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REMARK ON THE GROSS PROPERTY

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1. Let $w=f(z)$ be a non-constant single-valued meromorphic function in a domain D on the complex z-plane and let Φ_f be the covering Riemann surface generated by the inverse function of $w=f(z)$ over the extended complex w-plane. Take a regular point $q_0 \in \Phi_f$ lying over the basic point $w_0=f(z_0)$ $(\neq \infty)$ and consider the longest segment l_θ on Φ_f which starts from q_0, consists of only regular points of Φ_f and lies over the half straight line $\arg(w-w_0)=\theta$ $(0 \leqq \theta < 2\pi)$ on the w-plane. Here a regular point of Φ_f is a point of Φ_f not being an algebraic branch point. If l_θ has finite length, then l_θ is said to be a singular segment with its argument θ of Φ_f. The set of union $\bigcup_{0 \leqq \theta < 2\pi} l_\theta$ is clearly a domain and is called a Gross' star region with the centre q_0 on Φ_f.

If for any Gross' star region on Φ_f the measure of the set of arguments of all singular segments equals zero, then we say that the function $f(z)$ or Φ_f has the Gross property. Further, if any non-constant single-valued meromorphic function in D has the Gross property, then we say that the domain D has the Gross property. This was first discussed by Gross [2] for meromorphic functions in the finite z-plane $|z| < +\infty$ and, later, Yûjôbô [6] extended Gross' theorem in the following form (cf. Noshiro [5]):

If the boundary of D is of logarithmic capacity zero, then D has the Gross property.

2. Suppose that a domain D has an exhaustion $\{D_n\}_{n=1}^{\infty}$ which satisfies the following conditions;

i) the domain D_n is compact relative to D and the boundary C_n of D_n consists of a finite number of closed analytic curves,

ii) $\overline{D}_n = D_n \cup C_n \subset D_{n+1}$, $\bigcup_{n=1}^{\infty} D_n = D$,

iii) the open set $D_{n+1} - \overline{D}_n$ consists of a finite number of doubly connected domains D_n^j $(j = 1, \cdots, N(n))$,

iv) each connected component of $D - \overline{D}_n$ is non-compact with respect to D and

v) each connected component of $D - \overline{D}_n$ contains at most ρ domains D_{n+1}^j.

Every domain D_n^j can be mapped onto an annulus $1 < |\omega| < R_n^j$ on the ω-plane in a one-to-one conformal manner. We denote by Γ_n^j the inverse image of the circle $|\omega| = \sqrt{R_n^j}$. The quantity R_n^j is called the harmonic modulus of D_n^j. We put $R_n = \operatorname{Min}_{1 \leq j \leq N(n)} R_n^j$.

Let $f(z)$ be a single-valued meromorphic function in the domain D and suppose that each boundary point of D is an essential singularity of $f(z)$. On the exceptional values of $f(z)$, Matsumoto [3] proved the very interesting theorem which can be stated as follows (cf. Carleson [1]):

If $R_n \to +\infty$ $(n \to \infty)$, then the number of exceptional values of $f(z)$ in Picard's sense in any neighborhood of every essential singularity is at most $\rho+1$.

In the proof of this theorem, it plays an important role that spherical length of the image curve of Γ_n^j by $w = f(z)$ tends to zero as $n \to \infty$. This follows from the assumption $R_n \to +\infty$ $(n \to \infty)$ in the theorem and from the following lemma (cf. [3]).

LEMMA. *Let $g(z)$ be a single-valued meromorphic function in an annulus $1 \leqq |z| \leqq R$. If $g(z)$ does not take three values w_1, w_2 and w_3 in the annulus, then there exists a positive constant A depending only on w_1, w_2 and w_3 such that spherical length of the image of $|z| = \sqrt{R}$ by $w = g(z)$ does not exceed $A/\sqrt{R}$.*

3. It seems to be of some interest to discuss the Gross property of a given function in connection with the above Matsumoto's theorem. Here we prove the following theorem from this point of view.

THEOREM. *Suppose that the domain D in the z-plane has an exhaustion $\{D_n\}_{n=1}^{\infty}$ satisfying* i), ii), iii), iv) *and*

vi) $$\lim_{\overline{n \to \infty}} \frac{N(n)}{\sqrt{R_n}} = 0 .$$

Let $f(z)$ be a single-valued meromorphic function in D with an essential singularity at every boundary point of D. If $f(z)$ has at least three *exceptional values in Picard's sense in some neighborhood of every essential singularity, then $f(z)$ has the Gross property.*

PROOF. First we note that the assumption vi) implies that the boundary E of D contains no non-degenerate continuum. For every $\zeta \in E$ we can find a positive integer m_ζ and a connected component G_{m_ζ} of the open set $D-\overline{D}_{m_\zeta}$ such that the boundary of G_{m_ζ} contains the point ζ and such that the function $f(z)$ does not take at least three values in G_{m_ζ}. Denote by $\widetilde{G}_{m_\zeta}$ the open set of union of G_{m_ζ} and its boundary contained in E. Letting $\widetilde{G}_{m_\zeta}$ correspond to the point $\zeta \in E$, we get an open covering $\{\widetilde{G}_{m_\zeta}\}_{\zeta \in E}$ of the set E and can choose a finite number of points $\zeta_1, \cdots, \zeta_\nu$ of E so that the union $\bigcup_{k=1}^{\nu} \widetilde{G}_{m_{\zeta_k}}$ covers E. Put $m_0 = \mathrm{Max}(m_{\zeta_1}, \cdots, m_{\zeta_\nu})$. Clearly $f(z)$ does not take at least three values in each connected component $F^j_{m_0}$ $(j=1, \cdots, N(m_0))$ of $D-\overline{D}_{m_0}$. We denote by w^j_i $(i=1, 2, 3)$ the three values not taken by $f(z)$ in $F^j_{m_0}$ and by $\{w_k\}^l_{k=1}$ $(l \geqq 3)$ the set of all points w^j_i $(1 \leqq i \leqq 3,\ 1 \leqq j \leqq N(m_0))$. It is obvious that $f(z)$ does not take at least three values among $w_1, \cdots, w_l$ in any connected component of $D-\overline{D}_n$ for $n \geqq m_0$, so in any D^j_n $(1 \leqq j \leqq N(n))$ for $n \geqq m_0$. From Lemma stated in §2, spherical length $L(n)$ of the image of $\bigcup_{j=1}^{N(n)} \Gamma^j_n$ by $w=f(z)$ does not exceed $AN(n)/\sqrt{R_n}$ for $n \geqq m_0$, where A is a constant depending only on $w_1, \cdots, w_l$.

Consider any Gross' star region S on the covering Riemann surface Φ_f generated by the inverse function of $w=f(z)$ on the extended w-plane. It suffices to show that the set of arguments of all singular segments of S, which end at accessible boundary points of Φ_f, is of outer measure zero. This can be easily seen from vi) and from the fact $L(n) \leqq AN(n)/\sqrt{R_n}$ for $n \geqq m_0$. Thus we get our Theorem.

4. Here we shall show the existence of a domain D and a function $w=f(z)$ satisfying conditions of Theorem by giving an example.

Consider a general Cantor set $E(p_1, p_2, \cdots)$ on the w-plane. This set is constructed as follows. Let p_n $(n \geqq 1)$ be a positive number greater than 1 and delete an open interval with length $1-1/p_1$ from the closed interval $I_0 = [-1/2, 1/2]$ on the real axis of the w-plane so that there remains the closed set I_1 which consists of two closed intervals I^i_1 $(i=1, 2)$ with equal length $l_1 = 1/2p_1$. In general, if I_n consists of closed intervals I^i_n $(i=1, \cdots, 2^n)$ of equal length $l_n = 1/(2^n p_1 \cdots p_n)$, we delete an open interval of length $l_n(1-1/p_{n+1})$ from every I^i_n so that there remain two closed intervals I^{2i-1}_{n+1}, I^{2i}_{n+1} $(i=1, \cdots, 2^n)$ with equal length $1/(2^{n+1} p_1 \cdots p_{n+1})$. The set $E(p_1, p_2, \cdots)$

is the set of intersection $\bigcap_{n=1}^{\infty} I_n$. It is known that $E(p_1, p_2, \cdots)$ is of positive logarithmic capacity if and only if

$$(1) \qquad \sum_{n=1}^{\infty} \frac{\log p_n}{2^n} < +\infty .$$

(cf. Nevanlinna [4]).

Denote by F the complementary domain of $E(p_1, p_2, \cdots)$ with respect to the extended w-plane. We describe circles

$$K_0^1 : \ |w| = 1, \ K_n^i : \ |w - w_n^i| = r_n \ \ (n \geqq 1, \ 1 \leqq i \leqq 2^n)$$

in F, where w_n^i is the middle point of I_n^i, $r_n = \dfrac{1}{2^n \, p_0 p_1 \cdots p_{n-1}} \Big(1 - \dfrac{1}{2p_n}\Big)$ and $p_0 = 1$. Clearly K_n^{2i-1} and K_n^{2i} are tangent outside each other and if

$$(2) \qquad 1 + 2p_{n-1}p_n > 3p_n \ \ (n \geqq 2) ,$$

then K_n^{2i-1} and K_n^{2i} are enclosed by K_{n-1}^i $(1 \leqq n, \ 1 \leqq i \leqq 2^{n-1})$. Let F_n^i be the doubly connected domain surrounded by three circles K_n^{2i-1}, K_n^{2i} and K_{n-1}^i $(n \geqq 1)$ and let F_n be the domain bounded by $\bigcup_{i=1}^{2^n} K_n^i$ and containing the point $z = \infty$ in its interior. We make a slit L_n^i in every $\overline{F}_n^i$ such that L_n^i is contained in $|w - w_{n-1}^i| \leqq 2r_n$ and only one end point of L_n^i lies on $K_n^{2i-1} \cup K_n^{2i}$ and we put

$$F^0 = F - \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^n} L_n^i - L_0^1 ,$$

$$F_k^1 = F - \bigcup_{n=2}^{\infty} \bigcup_{i=1}^{2^n} L_n^i - L_1^k , \ \ (k = 1, 2) ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$F_k^m = F - \bigcup_{n=m+1}^{\infty} \bigcup_{i=1}^{2^n} L_n^i - L_m^k , \ \ (k = 1, \cdots, 2^m) ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

First we connect two replicas of F^0 with each other crosswise across the slit L_0^1 and denote by $\widehat{F^0}$ the resulting surface which has two free slits corresponding to every L_1^k $(k=1,2)$. Next we take a replica of F_k^1, connect it with $\widehat{F}_0$ crosswise across a free slit corresponding to L_1^k and proceed this process for all free slits of $\widehat{F}_0$ corresponding to L_1^k $(k=1,2)$. Thus we get the resulting surface $\widehat{F}^1$ which has $2(1+2)$ sheets and $2(1+2)$ free slits corresponding to each L_2^k $(k=1,\cdots,2^2)$. In general, we connect a replica of F_k^n with $\widehat{F}^{n-1}$ crosswise across a free slit corresponding to L_n^k and proceed this for all slits of $\widehat{F}^{n-1}$ corresponding to L_n^k $(k=1,\cdots,2^n)$. Thus we get the surface $\widehat{F}^n$ with $\prod_{i=0}^{n}(1+2^i)$ sheets. Continuing the procedure indefinitely, we obtain the surface $\widehat{F}$ of planar character which covers no point of the set $E(p_1, p_2, \cdots)$. This surface $\widehat{F}$ is considered as the limiting surface of $\widehat{F}^n$ and every $\widehat{F}^n$ is a subdomain of $\widehat{F}$. Denote by $\widehat{F}_n$ the part of $\widehat{F}^n$ lying over F_{n+1}. It is not so difficult to see that $\{\widehat{F}_n\}_{n=1}^{\infty}$ is an exhaustion of $\widehat{F}$ and that the number of doubly connected components $\widehat{F}_n^i$ of $\widehat{F}_{n+1}-\overline{\widehat{F}}_n$ equals $2^n\prod_{i=0}^{n-1}(1+2^i)$. Clearly the harmonic modulus R_n^i of $\widehat{F}_n^i$ is independent of i. Putting $R_n=R_n^i$, we easily have

$$R_n > p_{n+1}\frac{1-\dfrac{1}{2p_{n+1}}}{1-\dfrac{1}{2p_{n+2}}} = \frac{r_{n+1}}{2r_{n+2}},$$

because $\widehat{F}_n^i$ contains the univalent annulus lying over $2r_{n+2}<|w-w_{n+1}^i|<r_{n+1}$.

Now we map $\widehat{F}$ onto a domain on the z-plane in a one-to-one conformal manner and denote by $w=f(z)$ the inverse function of this conformal mapping. If we denote by D_n the subdomain of D which is mapped onto $\widehat{F}_n$ by $w=f(z)$, then it is evident that $\{D_n\}_{n=1}^{\infty}$ forms an exhaustion of D and each doubly connected component of $D_{n+1}-\overline{D}_n$ is of harmonic modulus R_n^i and the number $N(n)$ of these components is equal to $2^n\prod_{i=0}^{n-1}(1+2^i)$.

So, if we take p_n such that

$$p_n \geqq 2^{(n+1)^2},$$

then (1) and (2) are valid and

$$\lim_{n\to\infty}\frac{N(n)}{\sqrt{R_n}}=0.$$

It is easy to see that $w=f(z)$ has an essential singularity at every boundary point of D and has $E(p_1, p_2, \cdots)$ as the set of exceptional values in Picard's sense in any neighborhood of its essential singularity. Thus we get an example which guarantees the existence of a domain D and a meromorphic function $f(z)$ in D satisfying the assumption in our Theorem.

Further, as mentiond already, (1) implies that the set $E(p_1, p_2, \cdots)$ is of positive logarithmic capacity, so we see from Nevanlinna's theorem [4] that the boundary of D is also of positive logarithmic capacity. Hence Gross-Yûjôbô's theorem stated in §1 can not imply the assertion of our Theorem.

It is still open whether the condition for the number of exceptional values of $f(z)$ in Theorem may be dropped or not.

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