are sub-σ-fields of $\mathfrak{F}$. Denote $d_n = x_n - x_{n-1}$, $x_0 \equiv 0$, $s_n = (x_1 + \cdots + x_n)/n$ for $n=1, 2, \cdots$.

The classical martingale convergence theorem asserts that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(d_1 + \cdots d_n) = d_1 + d_2 + \cdots \tag{1}$$

exists (and is finite) almost surely, if X is L^1-bounded: $\sup_n \mathrm{E}\{|x_n|\} < \infty$. D. G. Austin [1] (see also Burkholder [2]) proved that the series

$$d_1^2 + d_2^2 + \cdots \tag{2}$$

converges almost surely under the same assumption on X. And, D. L. Burkholder [2] obtained further remarkable properties of the Austin series (2).

In this note we shall prove, along the Burkholder way, some results on the following two series

$$\lambda(X) = \sum_{n=1}^{\infty} \frac{|x_n - s_n|^2}{n} \tag{3}$$

and

$$\pi(X) = \sum_{k=1}^{\infty} |x_{n_k} - s_{n_k}|^2, \tag{4}$$

where $n_1 < n_2 < \cdots$ are positive integers such that $q_2 \geqq n_{k+1}/n_k \geqq q_1 > 1$ $(k=1, 2, \cdots)$ with constants q_1 and q_2.

In the case of L^2-bounded martingale, the sequence $\{d_n\}$ forms an orthogonal system, and the almost sure convergence of the three series (2), (3) and (4) will be easily obtained (see for example [6] II, Chap. XIII, XIV, XV). The last two series correspond to the so-called Littlewood-Paley functions for Walsh and trigonometric series.

Throughout the paper we denote by A an absolute positive constant and by A_p a positive constant depending on the indexed parameter p, both letters are not necessarily the same in each occurrence. Further, the martingales treated in this paper are assumed to be real; this convention does not restrict the generality of the results.

2. Convergence of martingales and the series $\lambda(X)$ and $\pi(X)$.

THEOREM 2.1. *If $X=\{x_n, \mathfrak{F}_n, n\geqq 1\}$ is an L^1-bounded martingale, then the series $\lambda(X)$ and $\pi(X)$ converge almost surely.*

PROOF. Denote by $\lambda_n=\lambda_n(X)$ and $\pi_n=\pi_n(X)$ the n-th partial sums of $\lambda(X)$ and $\pi(X)$ respectively. Then

$$\begin{aligned}\lambda_N &= \sum_{n=1}^{N}\frac{1}{n}(x_n-s_n)^2\\ &= \sum_{n=1}^{N}\frac{1}{n^3}\left(\sum_{k=1}^{n}(k-1)\,d_k\right)^2\\ &= \sum_{n=1}^{N}\frac{1}{n^3}\sum_{k=1}^{n}(k-1)^2 d_k^2 + 2\sum_{n=1}^{N}\frac{1}{n^3}\sum_{k=1}^{n}\sum_{j=1}^{k-1}(j-1)(k-1)\,d_j\,d_k\\ &= S_N^{(1)}+2S_N^{(2)}\end{aligned}$$

say. Clearly,

$$S_N^{(1)}=\sum_{k=1}^{N}(k-1)^2 d_k^2\sum_{n=k}^{N}n^{-3}\leqq A\sum_{k=1}^{N}d_k^2$$

which is bounded as $N\to\infty$, almost surely by the Austin theorem stated in §1. On the other hand, as $\sum_{j=1}^{k-1}(j-1)\,d_j=(k-1)(x_{k-1}-s_{k-1})$,

$$\begin{aligned}S_N^{(2)} &= \sum_{n=1}^{N}n^{-3}\sum_{k=1}^{n}(k-1)^2 d_k(x_{k-1}-s_{k-1})\\ &= \sum_{k=1}^{N}(k-1)^2(x_{k-1}-s_{k-1})\,d_k\sum_{n=k}^{N}n^{-3}\\ &= \sum_{k=1}^{N}(k-1)^2(x_{k-1}-s_{k-1})(C_{k-1}-C_N)\,d_k\end{aligned}$$

where $C_{j-1}=\sum_{n=j}^{\infty} n^{-3} \sim j^{-2}$ as $j \to \infty$. The sums

$$S_N^{(3)} \equiv \sum_{k=1}^{N} (k-1)^2(x_{k-1}-s_{k-1})\, C_{k-1}\, d_k \quad (N=1,2,\cdots)$$

form a martingale transform defined by Burkholder [2], since the k-th multiplier $(k-1)^2(x_{k-1}-s_{k-1})\,C_{k-1}$ is clearly measurable $\mathfrak{F}_{k-1}$. As $\lim_{n\to\infty} x_n = \lim_{n\to\infty} s_n$ exists almost surely by the convergence theorem, the multipliers are bounded almost surely. Hence by the Burkholder theorem ([2] Theorem 1), $\lim_{N\to\infty} S_N^{(3)}$ exists almost surely. Now, by the Abel transformation,

$$\begin{aligned} S_N^{(4)} &\equiv C_N \left| \sum_{k=1}^{N} (k-1)^2(x_{k-1}-s_{k-1})\, d_k \right| \\ &\leqq C_N \sum_{k=1}^{N} (2k-1) \left| \sum_{j=1}^{k} (x_{j-1}-s_{j-1})\, d_j \right| + C_N N^2 \left| \sum_{j=1}^{N} (x_{j-1}-s_{j-1})\, d_j \right|. \end{aligned}$$

Since $x_{j-1}-s_{j-1}$ is measurable $\mathfrak{F}_{j-1}$ and bounded in j almost surely, $\lim_{k\to\infty} \sum_{j=1}^{k} (x_{j-1}-s_{j-1})\, d_j$ also exists almost surely by the same theorem of Burkholder, therefore

$$S_N^{(4)} \leqq C_N \sum_{k=1}^{N} (2k-1)\, O(1) + C_N N^2 O(1) = O(1)$$

almost surely, where $O(1)$ means a finite valued random variable independent of N. So that, $|S_N^{(2)}| \leqq |S_N^{(3)}| + |S_N^{(4)}| = O(1)$ almost surely, and hence $|\lambda_N| \leqq |S_N^{(1)}| + 2|S_N^{(2)}| = O(1)$ almost surely. This proves the almost sure convergence of $\lambda(X)$.

Similar estimation will be applied to the series $\pi(X)$. We notice that, as $n_1 < n_2 < \cdots$ form a lacunary sequence of integers,

$$D_{j-1} = \sum_{k;\, n_k \geqq j} n_k^{-2} \leqq A j^{-2} \quad (j=1,2,\cdots).$$

Now, expanding as before,

$$\begin{aligned} \pi_N(X) &= \sum_{k=1}^{N} (x_{n_k}-s_{n_k})^2 \\ &= \sum_{k=1}^{N} \frac{1}{n_k^2} \sum_{j=1}^{n_k} (j-1)^2\, d_j^2 + 2 \sum_{k-1}^{N} \frac{1}{n_k^2} \sum_{j=1}^{n_k} \sum_{i=1}^{j-1} (i-1)(j-1)\, d_i\, d_j \end{aligned}$$

$$= T_N^{(1)} + 2T_N^{(2)}$$

say. Then,

$$T_N^{(1)} = \sum_{j=1}^{n_N} (j-1)^2 d_j^2 \sum_{k;n_k \geqq j} n_k^{-2}$$

$$= A \sum_{j=1}^{n_N} d_j^2$$

which is bounded in N almost surely by the Austin theorem. And,

$$T_N^{(2)} = \sum_{k=1}^{N} n_k^{-2} \sum_{j=1}^{n_k} (j-1)^2(x_{j-1}-s_{j-1})\, d_j$$

$$= \sum_{j=1}^{n_N} (j-1)^2(x_{j-1}-s_{j-1})(D_{j-1}-D_{n_N})\, d_j\,.$$

Since the multiplier $(j-1)^2(x_{j-1}-s_{j-2})\,D_{j-1}$ is measurable $\mathfrak{F}_{j-1}$ and is $O(1)$ as $j \to \infty$ almost surely, the martingale transform

$$T_N^{(3)} = \sum_{j=1}^{n_N} (j-1)^2(x_{j-1}-s_{j-1})\, D_{j-1}\, d_{j-1}$$

converges as $N \to \infty$ almost surely, and as we have just shown

$$T_N^{(4)} \equiv D_{n_N} \sum_{j=1}^{n_N} (j-1)^2(x_{j-1}-s_{j-1})\, d_j = S_{n_N}^{(4)} = O(1)$$

as $N \to \infty$, almost surely. Therefore

$$|T_N^{(2)}| \leqq |T_N^{(3)}| + |T_N^{(4)}| = O(1)$$

as $N \to \infty$, and $\pi_N(X) \leqq |T_N^{(1)}| + 2|T_N^{(2)}| = O(1)$ as $N \to \infty$ almost surely. q.e.d.

This Theorem can be carried over to the submartingale case.

COROLLARY 2.2. *If* $X = \{x_n, \mathfrak{F}_n, n \geqq 1\}$ *is an* L^1*-bounded submartingale, then both series* $\lambda(X)$ *and* $\pi(X)$ *converge almost surely.*

PROOF. The submartingale can be written as

$$x_n = x_n' + \sum_{j=1}^{n} \Delta_j \qquad (n = 1, 2, \cdots, \Delta_1 = 0)$$

where $\{x_n', \mathfrak{F}_n, n \geqq 1\}$ is a martingale, $\Delta_j \geqq 0$ and Δ_j is measurable $\mathfrak{F}_{j-1}$ $(j=1, 2, \cdots)$. Denote

$$d_n = x_n - x_{n-1}, \quad d_n' = x_n' - x_{n-1}',$$

$$s_n = \frac{1}{n}(x_1 + \cdots + x_n), \quad s_n' = \frac{1}{n}(x_1' + \cdots + x_n'),$$

then

$$d_n = d_n' + \Delta_n,$$

$$s_n = s_n' + \frac{1}{n}\sum_{j=1}^{n}(n-j+1)\Delta_j,$$

and

$$x_n - s_n = (x_n' - s_n') + \frac{1}{n}\sum_{j=1}^{n}(j-1)\Delta_j.$$

As the sequence $\{x_n\}$ is L^1-bounded, so is the sequence $\{x_n'\}$, and $\sum_{j=1}^{\infty}\Delta_j < \infty$ almost surely (Doob [3] p. 297, Theorem 1.2). Therefore

$$\lambda(X) = \sum_{n=1}^{\infty}\frac{(x_n - s_n)^2}{n} \leqq 2\sum_{n=1}^{\infty}\frac{(x_n' - s_n')^2}{n} + 2\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{j=1}^{n}(j-1)\Delta_j\right)^2,$$

and the first series in the last hand side is convergent almost surely by Theorem 2.1; the last series is also convergent, in fact, by the Minkowski inequality,

$$\left[\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\sum_{j=1}^{n}(j-1)\Delta_j\right)^2\right]^{1/2} \leqq \sum_{j=1}^{\infty}\left[\sum_{n=j}^{\infty}\frac{(j-1)^2\Delta_j^2}{n^3}\right]^{1/2}$$

$$\leqq \sum_{j=1}^{\infty} j\Delta_j\left(\sum_{n=j}^{\infty}\frac{1}{n^3}\right)^{1/2}$$

$$\leqq A\sum_{j=1}^{\infty}\Delta_j$$

which is convergent almost surely.

The series $\pi(X)$ will be treated similarly.

LEMMA 2.3. *If* $\{x_n, \mathfrak{F}_n, n \geqq 1\}$ *is a martingale and* $\mathrm{E}\{\lambda(X)^{1/2}\} < \infty$, *then* $\sup_N \mathrm{E}\{|x_N - s_N|\} \leqq A\,\mathrm{E}\{\lambda(X)^{1/2}\}$.

PROOF. As we see easily $\{n(x_n - s_n)\}$ forms a martingale, the sequence $\{n|x_n - s_n|\}$ is a submartingale and the submartingale inequality shows that, if $N < n$, then

$$\mathrm{E}\{N|x_N - s_N|\} \leqq \mathrm{E}\{n|x_n - s_n|\}\,.$$

Hence

$$\begin{aligned}
\mathrm{E}\{|x_N - s_N|\} &= \frac{1}{N}\mathrm{E}\{N|x_N - s_N|\} \\
&\leqq \frac{1}{N^2}\sum_{n=N+1}^{2N} \mathrm{E}\{n|x_n - s_n|\} \\
&\leqq \frac{1}{N^2}\mathrm{E}\left\{\sum_{n=N+1}^{2N} n|x_n - s_n|\right\} \\
&\leqq \frac{1}{N^2}\mathrm{E}\left\{\left[\sum_{n=N+1}^{2N} n^2\right]^{1/2}\left[\sum_{n=N+1}^{2N}(x_n - s_n)^2\right]^{1/2}\right\} \\
&\leqq A\,\mathrm{E}\left\{\left[\frac{1}{N}\sum_{n=N+1}^{2N}(x_n - s_n)^2\right]^{1/2}\right\} \\
&\leqq A\,\mathrm{E}\left\{\left[\sum_{n=N+1}^{2N}\frac{(x_n - s_n)^2}{n}\right]^{1/2}\right\} \\
&\leqq A\,\mathrm{E}\{\lambda(X)^{1/2}\}\,, \qquad \text{q. e. d.}
\end{aligned}$$

REMARK. In the case $p \geqq 1$, the inequalities $\sup_N \mathrm{E}\{|x_N|^p\} < \infty$ and $\sup_N \mathrm{E}\{|s_N|^p\} < \infty$ are equivalent (see [5]), and then by Theorem 3.1 in the next §, the conclusion of Lemma 2.3 will be extended to

$$\sup_N \mathrm{E}\{|x_N - s_N|^p\} \leqq A_p \mathrm{E}\{\lambda(X)^{p/2}\}\,.$$

THEOREM 2.4. *Let* $X = \{x_n, \mathfrak{F}_n, n \geqq 1\}$ *be a martingale, and suppose that* $\mathrm{E}\{\sup_n |d_n|\} < \infty$. *If one of the expectations* $\mathrm{E}\{\lambda(X)^{1/2}\}$ *and* $\mathrm{E}\{\pi(X)^{1/2}\}$ *is finite, then* $\lim_{n\to\infty} x_n$ *exists almost surely.*

PROOF. (i) Let $\mathrm{E}\{\lambda(X)^{1/2}\}$ be finite. There exists a finite constant K such that

$$\mathrm{E}\{\lambda_N(X)^{1/2}\} = \mathrm{E}\left\{\left[\sum_{n=1}^{N} \frac{1}{n^3}\left(\sum_{k=1}^{n}(k-1)\,d_k\right)^2\right]^{1/2}\right\} < K$$

for all N. By the Khinchin inequality of the Rademacher system (see for example [6] I, p.213), the second side is not less than

$$\begin{aligned}
&A\,\mathrm{E}\left\{\int_0^1 \left|\sum_{n=1}^{N}\frac{r_n(t)}{n^{3/2}}\sum_{k=1}^{n}(k-1)\,d_k\right| dt\right\} \\
&\geqq A\int_0^1 \mathrm{E}\left\{\left|\sum_{k=1}^{N}(k-1)\,d_k\sum_{n=k}^{N}\frac{r_n(t)}{n^{3/2}}\right|\right\} dt \\
&\geqq A\int_0^1 \mathrm{E}\left\{\left|\sum_{k=1}^{N}(k-1)\,d_k\,R_{k-1}(t)\right|\right\}dt - A\int_0^1 \mathrm{E}\left\{\left|\sum_{k=1}^{N}(k-1)\,d_k\,R_N(t)\right|\right\}dt \\
&= A\,I_N^{(1)} - A I_N^{(2)}
\end{aligned}$$

say, where $R_{j-1}(t) = \sum_{n=j}^{\infty} n^{-3/2}\,r_n(t)$. Hence

$$I_N^{(1)} \leqq AK + A I_N^{(2)}$$

for all N, and

$$\begin{aligned}
I_N^{(2)} &= A\,\mathrm{E}\left\{\left|\sum_{k=1}^{N}(k-1)\,d_k\right|\right\}\int_0^1 |R_N(t)|\,dt \\
&\leqq A\,\mathrm{E}\left\{\left|\sum_{k=1}^{N}(k-1)\,d_k\right|\right\}\left(\sum_{n=N+1}^{\infty} n^{-3}\right)^{1/2} \\
&\leqq A\,\mathrm{E}\left\{\left|\frac{1}{N}\sum_{k=1}^{N}(k-1)\,d_k\right|\right\} \\
&\leqq A\,\mathrm{E}\{|x_N - s_N|\} \\
&\leqq AK
\end{aligned}$$

by Lemma 2.3. Therefore $I_N^{(1)} \leqq AK$ for all N, that is,

$$\int_0^1 \mathrm{E}\left\{\left|\sum_{k=1}^{N}(k-1)\,R_{k-1}(t)\,d_k\right|\right\} dt < AK$$

for all N. Since $\left\{\sum_{k=1}^{N}(k-1)R_{k-1}(t)\,d_k\right\}$ is a martingale for any fixed t, the integrand of the left hand side is nondecreasing in N, so that

$$\int_0^1 \sup_N \mathrm{E}\left\{\left|\sum_{k=1}^{N}(k-1)R_{k-1}(t)\,d_k\right|\right\} dt < AK,$$

and then $\sup_N \mathrm{E}\left\{\left|\sum_{k=1}^{N}(k-1)R_{k-1}(t)\,d_k\right|\right\} < \infty$ for almost all t. Applying the Austin theorem,

$$\sum_{k=1}^{\infty}(k-1)^2 R_{k-1}(t)^2 d_k^2 < \infty$$

for almost all $(t, \omega) \in [0, 1[\times \Omega$, and then for almost all fixed $\omega \in \Omega$, the above inequality holds for all $t \in \mathrm{E}_\omega \subset [0, 1[$, where E_ω is a full set. Hence by the Egorov theorem, for such ω, there exist a constant $K \equiv K_\omega$ and a set $F \equiv F_\omega \subset \mathrm{E}_\omega$, $|F| > 0$ such that

$$\sum_{k=1}^{\infty}(k-1)^2 R_{k-1}(t)^2 d_k^2 < K$$

for all $t \in F$. Integrating both sides on F and using the Khinchin inequality,

$$\begin{aligned} K|F| &\geqq \sum_{k=1}^{\infty}(k-1)^2 d_k^2 \int_F R_{k-1}(t)^2\,dt \\ &\geqq A\sum_{k=1}^{\infty}(k-1)^2 d_k^2 \sum_{n=k}^{\infty} n^{-3} \\ &\geqq A\sum_{k=1}^{\infty} d_k^2, \end{aligned}$$

that is, $\sum_{k=1}^{\infty} d_k^2 < \infty$ almost surely. By a theorem of Burkholder ([2] Theorem 4), $\lim_n x_n$ exists almost surely.

(ii) Now, let $\mathrm{E}\{\pi(X)^{1/2}\} \equiv M < \infty$. Then for all N, as in (i),

$$M \geqq \mathrm{E}\left\{\left[\sum_{k=1}^{N}(x_{n_k}-s_{n_k})^2\right]^{1/2}\right\}$$

$$\geqq A\mathrm{E}\left\{\int_0^1\left|\sum_{k=1}^{N} r_k(t)(x_{n_k}^- - s_{n_k})\right|dt\right\}$$

$$= A\int_0^1 \mathrm{E}\left\{\left|\sum_{j=1}^{n_N}(j-1)\,d_j \sum_{\substack{k\\ j\leqq n_k\leqq n_N}} \frac{r_k(t)}{n_k}\right|\right\}dt$$

$$\geqq A\int_0^1 \mathrm{E}\left\{\left|\sum_{j=1}^{n_N}(j-1)\,d_j S_{j-1}(t)\right|\right\}dt - A\int_0^1 \mathrm{E}\left\{\left|\sum_{j=1}^{n_N}(j-1)\,d_j S_{n_N}(t)\right|\right\}dt$$

$$= AJ_N^{(1)} - AJ_N^{(2)}$$

say, where $S_{j-1}(t) = \sum_{\substack{k\\ j\leqq n_k}} r_k(t)/n_k$.

For any positive integer j, let k_0 be such that $n_{k_0}\leqq j < n_{k_0+1}$, then by the condition $n_{k+1}/n_k \leqq q_2$ it is obtained that

$$\sum_{\substack{k\\ j\leqq n_k}} n_k^{-2} \geqq \sum_{k=k_0+1}^{\infty} n_k^{-2} \geqq n_{k_0+1}^{-2} > q_2^{-2} j^{-2}\,.$$

It follows that

$$J_N^{(2)} \leqq A\mathrm{E}\left\{\left|\sum_{j=1}^{n_N}(j-1)\,d_j\right|\right\}\int_0^1 |S_{n_N}(t)|\,dt$$

$$\leqq A\mathrm{E}\left\{\left|\sum_{j=1}^{n_N}(j-1)\,d_j\right|\right\}\left(\sum_{\substack{k\\ n_N\leqq n_k}}\frac{1}{n_k^2}\right)^{1/2}$$

$$\leqq A\mathrm{E}\left\{\left|\frac{1}{n_N}\sum_{j=1}^{n_N}(j-1)\,d_j\right|\right\}$$

$$= A\mathrm{E}\{|x_{n_N} - s_{n_N}|\}$$

$$\leqq A\mathrm{E}\{\pi(X)^{1/2}\}$$

$$\leqq AM,$$

and $J_N^{(1)} \leqq AM + AM \leqq AM$ for all N. As in the case of $\lambda(X)$, the sequence

$$\left\{\sum_{j=1}^{n_N}(j-1)\,d_j S_{j-1}(t)\right\} \qquad (N=1,2,\cdots)$$

forms an L^1-bounded martingale for almost all t, and the Austin theorem implies that

$$\sum_{j=1}^{\infty}(j-1)^2 d_j^2 S_{j-1}(t)^2 < \infty$$

for almost all $(t, \omega) \in [0, 1[\times \Omega$. Hence, just as for $\lambda(X)$, $\sum_{j=1}^{\infty} d_j^2 < \infty$ almost surely, and $\lim_{n\to\infty} x_n$ exists almost surely. q. e. d.

THEOREM 2.5. *Let* $X = \{x_n, \mathfrak{F}_n, n \geqq 1\}$ *be a martingale and let* $\mathrm{E}\{\sup_n |d_n|\} < \infty$. *Then* (i) $\lim_{n\to\infty} x_n$ *exists almost surely on the set* $\{\lambda(X) < \infty$ *and* $\sup_n |x_n - s_n| < \infty\}$, (ii) $\lambda(X) < \infty$ *almost surely on* $\{\sup_n x_n < \infty\}$, (iii) $\lim_{n\to\infty} x_n$ *exists almost surely on* $\{\pi(X) < \infty\}$ *and* (iv) $\pi(X) < \infty$ *almost surely on* $\{\sup_n x_n < \infty\}$.

PROOF. (i) Let c be a positive constant. Define

$$m = m(\omega) = \inf\{n\,;\ \lambda_n(X) \geqq c^2 \text{ or } |x_n - s_n| \geqq c\}$$

where put $\inf \phi = \infty$. Since $m(\omega)$ is a stopping time, if $\hat{x}_n = x_{m\wedge n}$, the sequence $\hat{X} = \{\hat{x}_n, \mathfrak{F}_n, n \geqq 1\}$ forms a martingale (see Doob [3]). Denote $\hat{d}_n = \hat{x}_n - \hat{x}_{n-1}$ $(\hat{x}_0 = 0)$ and $\hat{s}_n = (\hat{x}_1 + \cdots + \hat{x}_n)/n$. On the set $\{m(\omega) < \infty\}$,

$$\begin{aligned}
\lambda(\hat{X}) &= \sum_{n=1}^{\infty} \frac{|\hat{x}_n - \hat{s}_n|^2}{n} \\
&= \sum_{n=1}^{m-1} \frac{|x_n - s_n|^2}{n} + \sum_{n=m}^{\infty} \frac{1}{n^3}\left[\sum_{j=1}^{n}(j-1)\hat{d}_j\right]^2 \\
&\leqq c^2 + \sum_{n=m}^{\infty} \frac{1}{n^3}\left[\sum_{j=1}^{m}(j-1)\, d_j\right]^2 \\
&\leqq c^2 + A\left[\frac{1}{m}\sum_{j=1}^{m}(j-1)\, d_j\right]^2 .
\end{aligned}$$

Since

$$\begin{aligned}
|x_m - s_m| &= \left|\frac{1}{m}\sum_{j=1}^{m}(j-1)\, d_j\right| \leqq \frac{1}{m}\left|\sum_{j=1}^{m-1}(j-1)\, d_j\right| + \frac{m-1}{m}|d_m| \\
&\leqq |x_{m-1} - s_{m-1}| + |d_m| \\
&\leqq c + \sup_n |d_n| ,
\end{aligned}$$

it follows that

$$\lambda(\widehat{X}) \leqq c^2 + A(c + \sup_n |d_n|)^2 \leqq A(c + \sup_n |d_n|)^2 .$$

On $\{m(\omega)=\infty\}$, clearly $\lambda(\widehat{X}) < c^2$. Hence $E\{\lambda(\widehat{X})^{1/2}\} < \infty$. From Theorem 2.4 $\lim_{n\to\infty} \hat{x}_n$ exists almost surely, so that $\lim_{n\to\infty} x_n$ exists almost surely on $\{\lambda(x) < c$ and $\sup_n |x_n - s_n| < c\}$, and hence on $\{\lambda(x) < \infty$ and $\sup_n |x_n - s_n| < \infty\}$ as c is arbitrary.

(ii) Let $c > 0$. If $m = \inf\{n\,;\,|x_n| \geqq c\}$, then m is a stopping time, and so if we put $\hat{x}_n = x_{m\wedge n}$, $\widehat{X} = \{\hat{x}_n\}$ forms a martingale. Moreover $\widehat{X}$ is L^1-bounded, since

$$|\hat{x}_n| \leqq |x_{m\wedge n} - x_{(m\wedge n)-1}| + |x_{(m\wedge n)-1}| \leqq \sup_n |d_n| + c .$$

Therefore by Theorem 2.1 $\lambda(\widehat{X}) < \infty$ almost surely. On the set $\{\sup |x_n| < c\}$, $\lambda(\widehat{X}) = \lambda(X)$ and as c is arbitrary, $\lambda(X) < \infty$ almost surely on $\{\sup_n |x_n| < \infty\}$. But, by the Doob theorem ([3] Theorem 4.1 (iv), p. 320) the last set is equivalent to the set $\{\sup_n x_n < \infty\}$.

(iii) Let $c > 0$, and let $m = \inf\{k\,;\,\pi_k(X) \geqq c^2\}$. Then $n_{m(\omega)}$ is a stopping time; if we define $\hat{x}_n = x_n$ for $n < n_{m-1}$ and $\hat{x}_n = x_{n_{m-1}}$ for $n \geqq n_{m-1}$, then $\widehat{X} = \{\hat{x}_n\}$ forms a martingale. Let $\widehat{d}_k = \hat{x}_k - \hat{x}_{k-1}$ ($\hat{x}_0 \equiv 0$) and $\widehat{s}_k = (\hat{x}_1 + \cdots + \hat{x}_k)/k$. Since $\pi(\widehat{X}) \leqq c$ on $\{m=\infty\}$, and on $\{m < \infty\}$,

$$\begin{aligned}
\pi(\widehat{X}) &= \sum_{j=1}^{m-1} (x_{n_j} - s_{n_j})^2 + \sum_{j=m}^{\infty} (\hat{x}_{n_j} - \widehat{s}_{n_j})^2 \\
&\leqq c^2 + \sum_{j=m}^{\infty} \frac{1}{n_j^2} \left[\sum_{i=1}^{n_j} (i-1)\,\widehat{d}_i \right]^2 \\
&= c^2 + \sum_{j=m}^{\infty} \frac{1}{n_j^2} \left[\sum_{i=1}^{n_{m-1}} (i-1)\,d_i \right]^2 \\
&\leqq c^2 + A(x_{n_{m-1}} - s_{n_{m-1}})^2 \\
&\leqq c^2 + Ac^2 ,
\end{aligned}$$

we have $E\{\pi(\widehat{X})^{1/2}\} < \infty$, hence $\lim_{n\to\infty} \hat{x}_n$ exists almost surely by Theorem 2.4.

As $\hat{x}_n = x_n$ on $\{m=\infty\}$, $\lim_{n\to\infty} x_n$ exists almost surely on $\{\pi(X) < c^2\}$. Let $c\to\infty$ and the results follows.

(iv) Let $c > 0$, $m = \inf\{n, |x_n| \geqq c\}$, $\hat{x}_n = x_{m\wedge n}$. Then clearly $|\hat{x}_n| \leqq c + \sup_n |d_n|$ and $\widehat{X} = \{\hat{x}_n\}$ is an L^1-bounded martingale. Therefore, by Theorem 2.1 $\pi(\widehat{X}) < \infty$ almost surely. Hence $\pi(X) = \pi(\widehat{X}) < \infty$ almost surely on $\{m = \infty\} = \{\sup_n |x_n| < c\}$. Let $c \to \infty$, and noting that the sets $\{\sup_n |x_n| < \infty\}$ and $\{\sup_n x_n < \infty\}$ are equivalent, we get the conclusion. q.e.d.

3. Inequalities.

THEOREM 3.1. *Let $\{x_n, \mathfrak{F}_n, n \leqq 1\}$ be a martingale, then for $1 < p < \infty$ the following inequalities hold for $N = 1, 2, \cdots$*

(i) $$A_p \mathrm{E}\{|x_N|^p\} \leqq \mathrm{E}\{\lambda_n(x)^{p/2}\} \leqq A_p \mathrm{E}\{|x_N|^p\}\,,$$

(ii) $$A_p \mathrm{E}\{|x_{n_N}|^p\} \leqq \mathrm{E}\{\pi_N(X)^{p/2}\} \leqq A_p \mathrm{E}\{|x_{n_N}|^p\}\,.$$

For the proof we use the following lemma concerning the Rademacher functions $r_n(t)$ $(n=1, 2, \cdots)$.

LEMMA 3.2. *Let a_j, $b_{j,n}$ $(j, n = 1, 2, \cdots)$ be any constants. Then for $p \geqq 1$,*

$$A_p\left(\sum_{j=1}^N a_j^2 \sum_{n=1}^N b_{j,n}^2\right)^{p/2} \leqq \int_0^1 \left[\sum_{j=1}^N a_j^2 \left(\sum_{n=1}^N b_{j,n} r_n(t)\right)^2\right]^{p/2} dt \leqq A_p\left(\sum_{j=1}^N a_j^2 \sum_{n=1}^N b_{j,n}^2\right)^{p/2}.$$

PROOF. In the case $p \geqq 2$, using the Hölder inequality, the second side of the conclusion is not smaller than

$$\left[\int_0^1 \sum_{j=1}^N a_j^2 \left(\sum_{n=1}^N b_{j,n} r_n(t)\right)^2 dt\right]^{p/2}$$

which is, by the orthonormality of the Rademacher system, equal to

$$\left(\sum_{j=1}^N a_j^2 \sum_{n=1}^N b_{j,n}^2\right)^{p/2}.$$

Hence the first inequality of the conclusion holds with $A_p = 1$.

In the case $1 \leqq p \leqq 2$, if we write

$$\int_0^1 \left[\sum_{j=1}^N a_j^2 \left(\sum_{n=1}^N b_{j,n} r_n(t)\right)^2\right] dt = \int_0^1 S^{\frac{p}{4-p}} S^{\frac{4-2p}{4-p}} dt$$

where, for simplicity, S denote the integrand in the left hand side; then by the Hölder inequality,

$$\int_0^1 S\,dt \leqq \left(\int_0^1 S^{\frac{p}{4-p}\frac{4-p}{2}} dt\right)^{\frac{2}{4-p}} \left(\int_0^1 S^{\frac{4-2p}{4-p}\frac{4-p}{2-p}} dt\right)^{\frac{2-p}{4-p}}$$

$$= \left(\int_0^1 S^{\frac{p}{2}} dt\right)^{\frac{2}{4-p}} \left(\int_0^1 S^2\,dt\right)^{\frac{2-p}{4-p}}.$$

Expanding the integrand S^2 of the second integral in the last side and using the independence property or multiplicative orthonormality of the Rademacher system, we get easily

$$\int_0^1 S^2\,dt \leqq A\left(\sum_{j=1}^N a_j^2 \sum_{n=1}^N b_{j,n}^2\right)^2.$$

On the other hand

$$\int_0^1 S\,dt = \sum_{j=1}^N a_j^2 \int_0^1 \left(\sum_{n=1}^N b_{j,n} r_n(t)\right)^2 dt$$

$$= \sum_{j=1}^N a_j^2 \sum_{n=1}^N b_{j,n}^2$$

and combining the above inequalities, we get

$$\sum_{j=1}^N a_j^2 \sum_{n=1}^N b_{j,n}^2 \leqq \left(\int_0^1 S^{p/2}\,dt\right)^{\frac{2}{4-p}} A_p \left(\sum_{j=1}^N a_j^2 \sum_{n=1}^N b_{j,n}^2\right)^{\frac{2(2-p)}{4-p}},$$

from which it follows the first inequality of the conclusion.

The second inequality of the conclusion is also easily proved. In fact for $1 \leqq p \leqq 2$,

$$\int_0^1 S^{p/2}\,dt \leqq \left(\int_0^1 S\,dt\right)^{p/2} = \left(\sum_{j=1}^N a_j^2 \sum_{n=1}^N b_{j,n}^2\right)^{p/2},$$

and for $2 \leqq p$, by the Minkowski inequality and the Khinchin inequality,

$$\left(\int_0^1 S^{p/2}\,dt\right)^{2/p} \leqq \sum_{j=1}^N \left(\int_0^1 |a_j|^p \left|\sum_{n=1}^N b_{j,n} r_n(t)\right|^p dt\right)^{2/p}$$
$$\leqq \sum_{j=1}^N a_j^2 A_p \sum_{n=1}^N b_{j,n}^2 \,.$$

Thus the Lemma was proved.

PROOF OF THEOREM 3.1. (i) Since, for $p > 1$

$$A_p \mathrm{E}\{|x_N|^p\} \leqq \mathrm{E}\left\{\left(\sum_{j=1}^N d_j^2\right)^{p/2}\right\} \leqq A_p \mathrm{E}\{|x_N|^p\} \quad (N=1, 2, \cdots)$$

by the Burkholder theorem ([2] Theorem 9), it is sufficient to prove (i) and (ii) replacing $\mathrm{E}\{|x_N|^p\}$ by $\mathrm{E}\left\{\left(\sum_{j=1}^N d_j^2\right)^{p/2}\right\}$. Now

$$\mathrm{E}\{\lambda_N(X)^{p/2}\} = \mathrm{E}\left\{\left[\sum_{n=1}^N \frac{1}{n^3}\left(\sum_{j=1}^n (j-1)\,d_j\right)^2\right]^{p/2}\right\}$$
$$\leqq A_p \mathrm{E}\left\{\int_0^1 \left|\sum_{n=1}^N \frac{r_n(t)}{n^{3/2}} \sum_{j=1}^n (j-1)\,d_j\right|^p dt\right\}$$

by the Khinchin inequality, and

$$= A_p \int_0^1 \mathrm{E}\left\{\left|\sum_{j=1}^N (j-1)\,d_j \sum_{n=j}^N \frac{r_n(t)}{n^{3/2}}\right|^p\right\} dt$$
$$\leqq A_p \int_0^1 \mathrm{E}\left\{\left[\sum_{j=1}^N (j-1)^2 d_j^2 \left(\sum_{n=j}^N \frac{r_n(t)}{n^{3/2}}\right)^2\right]^{p/2}\right\} dt$$

by the Burkholder theorem quoted above, since the sequence

$$\left\{\sum_{j=1}^k (j-1)\,d_j \sum_{n=j}^N \frac{r_n(t)}{n^{3/2}}\right\} \quad (k=1, 2, \cdots, N)$$

forms a martingale for all t. The last integral is equal to

$$\mathrm{E}\left\{\int_0^1\left[\sum_{j=1}^N (j-1)^2\, d_j^2\left(\sum_{n=j}^N \frac{r_n(t)}{n^{3/2}}\right)^2\right]^{p/2} dt\right\}$$

and by Lemma 3.2 this is not less than

$$A_p\mathrm{E}\left\{\left[\sum_{j=1}^N (j-1)^2\, d_j^2\left(\sum_{n=j}^N \frac{1}{n^3}\right)^2\right]^{p/2}\right\} \leqq A_p\mathrm{E}\left\{\left(\sum_{j=1}^N d_j^2\right)^{p/2}\right\}.$$

The second inequality of (i) was proved.

Similarly, the first inequality of (i) is shown, because the inverse inequalities in the above argument are also true.

(ii) This case is also treated along the same line. In fact,

$$\begin{aligned}
\mathrm{E}\left\{\left[\sum_{k=1}^N (x_{n_k}-s_{n_k})^2\right]^{p/2}\right\} &\leqq A_p\mathrm{E}\left\{\int_0^1\left|\sum_{k=1}^N r_k(t)(x_{n_k}-s_{n_k})\right|^p dt\right\}\\
&= A_p\int_0^1 \mathrm{E}\left\{\left|\sum_{k=1}^N r_k(t)\frac{1}{n_k}\sum_{j=1}^{n_k}(j-1)\, d_j\right|^p\right\} dt\\
&= A_p\int_0^1 \mathrm{E}\left\{\left|\sum_{j=1}^{n_N}(j-1)\, d_j \sum_{\substack{k\\ j\leqq n_k\leqq n_N}} \frac{r_k(t)}{n_k}\right|^p\right\} dt\\
&\leqq A_p\int_0^1 \mathrm{E}\left\{\left[\sum_{j=1}^{n_N}(j-1)^2\, d_j^2\left(\sum_{\substack{k\\ j\leqq n_k\leqq n_N}} \frac{r_k(t)}{n_k}\right)^2\right]^{p/2}\right\} dt\\
&\leqq A_p\mathrm{E}\left\{\left[\sum_{j=1}^{n_N}(j-1)^2\, d_j^2 \sum_{\substack{k\\ j\leqq n_k\leqq n_N}} \frac{1}{n_k^2}\right]^{p/2}\right\}\\
&\leqq A_p\mathrm{E}\left\{\left(\sum_{j=1}^{n_N} d_j^2\right)^{p/2}\right\},
\end{aligned}$$

and all the inequalities can be reversed.

The Theorem was proved completely.

Recently R. F. Gundy [4] introduced a mapping of class B, and gave an elegant proof of the weak type inequalities of Burkholder ([2] Theorem 8). If the series $\lambda(X)$ and $\pi(X)$ are considered mappings of a martingale X, as we may checked easily, they are of class B of Gundy. Hence it follows the following result and from which the second inequalities of (i) and (ii) of Theorem 3.1 may be deduced by the Marcinkiewicz interpolation theorem.

THEOREM 3.3. *If X is a martingale, then for all $a > 0$*

$$a\mathrm{P}\{\lambda(X) > a\} \leqq A \sup_n \mathrm{E}\{|x_n|\},$$

and

$$a\mathrm{P}\{\pi(X) > a\} \leqq A \sup_n \mathrm{E}\{|x_n|\}.$$

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