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INVARIANT SUBSPACES OF SOME NON-SELFADJOINT OPERATORS

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1. In [3] J. Schwartz has proved the following result.

Let T be an operator on a Hilbert space $\mathfrak{H}$ of more than one dimension. Write $T = A+iB$, where A and B are selfadjoint, and suppose that the imaginary part B of T belongs to one of the classes C_p, where $1 \leqq p < \infty$. Then the Hilbert space admits a proper closed subspace which is invariant under T.

The purpose of this note is to show that the above theorem may be generalized to an operator such as T is the sum of a normal operator A with some spectrum condition and a compact operator B with some condition. In fact, we shall show the following theorem:

THEOREM. *Let T be an operator on a Hilbert space $\mathfrak{H}$ of more than one dimension. Write $T=A+B$, where T is the sum of a normal operator A, whose spectrum lies on a Jordan curve J, which consists of a finite number of rectifiable smooth arcs, (it may well be the case that the spectrum separates the plane), and a compact operator B, which belongs to one of the classes C_p, where $1 \leqq p < \infty$. Then the Hilbert space admits a proper closed subspace which is invariant under T.*

Throughout the present note, an operator means a bounded linear operator on a Hilbert space $\mathfrak{H}$ which we assume to be separable. We denote by $\sigma(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$ and $\rho(T)$ the spectrum, the point spectrum, the continuous spectrum, the residual spectrum and the resolvent set of an operator T respectively. For the sake of convenience, we shall list some results on the classes C_p ([2], Part II, Section XI. 9).

Let T be a compact operator on a Hilbert space and $H = (T^*T)^{1/2}$. Let $\mu_1, \mu_2, \cdots, \mu_n, \cdots$ be the eigenvalues of H, arranged in decreasing order and repeated according to multiplicity. We write $\mu_n(T)$ for the n-th eigenvalue of H. we write

$$\|T\|_p = \left\{\sum (\mu_n(T))^p\right\}^{1/p} \quad 0 < p \leqq \infty ,$$

in case $p = \infty$, $\|T\|_p$, as usual, has the meaning $\|T\|_\infty = \sup_j \mu_j(T) = \mu_1(T)$. The class C_p is the set of all compact operators T such that $\|T\|_p$ is finite.

THEOREM A. *Let $T \in C_p$, where $1 \leqq p < \infty$, and let $\lambda_i = \lambda_i(T)$ be the eigenvalues of T, repeated according to multiplicity. Then*

(a) $$\left\{\sum |\lambda_i(T)|^p\right\}^{1/p} \leqq \|T\|_p ,$$

(b) *if $k \geqq p$, the infinite product*

$$\delta_k(T) = \prod_{i=1}^{\infty} \left[(1+\lambda_i) \exp\left\{-\lambda_i + \frac{\lambda_i^2}{2} - \cdots + \frac{(-1)^{k-1}\lambda_i^{k-1}}{k-1}\right\}\right]$$

converges absolutely,

(c) *if $k \geqq p \geqq k-1$, there exists a finite constant K_1 depending only on p, such that*

$$|\delta_k(T)| \leqq \exp\{K_1\|T\|_p^p\} ,$$

(d) *for each $T_1 \in C_p$, the function $\delta_k(T+zT_1)$ is an analytic function of z.*

THEOREM B. *Let $1 \leqq p < \infty$ and $T \in C_p$. Let $k \geqq p \geqq k-1$, and let $\delta_k(T)$ denote the infinite product of the preceding theorem. Then the operator $\delta_k(T)(I+T)^{-1}$ depends continuously on T, and satisfies the inequality*

$$\|\delta_k(T)(I+T)^{-1}\| \leqq \exp\{K_2\|T\|_p^p\} ,$$

where K_2 is a finite constant depending only on p.

Before going into discussions, the author wishes to express his hearty thanks to Professor M. Fukamiya and Professor M. Takesaki for their valuable suggestions and constant encouragement.

2. In order to prove our theorem, we need the following lemmas.

Now, we shall understand a smooth arc to be such that it has a continuous second derivative when parametrized with respect to arc length. We assume that the curve J is positively oriented and, for a fixed λ_0 on J, J has a parametrization $\lambda=g(s)$, $0\leqq s\leqq l(J)$, in terms of arc length s from λ_0, $g(0)=\lambda_0$, $g(s)=g(s+l(J))$, and $g(s)$ is continuous on J and $g'(s)$, $g''(s)$ are continuous except the points $\lambda_k=g(s_k)$, $s_k<s_{k+1}$, $k=1,2,\cdots,n$ on J, where $l(J)$ denotes the whole length of J.

LEMMA 2.1. ([4]) *Let J be as above. Then for each pair of the points $\lambda_\alpha=g(s_\alpha)$, $s_j<s_\alpha<s_{j+1}$, $\lambda_\beta=g(s_\beta)$, $s_k<s_\beta<s_{k+1}$, $s_\alpha<s_\beta$ on J and for any sufficiently small number ε, we have a closed simply connected domain $D(s_\alpha, s_\beta)$ containing the subarc $(g(s_\alpha), g(s_\beta))$ of J in its interior such that*

(a) *$\partial D(s_\alpha, s_\beta)$, boundary of $D(s_\alpha, s_\beta)$, is a rectifiable Jordan curve traversing J at λ_α and λ_β only,*

(b) *for each $\lambda\in\partial D(s_\alpha,s_\beta)\cap\{\lambda\,;\,|\lambda-g(s_\alpha)|<\varepsilon/4\}$, $d(\lambda,J)=|\lambda-g(s_\alpha)|$ and also for each $\lambda\in\partial D(s_\alpha,s_\beta)\cap\{\lambda\,;\,|\lambda-g(s_\beta)|<\varepsilon/4\}$, $d(\lambda,J)=|\lambda-g(s_\beta)|$,*

(c) $$\max_{\lambda\in\partial D(\cdot)} d(\lambda, \operatorname{arc}[g(s_\alpha), g(s_\beta)])<\varepsilon\,.$$

LEMMA 2.2. (due to T. Yoshino) *If an operator T is the sum of a compact operator C and an operator S, then*

$$\sigma_c(T)\subset\sigma(S)\,.$$

PROOF. Suppose $\lambda\in\sigma_c(T)$ and $\lambda\in\rho(S)$. Then there exist unit vectors $x_n\in\mathfrak{H}$ such that $Tx_n-\lambda x_n\to 0$, because $\lambda\in\sigma_c(T)$. Here by the compactness of C, there exists a non-zero vector $y\in\mathfrak{H}$ such that $Cx_n-y\to 0$, because if y is a zero vector, then

$$\|Sx_n-\lambda x_n\|\leqq\|Tx_n-\lambda x_n\|+\|Cx_n\|\to 0\,,$$

i.e., $\lambda\in\sigma(S)$ and this contradicts with $\lambda\in\rho(S)$. On the other hand, $\lambda\in\rho(S)$ implies $S-\lambda$ is invertible. Let $z=(S-\lambda)^{-1}y$, then clearly z is a non-zero vector and

$$\begin{aligned}\|x_n+z\|&\leqq\|(S-\lambda)^{-1}\|\;\|(S-\lambda)\,x_n+y\|\\&=\|(S-\lambda)^{-1}\|\;\|(T-\lambda)\,x_n-(Cx_n-y)\|\\&\leqq\|(S-\lambda)^{-1}\|\{\|Tx_n-\lambda x_n\|+\|Cx_n-y\|\}\to 0\,.\end{aligned}$$

Therefore,

$$\|Tz-\lambda z\| \leqq \|Tz+Tx_n\| + \|Tx_n-\lambda x_n\| + \|\lambda x_n+\lambda z\|$$
$$\leqq (\|T\|+|\lambda|)\|x_n+z\| + \|Tx_n-\lambda x_n\| \to 0\,,$$

i.e., $\lambda \in \sigma_p(T)$, this contradicts with $\lambda \in \sigma_c(T)$. Thus $\sigma_c(T) \subset \sigma(S)$.

LEMMA 2.3. *Let T be a quasi-nilpotent operator which is the sum of a normal operator and a compact operator, then T is compact.*

PROOF. By virture of that a normal quasi-nilpotent operator is zero. This lemma may be proved just in the same way as in the proof of Lemma 2.2 of [3].

LEMMA 2.4. *Let $f(z)$ be a function analytic in the unit disc and for some $p \geqq 1$ the following condition be satisfied*

$$\operatorname{Re} f(z) \leqq (1-|z|)^{-p} \qquad \textit{for} \quad |z|<1\,.$$

Then, there exists a finite constant K, independent of z, such that

$$|f(z)| \leqq K(1-|z|)^{-p-1} \qquad \textit{for} \quad |z|<1\,.$$

PROOF. Let r be an arbitrary real value, satisfying $1/2 \leqq r < 1$, then using the Carathéodory inequality for the disc $|z|<r$, we have

$$|f(z)-f(0)| \leqq 2\{(1-r)^{-p} - \operatorname{Re} f(0)\} \frac{|z|}{r-|z|}$$

for $|z|<r$. Thus we have

$$|f(z)| \leqq \frac{K_1(1-r)^{-p}}{r-|z|} + \frac{K_2}{r-|z|} + K_3 \tag{1}$$

for all $|z|=2r-1$, where K_i ($i=1,2,3$) are finite constants. On the other hand, let z be an arbitrary point of the unit disc $|z|<1$, then there exists the real value r, satisfying $|z|=2r-1$, therefore, it follows that the right hand side of the inequality (1) is equal to $2^{p+1}K_1(1-|z|)^{-p-1} + 2K_2(1-|z|) + K_3$. By $p \geqq 1$, we obtain the following inequality

$$|f(z)| \leqq K(1-|z|)^{-p-1}\,,$$

where K is a finite constant independent of z. The proof of Lemma 2.4 is now completed.

3. We return to the proof of our theorem. We shall divide the proof into portions. Theorem follows immediately in each case where the following condition is satisfied: (i) $\sigma_p(T)\neq\emptyset$; (ii) $\sigma_r(T)\neq\emptyset$; (iii) $\sigma(T)$ is disconnected. Therefore, we may prove this theorem under the condition that $\sigma(T)$ coincide with $\sigma_c(T)$ and is connected. Since the operator T may be translated, we may assume without loss of generality that either $\sigma(T)=\{0\}$ or $\sigma(T)$ is a connected subarc of J. In the case $\sigma(T)=\{0\}$, from Lemma 2.3, it follows that the Hilbert space admits a proper closed invariant subspace under T, using Aronszajn-Smith theorem ([1]). Therefore, we have only to show that there can exist no operator $T=A+B$, where A is normal such that $\sigma(A)\subset J$ and $B\in C_p$, for some finite $p\geqq 1$, and $\sigma(T)$ is a connected subarc of J and $\mathfrak{H}$ admits no proper closed invariant subspace under T.

Let k be an integer such that $k\geqq p\geqq k-1$. Let $\delta(\lambda)=\delta_k(-B(\lambda-A)^{-1})$. By Theorem A, $\delta(\lambda)$ is defined and analytic for all $\lambda\in\rho(A)$ and satisfies the inequality

$$|\delta(\lambda)|\leqq\exp[K\{d(\lambda,\sigma(A))\}^{-p}],$$

where $d_\lambda=d(\lambda,\sigma(A))$ denotes the distance from λ to the spectrum $\sigma(A)$ and K denotes some finite constant. By Theorem A, we have $\delta(\lambda)\neq 0$ for every $\lambda\in\rho(A)$. Thus we have $\delta(\lambda)=\exp\{\alpha_j(\lambda)\}$, where $\alpha_1(\lambda)$ is defined and analytic in the interior of the Jordan curve J, which we denote by D_1, and $\alpha_2(\lambda)$ is defined and analytic in the exterior of the Jordan curve J, which we denote by D_2. Then $\alpha_j(\lambda)$, $(j=1,2)$ satisfy the inequalities

$$\operatorname{Re}\alpha_j(\lambda)\leqq Kd_\lambda{}^{-p}.$$

Let $z=\phi_j(\lambda)$ be the conformal mappings of the domain D_j onto the unit disc and let $\lambda=\psi_j(z)$ be its inverse for $j=1,2$ respectively. Then $\alpha_j(\psi_j(z))$ are functions analytic in the unit disc, which satisfy the inequality

$$\operatorname{Re}\alpha_j(\lambda)=\operatorname{Re}\alpha_j(\psi_j(z))\leqq K_j'(1-|z|)^{-p}$$

for some finite constants K_j' $(j=1,2)$. It follows by Lemma 2.4 that

$$|\alpha_j(\psi_j(z))|\leqq K_j''(1-|z|)^{-p-1}$$

for some finite constants K_j'' $(j=1,2)$. Thus, returning back to the domain D_1, D_2 respectively, we find that

$$|\alpha_j(\lambda)|\leqq K'd_\lambda{}^{-p-1}\qquad(j=1,2)$$

for some finite constant K'. In particular,

$$\operatorname{Re} \alpha_j(\lambda) \geqq -K' d_\lambda^{-p-1} \qquad (j=1,2)$$

so that

$$|\delta(\lambda)|^{-1} \leqq \exp\{K' d_\lambda^{-p-1}\} .$$

Since

$$(\lambda - T)^{-1} = \frac{(\lambda - A)^{-1}\delta(\lambda)(I - B(\lambda - A)^{-1})^{-1}}{\delta(\lambda)} ,$$

it follows by Theorem B that there exists a finite constant K_0 such that

(2) $$\|(\lambda - T)^{-1}\| \leqq \exp\{K_0 d_\lambda^{-p-1}\} .$$

This growth condition of the resolvent of T near its spectrum $\sigma(T)$ plays an important role in our considerations.

LEMMA 3.1. *Let T be as above. For each pair of the points $\lambda_\alpha = g(s_\alpha)$, $\lambda_\beta = g(s_\beta)$, where $s_\alpha < s_\beta$ on J and $s_\alpha \neq s_j$, $s_\beta \neq s_j$, $(j=1,2,\cdots,n)$ we put*

(3) $$\mathfrak{S}(s_\alpha, s_\beta) = \{x\,;\, (\lambda - T)^{-1}x \textit{ is continuable to a function which is analytic near the arc } (g(s_\alpha), g(s_\beta))\}$$

and

(4) $$\widetilde{\mathfrak{S}}(s_\alpha, s_\beta) = \{x\,;\, (\lambda - T)^{-1}x \textit{ is continuable to a function which is analytic near the arc } (g(s_\beta), g(s_\alpha + l(J)))\} .$$

Then both $\mathfrak{S}(s_\alpha, s_\beta)$ and $\widetilde{\mathfrak{S}}(s_\alpha, s_\beta)$ are closed linear subspaces of the Hilbert space $\mathfrak{H}$, invariant under T.

Proof of the first assertion of Lemma 3.1. Because both of the linearity of $\mathfrak{S}(s_\alpha, s_\beta)$ and the invariantness under T are plain, we have only to prove that $\mathfrak{S}(s_\alpha, s_\beta)$ is closed. Let $x_n \in \mathfrak{S}(s_\alpha, s_\beta)$, and $x_n \to x$. Let $R(\lambda, T : x_n)$ denote the analytic continuation of $(\lambda - T)^{-1}x_n$, then for every $\lambda \in \rho(T)$, $R(\lambda, T : x_n) \to (\lambda - T)^{-1}x$. For any sufficiently small positive number ε, let $D_0(s_\alpha + \varepsilon, s_\beta - \varepsilon)$ be a closed simply connected domain containing the subarc $(g(s_\alpha + \varepsilon), g(s_\beta - \varepsilon))$ of J indicated in Figure (*) in relation to given domain $D(\cdot)$ in Lemma 2.1 (we can consider that $D_0(\cdot)$ with sufficiently small ω has the same properties of $D(\cdot)$ in Lemma 2.1). Then $R(\lambda, T : x_n)$ are analytic in the interior of $D_0(s_\alpha + \varepsilon, s_\beta - \varepsilon)$. Here we define the function such that

$$m(\lambda) = \begin{cases} Q(\lambda : s_\alpha+\varepsilon, s_\beta-\varepsilon) & \\ \qquad\qquad \text{if} \quad \lambda \neq g(s_\alpha+\varepsilon) \text{ and } \lambda \neq g(s_\beta-\varepsilon) \\ 0 \qquad\qquad \text{if} \quad \lambda = g(s_\alpha+\varepsilon) \quad \text{or} \quad \lambda = g(s_\beta-\varepsilon) \end{cases}$$

where $Q(\lambda : s_\alpha+\varepsilon, s_\beta-\varepsilon) = \exp[-\exp\{(\lambda-g(s_\alpha+\varepsilon))^{-1} e^{i(\pi/2+\theta_1)}\} - \exp\{-(\lambda-g(s_\beta-\varepsilon))^{-1} e^{i(\pi/2+\theta_2)}\}]$, $\theta_1 = \arg i\cdot s\cdot g'(s_\alpha+\varepsilon)$ and $\theta_2 = \arg i\cdot s\cdot g'(s_\beta-\varepsilon)$. Then, $m(\lambda)$ is analytic and nonvanishing in the interior of $D_0(s_\alpha+\varepsilon, s_\beta-\varepsilon)$. We define the function $f_n(\lambda)$ on $D_0(s_\alpha+\varepsilon, s_\beta-\varepsilon)$ as follows:

$$f_n(\lambda) = \begin{cases} m(\lambda)\, R(\lambda, T : x_n) & \\ \qquad\qquad \text{if} \quad \lambda \neq g(s_\alpha+\varepsilon) \text{ and } \lambda \neq g(s_\beta-\varepsilon) \\ 0 \qquad\qquad \text{if} \quad \lambda = g(s_\alpha+\varepsilon) \quad \text{or} \quad \lambda = g(s_\beta-\varepsilon)\,. \end{cases}$$

Then $f_n(\lambda)$ are analytic in the interior of $D_0(s_\alpha+\varepsilon, s_\beta-\varepsilon)$ and strongly continuous on $\partial D_0(s_\alpha+\varepsilon, s_\beta-\varepsilon)$, which follows from the inequality (2). By the maximum modulus principle, $\{f_n(\lambda)\}$ is a uniform Cauchy sequence with respect to λ, hence the limit function $f_0(\lambda)$ is analytic in the interior of $D_0(s_\alpha+\varepsilon, s_\beta-\varepsilon)$ and so

$$x_\infty(\lambda) = f_0(\lambda)\, Q(\lambda : s_\alpha+\varepsilon,\ s_\beta-\varepsilon)^{-1}$$

is also analytic in the interior of $D_0(s_\alpha+\varepsilon,\ s_\beta-\varepsilon)$. It follows that $(\lambda-T)^{-1}x$ has an analytic continuation to a neighborhood of the arc $(g(s_\alpha+\varepsilon), g(s_\beta-\varepsilon))$ for each sufficiently small ε, and hence to a neighborhood of the arc $(g(s_\alpha), g(s_\beta))$. Thus $x \in \mathfrak{S}(s_\alpha, s_\beta)$. The assertion for the $\widetilde{\mathfrak{S}}(s_\alpha, s_\beta)$ may be proved in just the same way.

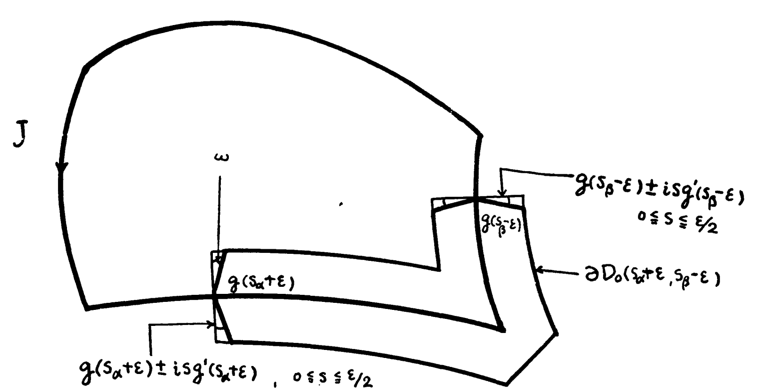

Figure (*)

LEMMA 3.2. *Let T be an operator as above, and let* $\mathfrak{S}(s_\alpha, s_\beta)$, $\tilde{\mathfrak{S}}(s_\alpha, s_\beta)$, $Q(\lambda : s_\beta, s_\alpha+l(J))$ *and* $D_0(s_\beta, s_\alpha+l(J))$ *be as the same as in Lemma* 3.1. *For any vector x in the Hilbert space, the function $f(\lambda)$ is defined such that*

$$f(\lambda) = \begin{cases} Q(\lambda : s_\beta, s_\alpha+l(J))(\lambda-T)^{-1}x \\ \qquad\qquad if \quad \lambda \in \partial D_0(s_\beta, s_\alpha+l(J)) - \{g(s_\beta), g(s_\alpha+l(J))\} \\ 0 \qquad\qquad if \quad \lambda = g(s_\beta) \ or \ \lambda = g(s_\alpha+l(J)) . \end{cases}$$

If $b(z)$ is any numerical-valued function, analytic in $|z|<1$ and continuous on $|z| \leqq 1$ and if τ is the conformal mapping from $D_0(s_\beta, s_\alpha+l(J))$ to the unit disc, then the contour integral

$$y = \int_{C_0} b(z) f(\tau^{-1}(z))\, dz \tag{5}$$

belongs to the space $\mathfrak{S}(s_\alpha, s_\beta)$ of (3), *where C_0 is the boundary of the unit disc. Moreover, unless x belongs to the space $\tilde{\mathfrak{S}}(s_\alpha, s_\beta)$ of* (4), *there exists a function $b(z)$ analytic in $|z|<1$ and continuous on $|z| \leqq 1$ such that the vector y defined by* (5) *is non-zero.*

Before proving Lemma 3.2, we notice that it implies our theorem. Indeed suppose that T were an operator satisfying the hypotheses of Lemma 3.2. By Lemma 3.1, we have only to prove that $\mathfrak{S}(s_\alpha, s_\beta)$ and $\tilde{\mathfrak{S}}(s_\alpha, s_\beta)$ are non-trivial. We may assume $\sigma(T)$ lies on both arcs $(g(s_\alpha), g(s_\beta))$ and $(g(s_\beta), g(s_\alpha+l(J)))$, because we can choose the pair of points $\lambda_\alpha = g(s_\alpha)$ and $\lambda_\beta = g(s_\beta)$ arbitrarily on J. This implies $\mathfrak{S}(s_\alpha, s_\beta) \neq \mathfrak{H}$ and $\tilde{\mathfrak{S}}(s_\alpha, s_\beta) \neq \mathfrak{H}$. Thus we have only to prove that $\mathfrak{S}(s_\alpha, s_\beta) \neq \{0\}$ and $\tilde{\mathfrak{S}}(s_\alpha, s_\beta) \neq \{0\}$. By Lemma 3.2, $\mathfrak{S}(s_\alpha, s_\beta) \neq \mathfrak{H}$, $\tilde{\mathfrak{S}}(s_\alpha, s_\beta) \neq \mathfrak{H}$ imply $\tilde{\mathfrak{S}}(s_\alpha, s_\beta) \neq \{0\}$, $\mathfrak{S}(s_\alpha, s_\beta) \neq \{0\}$ respectively.

PROOF OF LEMMA 3.2. Clearly, the function $f(\tau^{-1}(z))$ is continuous on C_0. Using the resolvent equation, we have

$$(\mu-T)^{-1} f(\lambda) = (\mu-\lambda)^{-1} f(\lambda) - (\mu-\lambda)^{-1}\, Q(\lambda : s_\beta, s_\alpha+l(J))(\mu-T)^{-1}x$$

for $\mu \in \rho(T) \cap \operatorname{Ext} D_0(s_\beta, s_\alpha+l(J))$, where $\operatorname{Ext} D_0(\cdot)$ denotes the exterior of the domain $D_0(\cdot)$, thus

$$(\mu-T)^{-1}y = \int_{C_0} \frac{b(z) f(\tau^{-1}(z))}{\mu-\tau^{-1}(z)}\, dz - \int_{C_0} \frac{b(z)\, Q(\lambda : s_\beta, s_\alpha+l(J))(\mu-T)^{-1}x}{\mu-\tau^{-1}(z)}\, dz$$

for the vector y of (5). By Cauchy's theorem, the second term is zero. Of course, we have used the inequality (2) to guarantee the convergence of our integrals. Therefore, we have

$$(6) \qquad (\mu - T)^{-1} y = \int_{C_0} \frac{b(z) f(\tau^{-1}(z))}{\mu - \tau^{-1}(z)} dz$$

for the vector y of (5). Since the equality (6) is plainly analytic in the exterior of $D_0(s_\beta, s_\alpha + l(J))$, it follows that $y \in \mathfrak{S}(s_\alpha, s_\beta)$. Next, we suppose that the vector y defined by (5) is zero for each $b(z)$ which is analytic in $|z| < 1$ and continuous on $|z| \leqq 1$, i.e., for all such $b(z)$

$$\int_{C_0} b(z) f(\tau^{-1}(z))\, dz = 0 .$$

Hence the vector-valued function $f(\tau^{-1}(z))$ defined on C_0 must be the boundary value of a vector-valued function analytic in $|z| < 1$ and continuous on $|z| \leqq 1$. Therefore $f(\lambda)$ must be analytically continuable in the interior of $D_0(s_\beta, s_\alpha + l(J))$. Thus $(\lambda - T)^{-1} x$ must be continuable onto the arc $(g(s_\beta), g(s_\alpha + l(J)))$. Thus, $x \in \widetilde{\mathfrak{S}}(s_\alpha, s_\beta)$. The proof of Lemma 3.2 is now completed.

As a immediate consequence of the above theorem and Theorem 9 of [4], we have the following corollary.

COROLLARY 3.3. *If T is the sum of a hyponormal operator A, whose spectrum lies on a Jordan curve J, which consists of a finite number of a rectifiable smooth arcs, and a compact operator B, which belongs to one of the classes C_p, for some finite $p \geqq 1$, then the Hilbert space admits a proper closed subspace which is invariant under T.*

In Mathematical Reviews (Vol. 26 (1963), #1759), L. de Branges states that the method of [3] may be applicable in other cases, for instance when $T^*T - I$ is compact. In this direction, we have the following corollary via the polar decomposition.

COROLLARY 3.4. *If $T^*T - I$ is an operator of the class C_p, for some finite $p \geqq 1$, then the Hilbert space admits a proper closed subspace which is invariant under T.*

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