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ON THE EQUIVALENCE OF q-MARTINGALES AND LOCALLY L^p-INTEGRABLE MARTINGALES

NORIHIKO KAZAMAKI

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K. E. Dambis studied q-martingales in [2]. Local martingales, which were defined somewhat differently from the definition of q-martingales, were discussed by K. Ito and S. Watanabe [1]. In the present paper we shall prove that the class of all q-martingales (resp. q-submartingales) concides with the class of all continuous locally L^p-integrable martingales (resp. locally L^p-integrable submartingales) for all $p \geqq 1$, under a general assumption.

As an application, we use this result to prove that the stochastic integral, which was defined for square integrable martingales by M. Motoo and S. Watanabe [6], can be extended to locally square integrable martingales.

1. In order to give a precise formulation of the theorems we need a series of definitions.

Let $(\Omega, \mathfrak{F}, P)$ be the basic P-complete probability space and $\{\mathfrak{F}(t)\}_{0 \leqq t < \infty}$ a family of Borel subfields of $\mathfrak{F}$ such that $\mathfrak{F}(s) \subset \mathfrak{F}(t)$ for $s < t$. We may, and do, suppose that each $\mathfrak{F}(t)$ contains all $\mathfrak{F}$-sets of P-measure zero. We write $a(b) = a_b$ and $a \wedge b = \min(a, b)$.

A submartingale (relative to the family $\mathfrak{F}(t)$) is a real valued process $\{x(t), \mathfrak{F}(t)\}$ such that

(i) $$\forall t \geqq 0,\ E[|x(t)|] < \infty$$

and

(ii) $$\forall (s,t),\ s \leqq t,\ x(s) \leqq E[x(t) \mid \mathfrak{F}(s)] \quad \text{a.s.}$$

If equality holds a.s. in (ii), the process is a martingale. If, moreover, $E[|x(t)|^p] < \infty$ holds, then the process is an L^p-integrable martingale. We shall be concerned here only with sample continuous (sub)martingales.

A stopping time with respect to the family $\mathfrak{F}(t)$ is a positive, possibly infinite, random variable $\tau(\omega)$ such that, for every $a \geqq 0$, $\{\tau \leqq a\} \in \mathfrak{F}(a)$. Given a stopping time τ we shall define $\mathfrak{F}(\tau)$ as the system of all sets $A \in \mathfrak{F}$ for which $A \cap \{\tau \leqq t\} \in \mathfrak{F}(t)$ for every $t \geqq 0$. To avoid constant repetition of qualifying phrases, we assume that τ, β, $\tau(n)$, $\beta(n)$, etc., denote stopping times.

We shall here assume that

$$\mathfrak{F}(t) = \bigcap_{h>0} \mathfrak{F}(t+h) \quad \text{for every } t \geqq 0$$

and

$$\mathfrak{F}(\tau(n)) \uparrow \mathfrak{F}(\tau) \quad \text{for} \quad \tau(n) \uparrow \tau\,, \quad \text{a.s.}$$

We sketch several concepts from [2]. By a φ-process, we mean a non-negative right continuous nondecreasing random process $\{\varphi(t), \mathfrak{F}(t)\}$ possibly assuming infinite values. By a τ-process we mean a family $\{\mathfrak{F}(t), \tau(t)\}$ where $\tau(t)=\tau(t, \omega)$ is right continuous and nondecreasing in t for each fixed ω. We call a τ-process $T = \{\mathfrak{F}(t), \tau(t)\}$ (resp. φ-process $\{\varphi(t), \mathfrak{F}(t)\}$) normal if it is continuous, finite and increases strictly from 0 to ∞.

For instance, let $X^a=\{x^a(t), \mathfrak{F}(t)\}$, $a \in A$, where A is an arbitrary set, be a collection of continuous random processes such that

$$\sup\{|x^a(s) - x^a(0)|;\ \ 0 \leqq s \leqq t,\ a \in A\}$$

is continuous (that is trivially satisfied if A is a finite set), and put

$$\Lambda = \{\lambda(t), \mathfrak{F}(\mathrm{t})\} \text{ where } \lambda(t) = t + \sup\{|x^a(s) - \boldsymbol{x}^a(0)|;\ 0 \leqq s \leqq t,\ a \in A\}\,,$$

then $\lambda(t)$ is $\mathfrak{F}(t)$-measurable, finite and increases strictly from 0 to ∞. Thus Λ is a normal φ-process. We call the τ-process $\Theta = \{\mathfrak{F}(t), \theta(t)\}$ where $\theta(t) = \inf\{u\,;\ \lambda(u) > t\}$, the stopping process for the processes X^a or the brake of the processes X^a. By the definition of Λ, $\theta(t)$ is continuous, finite and strictly increasing from 0 to ∞. Moreover, from the continuity of Λ, we have $\lambda(\theta(t))=t$ and so for each $t \geqq 0$,

$$^{\forall}s \geqq 0,\ [\theta(t) \leqq s] = [t \leqq \lambda(s)] \in \mathfrak{F}(s)\,.$$

Thus each $\theta(t)$ is a stopping time with respect to the family $\mathfrak{F}(t)$. In other words, $\Theta=\{\mathfrak{F}(t), \theta(t)\}$ is a normal τ-process. As $[\lambda(t) \leqq s]=[t \leqq \theta(s)] \in \mathfrak{F}(\theta(s))$, $\{\mathfrak{F}(\theta(t)), \lambda(t)\}$ is a normal τ-process.

Let $X=\{x(t), \mathfrak{F}(t)\}$ be a process that is continuous from the right and let $T=\{\mathfrak{F}(t), \tau(t)\}$ be a τ-process such that $x(\infty)=\lim_{t\to\infty} x(t)$ is defined for all ω for which $\tau(t, \omega)=\infty$ for some t, $0 \leqq t < \infty$.

Put $TX=\{x(\tau(t)), \mathfrak{F}(\tau(t))\}$. Then we say that the process TX is obtained from X by means of a random time change. If T is normal, then the random time change will be called normal.

DEFINITION 1. We call q-(sub)martingales those which are obtained by normal random time changes from continuous (sub)martingales.

DEFINITION 2. $X=\{x(t),\mathfrak{F}(t)\}$ is called a locally L^p-integrable (sub)-martingale if there exists a sequence $\tau(n)$ of stopping times with respect to the family $\mathfrak{F}(t)$ with $\mathrm{P}[\tau(n)<\infty,\ \tau(n)\uparrow\infty]=1$ such that each random process $\{x(t\wedge\tau(n)),\mathfrak{F}(t\wedge\tau(n))\}$ is an L^p-integrable (sub)martingale.

We shall denote by $(SM)^p$ (resp. M^p, $(SM)^p_{loc}$, M^p_{loc}, QSM and QM) the family of all continuous L^p-integrable submartingales (resp. L^p-integrable martingales, locally L^p-integrable submartingales, locally L^p-integrable martingales, q-submartingales and q-martingales).

2. In what follows, we may, and do, suppose that $x(0)=0$.

LEMMA 1. *Let $X=\{x(t),\mathfrak{F}(t)\}$ be a right continuous submartingale and $T=\{\mathfrak{F}(t),\tau(t)\}$ a τ-process. Then:*

(1) *If X is uniformly integrable or there exists a "constant process" c_t such that $\tau(t)\leqq c_t<\infty$, then TX is also a submartingale. If, moreover, X is a martingale, then TX is a martingale.*

(2) *If, for any $a\in[0,\infty)$, the random variable $x^+(t\wedge\tau(a))$ is uniformly integrable with respect to t, then TX is a submartingale. If $x(t\wedge\tau(a))$ is uniformly integrable and X is a martingale, then TX is a martingale.*

Part (1) of Lemma 1 is proved in [4] (see Theorem 11.8, Chapter 7) and for the proof of part (2), see Theorems 4.1. and 4.1.s of Chapter 7 in [4].

LEMMA 2. *For any random process $X=\{x(t),\mathfrak{F}(t)\}\in(SM)^1_{loc}$ (resp. M^1_{loc}), ΘX, where Θ is the brake of X, belongs to $(SM)^p_{loc}$ (resp. M^p_{loc}) for any $p\geqq 1$.*

PROOF. If $X=\{x(t),\mathfrak{F}(t)\}\in(SM)^1_{loc}$, there exists a sequence $\beta(n)$ of stopping times with respect to the family $\mathfrak{F}(t)$ such that

$$\mathrm{P}[\beta(n)<\infty,\beta(n)\uparrow\infty]=1 \quad\text{and}\quad \{x(t\wedge\beta(n)),\mathfrak{F}(t\wedge\beta(n))\}\in(SM)^1\,.$$

Put $\alpha(n)=\inf\{t\,;\theta(t)>\beta(n)\}$. It follows at once from the normality of Θ that $\mathrm{P}[\alpha(n)<\infty,\alpha(n)\uparrow\infty]=1$ and $\{\alpha(n)\leqq s\}=\{\beta(n)\leqq\theta(s)\}\in\mathfrak{F}(\theta(s))$ for any $s\geqq 0$. Therefore each $\alpha(n)$ is a stopping time with respect to the family $\mathfrak{F}(\theta(t))$. By the definition of $(SM)^p_{loc}$, we have only to give the proof of the fact $\{x(\theta(t\wedge\alpha(n))),\mathfrak{F}(\theta(t\wedge\alpha(n)))\}\in(SM)^p$. Since $t=\lambda(\theta(t))=\theta(t)+\sup\{|x(u)|;\ 0\leqq u\leqq\theta(t)\}$ from the definitions of $\lambda(t)$ and $\theta(t)$, we see

$$0 \leqq \theta(t) \leqq t \quad \text{and} \quad \sup\{|x(u)|\,;\, 0 \leqq u \leqq \theta(t)\} \leqq t\,.$$

On the other hand, since

$$x(\theta(t\wedge\alpha(n))) = x(\theta(t)\wedge\theta(\alpha(n))) = x(\theta(t)\wedge\beta(n)) = x(\{\theta(t)\wedge\beta(n)\}\wedge\beta(n))$$

and

$$\{\theta(t)\wedge\beta(n)\leqq s\} = \{\theta(t)\wedge\beta(n)\leqq s\wedge\beta(n)\}\in\mathfrak{F}(s\wedge\beta(n)) \quad \text{for all } s\geqq 0\,,$$

that is, $\theta(t)\wedge\beta(n)$ is a stopping time with respect to the family $\mathfrak{F}(t\wedge\beta(n))$, in view of Lemma 1 (1), each $\{x(\theta(t\wedge\alpha(n))), \mathfrak{F}(\theta(t\wedge\alpha(n)))\}$ belongs to $(SM)^p$. Hence ΘX belongs to $(SM)^p_{loc}$. This completes the proof.

LEMMA 3. *For any random process* $X=\{x(t), \mathfrak{F}(t)\}\in QSM$ *(resp.* QM*),* ΘX*, where* Θ *is the brake of* X*, belongs to* $(SM)^p$ *(resp.* M^p*) for any* $p\geqq 1$.

PROOF. From the definition of QSM, $X=TY$, where $Y=\{y(t), \mathfrak{G}(t)\}$ is a continuous submartingale and $T=\{\mathfrak{G}(t), \tau(t)\}$ is a normal τ-process. As $\sup\{|x(u)|;\, 0\leqq u\leqq\theta(a)\}\leqq a$ for any $a\geqq 0$, we have

$$\sup\{|y(t\wedge\tau(\theta(a)))|;\, 0\leqq t<\infty\}\leqq a\,.$$

This implies the uniform integrability of $\{y(t\wedge\tau(\theta(a)))\}_{0\leqq t<\infty}$, hence by Lemma 1 (2) $\Theta X = [\Theta T]Y$ belongs to $(SM)^p$. Thus the lemma is proved. (This proof is due to K. E. Dambis [2]).

THEOREM 1. *For any* $p\geqq 1$, $(SM)^p_{loc}$ *(resp.* M^p_{loc}*) coincides with* QSM *(resp.* QM*).*

PROOF. Let $X=\{x(t), \mathfrak{F}(t)\}$ be any random process of QSM and Θ be the brake of X. Lemma 3 implies $\Theta X\in(SM)^p$ for any $p\geqq 1$.

Put $X^n=\{x(t\wedge\theta(n)), \mathfrak{F}(t\wedge\theta(n))\}$. Then clearly

$$x(t\wedge\theta(n)) = x(\theta(\lambda(t))\wedge\theta(n)) = x(\theta(\lambda(t)\wedge n))$$

and so each $X^n\in(SM)^p$ by Lemma 1(1). Hence $X\in(SM)^p_{loc}$.

Conversely $X=\{x(t), \mathfrak{F}(t)\}$ is a locally L^p-integrable submartingale, then ΘX is a locally L^p-integrable submartingale by Lemma 2, that is, there exists a sequence $\tau(n)$ of stopping times with respect to the family $\mathfrak{F}(\theta(t))$, with $P[\tau(n)<\infty, \tau(n)\uparrow\infty]=1$ such that $\{x(\theta(t\wedge\tau(n))), \mathfrak{F}(\theta(t\wedge\tau(n)))\}$ belongs to $(SM)^1$ for each n. By the assumption on $\mathfrak{F}(t)$, as $\mathfrak{F}(\theta(s\wedge\tau(n)))\uparrow\mathfrak{F}(\theta(s))$, for any $A\in\mathfrak{F}(\theta(s))$ there exists $A^n\in\mathfrak{F}(\theta(s\wedge\tau(n)))$ such that $P(A\triangle A^n)$ converges to 0. Then for each n,

$$\int_{A^n} x(\theta(s \wedge \tau(n)))\, dP \leqq \int_{A^n} x(\theta(t \wedge \tau(n)))\, dP, \quad s \leqq t.$$

In view of the Lebesgue bounded convergence theorem, we have

$$\int_A x(\theta(s))\, dP \leqq \int_A x(\theta(t))\, dP.$$

Hence ΘX belongs to $(SM)^1$. Therefore as $X = \Lambda[\Theta X]$, X is a q-submartingale. This completes the proof.

COROLLARY. *Let $X = \{x(t), \mathfrak{F}(t)\}$ be a continuous local submartingale. Then $\lim_{t\to\infty} x(t)$ exists and is finite almost surely where $\limsup_{t\to\infty} x(t) < \infty$.*

PROOF. It is proved in [4] (see Theorem 3.1.s (iv) of Chapter 11) when X is a continuous submartingale. Then the proof is obvious from the fact $(SM)^1_{loc} = QSM$.

3. Next we shall show that the equivalence of q-martingales and locally L^p-integrable martingales is also true in R^n.

DEFINITION 3. We call a process $X = \{x(t), \mathfrak{F}(t)\}$ in R^n with $\{(h \circ x)(t), \mathfrak{F}(t)\}$ a continuous martingale, for each spherical harmonic polynomial h in R^n, a continuous martingale.

DEFINITION 4. We call q-martingales in R^n those processes which are obtained by normal random time changes from continuous martingales in R^n.

DEFINITION 5. A process $X = \{x(t), \mathfrak{F}(t)\}$ in R^n is called a locally L^p-integrable martingales in R^n if $h \circ X = \{(h \circ x)(t), \mathfrak{F}(t)\}$ is a locally L^p-integrable martingale for each spherical harmonic polynomial h in R^n.

We shall denote by $M^p(R^n)$ (*resp.* $M^p_{loc}(R^n)$ and $QM(R^n)$) the family of all continuous L^p-integrable martingales (resp. locally L^p-integrable martingales and q-martingales) in R^n.

THEOREM 2. *For any $p \geqq 1$, $M^p_{loc}(R^n)$ coincides with $QM(R^n)$.*

PROOF. If $X = \{x(t), \mathfrak{F}(t)\} \in QM(R^n)$, then $X = TY$, where $Y = \{y(t), \mathfrak{G}(t)\}$

is a continuous martingale in R^n and $T=\{\mathfrak{G}(t), \tau(t)\}$ is a normal τ-process.

Let h be any spherical harmonic polynomial in R^n. Put $h\circ X=\{(h\circ x)(t), \mathfrak{F}(t)\}$. Then $h\circ X = T[h\circ Y]$. As $h\circ Y$ is a continuous martingale, $h\circ X$ is a q-martingale. Theorem 1 implies that $h\circ X$ is a locally L^p-integrable martingale. Hence $X=\{x(t), \mathfrak{F}(t)\}\in M^p_{loc}(R^n)$ for any $p\geqq 1$.

Conversely if $X=\{x(t), \mathfrak{F}(t)\}\in M^p_{loc}(R^n)$, $h\circ X=\{(h\circ x)(t), \mathfrak{F}(t)\}$ is a locally L^p-integrable martingale for each spherical harmonic polynomial h *in* R^n. There exists a sequence $\beta(n)$ of stopping times with respect to the family $\mathfrak{F}(t)$ such that $P[\beta(n)<\infty, \beta(n)\uparrow\infty]=1$ holds and $\{(h\circ x)(t\wedge\beta(n)), \mathfrak{F}(t\wedge\beta(n))\}$ is a martingale for each n. This sequence $\{\beta(n)\}$ may depend on h. Let Θ be the brake of X. Then $\{(h\circ x)(\theta(t)), \mathfrak{F}(\theta(t))\}$ is a locally L^p-integrable martingale in view of Lemma 2. Therefore there exists a sequence $\alpha(n)$ of stopping times with respect to the family $\mathfrak{F}(\theta(t))$ such that $\{(h\circ x)(\theta(t\wedge\alpha(n))), \mathfrak{F}(\theta(t\wedge\alpha(n)))\}$ is a martingale with $P[\alpha(n)<\infty, \alpha(n)\uparrow\infty]=1$.

By assumption, as $\mathfrak{F}(\theta(s\wedge\tau(n)))\uparrow\mathfrak{F}(\theta(s))$, for any $A\in\mathfrak{F}(\theta(s))$ there exists $A^n\in\mathfrak{F}(\theta(s\wedge\tau(n)))$ such that $P(A\triangle A^n)$ converges to 0. Then for each n,

$$\int_{A^n}(h\circ x)(\theta(s\wedge\alpha(n)))dP=\int_{A^n}(h\circ x)(\theta(t\wedge\alpha(n)))\,dP\,,\quad s\leqq t\,.$$

As a harmonic function is bounded on every compact set, by the Lebesgue bounded convergence theorem we have

$$\int_A(h\circ x)(\theta(s))\,dP=\int_A(h\circ x)(\theta(t))\,dP.$$

This implies $h\circ\Theta X\in M^1$. Hence $\Theta X\in M^1(R^n)$. As $X=\Lambda[\Theta X]$, X belongs to $QM(R^n)$. This completes the proof.

4. We shall now apply the result obtained above to a generalization of the stochastic integral defined in [6].

Let N^+ be the set of all natural increasing processes $A(t)$ defined for $t\in[0,\infty)$ and write

$$N=\{A(t)=A^1(t)-A^2(t)\,;\ A^i(t)\in N^+,\ i=1,2\}\,.$$

Let Ψ be the class of all (t,ω)-measurable real-valued processes $\psi(t,\omega)$ that are $\mathfrak{F}(\tau)$-measurable for each stopping time τ with respect to the family $\mathfrak{F}(t)$ and let Ψ_{rc} be the class of all bounded right continuous process having left hand limits. We define for $\varphi\in N^+$ semi-norms $\|\ \ \|_\varphi(t)$ over Ψ by

$$\|\psi\|_\varphi(t)=E\left[\left(\int_0^t\psi(s)^2\,d\varphi(s)\right)^{1/2}\right],\quad \psi\in\Psi$$

and put $L_2(\varphi)=\Psi\cap\overline{\Psi}_{rc}$, where $\overline{\Psi}_{rc}$ is the closure of Ψ_{rc} with respect to semi-norms $\| \ \|_\varphi(t)$. Clearly $\|\psi\|_\varphi(s)\leqq\|\psi\|_\varphi(t)$ for $s<t$.

First we recall the followings.

THEOREM 3. *For* $X=\{x(t),\mathfrak{F}(t)\}$, $Y=\{y(t),\mathfrak{F}(t)\}\in M^2$, *there exists a unique (up to equivalence)* $<X,Y>\in N$ *such that for each* $t>s\in[0,\infty)$

$$E[(x(t)-x(s))(y(t)-y(s))\,|\,\mathfrak{F}(s)]=E[<X,Y>(t)-<X,Y>(s)\,|\,\mathfrak{F}(s)].$$

THEOREM 4. *For every* $X=\{x(t),\mathfrak{F}(t)\}\in M^2$ *and* $\psi\in L^2(<X>)$ *where* $<X>=<X,X>$, *there exists a unique* $Y=\{y(t),\mathfrak{F}(t)\}\in M^2$ *satisfying*

$$<Y,Z>(t)=\int_0^t\psi(s)\,d<X,Z>(s),\quad \text{P-}a.s.\ \ \textit{for any}\ Z=\{z(t),\mathfrak{F}(t)\}\in M^2.$$

For the proof of these two theorems, see [5] or [6]. In M. Motoo and S. Watanabe [4], Y of the above Theorem 4 is called the stochastic integral of ψ by X and is denoted by

$$y(t)=\int_0^t\psi(s)\,dx(s).$$

Now let $X=\{x(t),\mathfrak{F}(t))\}$ be any random process of M^2_{loc}. In view of Lemma 4, $\Theta X=\{x(\theta(t)),\mathfrak{F}(\theta(t))\}$ belongs to M^2 for $X=\{x(t),\mathfrak{F}(t)\}\in M^2_{loc}$. It is easy to see that $\psi(\theta(t))$ is (t,ω)-measurable. As $\{\mathfrak{F}(\theta(t)),\lambda(t)\}$ is a normal τ-process and $[\theta(\tau)\leqq s]=[\tau\leqq\lambda(s)]$ holds for any stopping time τ with respect to the family $\mathfrak{F}(\theta(t))$, we see $[\theta(\tau)\leqq s]\in\mathfrak{F}(\theta(\lambda(s)))=\mathfrak{F}(s)$, that is, $\theta(\tau)$ is a stopping time with respect to the family $\mathfrak{F}(t)$.

Therefore $\psi(\theta(\tau))$ is $\mathfrak{F}(\theta(\tau))$-measurable. Thus, from Theorem 4, there exists a unique $Y^*=\{y^*(t),\mathfrak{F}(\theta(t))\}\in M^2$ such that

$$<Y^*,Z>(t)=\int_0^t\psi(\theta(s))\,d<\Theta X,Z>(s),\quad \text{P-a.s. for any } Z=\{z(t),\mathfrak{F}(\theta(t))\}\in M^2.$$

Put $H(X,\psi)=\Lambda Y^*=\{y^*(\lambda(t)),\mathfrak{F}(t)\}$. Theorem 1 implies $\Lambda Y^*\in M^2_{loc}=QM$. This mapping H coincides with the stochastic integral on M^2. In fact, let $X=\{x(t),\mathfrak{F}(t)\}$ be an L^2-integrable martingale. First we consider the case that $\psi(t)$ is a step function, that is, there exists an increasing sequence $\{\tau(n)\}$ of stopping times with respect to the family $\mathfrak{F}(t)$ such that $\tau(n)\uparrow\infty$ and $\psi(s)=\psi(\tau(n-1))$ if $\tau(n-1)\leqq s<\tau(n)$. Then there exists an increasing sequence $\beta(n)$ such that $\theta(\beta(n))=\tau(n)$ and $\psi(\theta(s))=\psi(\theta(n-1)))$ if $\beta(n-1)\leqq s<\beta(n)$. As

$[\beta(n) \leqq s] = [\theta(\beta(n)) \leqq \theta(s)] = [\tau(n) \leqq \theta(s)] \in \mathfrak{F}(\theta(s))$ for all $s \geqq 0$, that is, each $\beta(n)$ is a stopping time with respect to the family $\mathfrak{F}(\theta(t))$, $\psi(\theta(t))$ is also a step function.

Since, by the definition of the stochastic integral in [6],

$$\int_0^t \psi(s)\, dx(s) = \sum_n \psi(t \wedge \tau(n-1))\{x(t \wedge \tau(n)) - x(t \wedge \tau(n-1))\}$$

and

$$\begin{aligned} y^*(t) &= \int_0^t \psi(\theta(s))\, dx(\theta(s)) \\ &= \sum_n \psi(\theta(t \wedge \beta(n-1)))\{x(\theta(t \wedge \beta(n)) - x(t \wedge \beta(n-1)))\} \\ &= \sum_n \psi(\theta(t) \wedge \tau(n-1))\{x(\theta(t) \wedge \tau(n)) - x(\theta(t) \wedge \tau(n-1))\}, \end{aligned}$$

we have

$$y^*(\lambda(t)) = \int_0^t \psi(s)\, dx(s).$$

In other words, $H(X, \psi)$ coincides with the stochastic integral of ψ by X if ψ is a step function.

Now let ψ be any element of $L^2(<X>)$. Then we can choose a sequence of step function $\{\psi^n\} \subset \Psi$ such that

$$\lim_{n \to \infty} \|\psi^n - \psi\|_{<x>}(t) = 0.$$

Put

$$y^n(t) = \int_0^t \psi^n(s)\, dx(s).$$

Then we get $\lim E[((y(t) - y^n(t))^2] = 0$ (see [5]). On the other hand, by the uniqueness of $<x>$ in Theorem 3, we have $<\Theta X>(t) = <X>(\theta(t))$ for every $t \geqq 0$. Since for any $\psi \in L^2(<X>)$

$$E\left[\left(\int_0^t \psi^2(\theta(s))\, d<\Theta X>(s)\right)^{\frac{1}{2}}\right] \leqq E\left[\left(\int_0^t \psi^2(s)\, d<X>(s)\right)^{\frac{1}{2}}\right],$$

we get $\lim_{n \to \infty} E[(y^*(t) - y^{*n}(t))^2] = 0$ where $y^{*n}(t) = \int_0^t \psi^n(\theta(s))\, dx(\theta(s))$. As

$\{(y^n(t)-y(t))^2, \mathfrak{F}(t)\}$ is a submartingale, we have

$$\begin{aligned} E[(y^*(t)-y(\theta(t)))^2] &\leqq \frac{1}{2}\{E[(y^*(t)-y^{*n}(t))^2]+E[(y^{*n}(t)-y(\theta(t)))^2]\} \\ &\leqq \frac{1}{2}\{E[(y^*(t)-y^{*n}(t))^2]+E[(y^n(\theta(t))-y(\theta(t)))^2]\} \\ &\leqq \frac{1}{2}\{E[(y^*(t)-y^{*n}(t))^2]+E[(y^n(t)-y(t))^2]\}\,. \end{aligned}$$

hence $y^*(t)=y(\theta(t))$. This implies $y(t)=y^*(\lambda(t))$. Therefore if $X=(x(t), \mathfrak{F}(t)) \in M^2$, $H(X,\psi)$ coincides with the stochastic integral of ψ by X for any $\psi \in L^2(<X>)$.

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MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN