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ON THE CENTRAL LIMIT THEOREM FOR LACUNARY TRIGONOMETRIC SERIES

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1. Introduction. Lacunary trigonometric series are series in which the terms that differ from zero are *very sparse*. Such series may be written in the form

$$\sum_{k=1}^{\infty} a_k \cos 2\pi(n_k x + \phi_k) \quad (a_k > 0,\ n_{k+1} > n_k \geqq 1). \tag{1.1}$$

Then we write

$$S_m(x) = \sum_{k=1}^{m} a_k \cos 2\pi(n_k x + \phi_k), \quad A_m = \left(\frac{1}{2}\sum_{k=1}^{m} a_k^2\right)^{1/2}. \tag{1.2}$$

It is well known that the behaviour of $S_m(x)$, as $m \to +\infty$, in many respect resembles that of series of independent random variables (cf. [2]). In fact, we proved the following

THEOREM [5]. *Suppose $\{n_k\}$ and $\{a_k\}$ satisfy the conditions*

$$n_{k+1}/n_k > 1 + ck^{-\alpha} \quad (c > 0 \text{ and } 0 \leqq \alpha \leqq 1/2), \tag{1.3}$$

$$A_m \to +\infty \text{ and } a_m = o(A_m m^{-\alpha}), \quad \text{as} \quad m \to +\infty, \tag{1.4}$$

then we have, for any set $E \subset [0, 1]$ of positive measure and any real number ω,

$$\lim_{m\to\infty} |\{x;\ x \in E,\ S_m(x) \leqq A_m\omega\}|/|E| = (2\pi)^{-1/2}\int_{-\infty}^{\omega} e^{-u^2/2}\,du \text{ [1]}. \tag{1.5}$$

This theorem was first proved by R. Salem and A. Zygmund for $\alpha = 0$ and in this case they also remarked that (1.4) is necessary. From this

1) For any measurable set E, $|E|$ denotes its Lebesgue measure.

theorem we can easily obtain, for any real number ω,

$$\lim_{m\to\infty}\left|\left\{x\,;\ x\in[0,\ 1),\ \sum_{k=1}^{m}\{\cos 2\pi(2^k-1)x+\cos 2\pi(2^k+1)x\}\leqq \omega m^{1/2}\right\}\right|$$

$$=(2\pi)^{-1/2}\int_0^1 dx\int_{-\infty}^{\omega/|\sqrt{2}\cos 2\pi x|}e^{-u^2/2}\,du\,.$$

Further, in [1] P. Erdös has pointed out that to every positive constant c there exists a sequence $\{n_k\}$ for which $n_{k+1}/n_k>1+ck^{-1/2}$ and (1.5) is not true for $a_k=1$ and $E=[0,\ 1)$ (cf. Theorem 3 in §3). The above facts show that for (1.5) the *lacunarity* condition of $\{n_k\}$ and the *magnitude* of a_k/A_k are closely related each other.

The purpose of the present note is to weaken the *lacunarity* condition (1.3). To state our theorem we need the following definitions and notations:

$$I(m)=\{k\,;\ 2^m<n_k\leqq 2^{m+1}\}\ ,\quad p(m)=\max\{k\,;\ n_k\leqq 2^m\}\ ,$$

$$q(m)=\begin{cases}\min\{g\,;\ g\text{ is a non-negative integer such that}\\ n_{k+1}/n_k\geqq 1+2^{-g}\text{ for all }k\in I(m)\},\ \text{if }I(m)\neq\emptyset\ ,\\ 0, \qquad\qquad\qquad\qquad\qquad\qquad \text{if }I(m)=\emptyset\ .\end{cases}$$

Therefore, if $I(m)\neq\emptyset$, $\{p(m+1)-p(m)\}$ denotes the number of elements of $I(m)$ and if $I(m)=\emptyset$, then $p(m+1)=p(m)$.

THEOREM 1. *If* $\{n_k\}$ *and* $\{a_k\}$ *satisfy the conditions*

$$(1.6)\qquad A_m\to+\infty\quad\textit{and}\quad \sum_{k=m-q(m)}^{m}\Big(\sum_{s\in I(k)}a_s\Big)^2=o(A^2_{p(m+1)})\,,\quad\textit{as}\quad m\to+\infty\,,$$

then (1.5) *holds for any set* $E\subset[0,\ 1]$ *of positive measure and any real number* ω.

For the proof of the theorem we use the following lemma which is a version of the central limit theorem for trigonometric series not necessarily lacunary.

LEMMA [4]. *Let* $T_m(x)=\sum_{k=1}^{m}c_k\cos 2\pi(kx+\alpha_k)$ *and* $C_m=\left(\frac{1}{2}\sum_{k=1}^{m}c_k^2\right)^{1/2}$, *then we put* $R_k(x)=T_{2^{k+1}}(x)-T_{2^k}(x)$ *and* $D_m=C_{2^{m+1}}$.

Suppose

$$C_m \to +\infty, \quad \sup_x |R_m(x)| = o(D_m),$$

and

$$\int_0^1 \left| D_m^{-2} \sum_{k=1}^m \{R_k^2(x) + 2R_k(x) R_{k+1}(x)\} - 1 \right| dx \to 0, \quad as \quad m \to +\infty,$$

then we have, for any set $E \subset [0, 1]$ *of positive measure and any real number* ω,

$$\lim_{m\to\infty} |\{x;\ x \in E,\ T_m(x) \leqq C_m\omega\}| / |E| = (2\pi)^{-1/2} \int_{-\infty}^{\omega} e^{-u^2/2}\, du.$$

2. Proof of Theorem 1. Let us put, for $m = 1, 2, \cdots$,

$$\Delta_m(x) = \sum_{k \in I(m)} a_k \cos 2\pi(n_k x + \phi_k),\ B_m^2 = A_{p(m+1)}^2 = \sum_{k=1}^m \|\Delta_k\|_2^2.$$

Then we have, by (1. 6),

$$\sup_x |\Delta_m(x)| = o(B_m), \tag{2.1}$$

and

$$\begin{aligned} &\sum_{k=1}^m \sum_{j=k-q(k)-1}^k \int_0^1 \Delta_k^2(x) \Delta_j^2(x)\, dx \\ &\leqq \sum_{k=1}^m \left\{ \sum_{j=k-q(k)}^k \left(\sum_{s \in I(j)} a_s \right)^2 + \max_{r \leqq k} \sup_x |\Delta_r(x)|^2 \right\} \int_0^1 \Delta_k^2(x)\, dx \\ &= o\left(\sum_{k=1}^m B_k^2 \|\Delta_k\|_2^2 \right) = o(B_m^4), \quad \text{as} \quad m \to +\infty. \end{aligned} \tag{2.2}$$

On the other hand from the definitions of $q(k)$ and $I(k)$, we obtain, for $s \in I(k)$, $r,\ t \in I(j)$ and $j \leqq k - q(k) - 2$,

$$n_{s+1} - n_s \geqq n_s 2^{-q(k)} \geqq 2^{k-q(k)} \geqq 2^{j+2} > n_r + n_t.$$

Thus if $j \leqq k - q(k) - 2$, then $(\Delta_k^2(x) - \|\Delta_k\|_2^2)$ and $(\Delta_j^2(x) - \|\Delta_j\|_2^2)$ are orthogonal. Hence we have, by (2. 2),

$$\int_0^1 \left| \sum_{k=1}^m \Delta_k^2 - B_m^2 \right|^2 dx = \int_0^1 \left| \sum_{k=1}^m (\Delta_k^2 - \|\Delta_k\|_2^2) \right|^2 dx \tag{2.3}$$

$$\leqq 2\sum_{k=1}^{m}\sum_{j=k-q(k)-1}^{k}\int_0^1 \Delta_k^2\Delta_j^2\,dx + 2\sum_{k=2}^{m}\sum_{j=1}^{k-q(k)-2}\int_0^1 (\Delta_k^2 - \|\Delta_k\|_2^2)(\Delta_j^2 - \|\Delta_j\|_2^2)\,dx$$

$$= o(B_m^4)\,, \quad \text{as} \quad m \to +\infty\,.$$

We have

$$\left|\sum_{k=1}^{m}\Delta_k\Delta_{k+1}\right|^2 = \sum_{k=1}^{m}\Delta_k^2\Delta_{k+1}^2 + 2\sum_{k=2}^{m}\sum_{j=1}^{k-1}\Delta_k\Delta_{k+1}\Delta_j\Delta_{j+1}$$

$$\leqq \sum_{k=1}^{m}\Delta_k^2(\Delta_{k+1}^2 + |2\Delta_{k-1}\Delta_{k+1}|) + 2\sum_{k=2}^{m}\sum_{j=1}^{k-q(k)-3}\Delta_k\Delta_{k+1}\Delta_j\Delta_{j+1}$$

$$+ \sum_{k=2}^{m}\sum_{j=k-q(k)-2}^{k-2}(\Delta_k^2 + \Delta_{k+1}^2)(\Delta_j^2 + \Delta_{j+1}^2)\,.$$

Then by (2.1) and (2.2) the integrals of the first and the last terms of the above formulas are $o(B_m^4)$, as $m \to +\infty$ [2]. Further, if $j \leqq k-q(k)-3$, then by the definition of $q(k)$, $\Delta_k(x)\Delta_{k+1}(x)$ and $\Delta_j(x)\Delta_{j+1}(x)$ are orthogonal. Hence, we have

$$(2.4) \qquad \int_0^1 \left(\sum_{k=1}^{m}\Delta_k\Delta_{k+1}\right)^2 dx = o(B_m^4)\,, \quad \text{as} \quad m \to +\infty.$$

From (2.3) and (2.4), we obtain

$$(2.5) \qquad \int_0^1 \left| B_m^{-2}\sum_{k=1}^{m}(\Delta_k^2 + 2\Delta_k\Delta_{k+1}) - 1\right|^2 dx = o(1)\,, \quad \text{as} \quad m \to +\infty\,.$$

Considering (2.1) and (2.5), we can apply the lemma in §1 to the series (1.1) and this completes the proof.

3. Applications of Theorem 1. In this paragraph we apply Theorem 1 to some typical cases. We use the same notation as before.

THEOREM 2. *If* $\{n_k\}$ *and* $\{a_k\}$ *satisfy the conditions*

$$(3.1) \qquad n_{k+1}/n_k \geqq 1 + c\log k/k \quad (c\log 2 \geqq 1)\,,$$

$$(3.2) \qquad A_m \to +\infty \quad \textit{and} \quad \left(\sum_{k\in I(m)} a_k\right)^2 \log p(m) = o(B_m^2)\,, \quad \textit{as} \quad m \to +\infty,$$

2) It is easily seen that $B_m^2/B_{m+1}^2 \to 1$, as $m \to +\infty$.

then (1.5) *holds for any set* $E\subset[0, 1]$ *of positive measure and any real number* ω.

PROOF. From the definition of $p(m)$ if $p(m+1)>p(m)+1$, then

$$2>n_{p(m+1)}/n_{p(m)+1}\geqq\prod_{k=p(m)+1}^{p(m+1)-1}(1+c\log k/k)$$

$$\geqq 1+\{p(m+1)-p(m)-1\}c\log p(m+1)/p(m+1),$$

and we obtain

$$\{p(m+1)-p(m)\}\leqq 1+\frac{p(m+1)}{c\log p(m+1)}=o(p(m)),\quad\text{as}\quad m\to+\infty. \tag{3.3}$$

Further in this case it is seen that

$$\begin{aligned} q(m) &\leqq |\log\{c\log p(m+1)/p(m+1)\}/\log 2|+1 \\ &< \frac{\log p(m)}{\log 2}-\log\log p(m),\quad\text{for}\quad m>m_0. \end{aligned} \tag{3.4}$$

Hence we have, by (3.3) and (3.4),

$$\begin{aligned} p(m)-p(m-q(m)) &= \sum_{k=m-q(m)}^{m-1}\{p(k+1)-p(k)\} \\ &< q(m)\{1+p(m)/c\log p(m)\} \\ &< p(m)+\frac{\log p(m)}{\log 2}-\frac{p(m)\log\log p(m)}{c\log p(m)}-\log\log p(m) \\ &< p(m)-\{p(m)/\log p(m)\},\quad\text{for}\quad m>m_0. \end{aligned}$$

The above relation implies that there exists a positive constant c_0 such that $m>m_0$ implies that

$$\log p(m-q(m))>c_0\log p(m).$$

Therefore we have, for any j such that $m\geqq j\geqq m-q(m)$,

$$\left(\sum_{m\in I(j)}a_m\right)^2\log p(m)=o(B_m^2),\quad\text{as}\quad m\to+\infty. \tag{3.5}$$

(3.4) and (3.5) imply (1.6) and we can complete the proof.

The following theorem shows that the validity of (1. 5) depends on the *numbers* and the *gaps* of $\{n_k\}$ in $(2^m, 2^{m+1}]$ for $a_k = 1$, $k \geqq 1$.

THEOREM 3. *Let* $\{n_k\}$ *and* $\{a_k\}$ *satisfy the conditions*:

$$\{p(m+1) - p(m)\}^2/p(m) = o(1/\log p(m)), \text{ as } m \to +\infty \tag{3.6}$$

$$n_{k+1}/n_k > 1 + k^{-s} \quad (s \geqq 0), \tag{3.7}$$

$$A_m \to +\infty \text{ and } a_m = O(A_m m^{-1/2}), \quad \text{as} \quad m \to +\infty\,^{3)}. \tag{3.8}$$

Then (1. 5) *holds for any set* $E \subset [0, 1]$ *of positive measure and any real number* ω.

PROOF. From (3. 7) and (3. 6) it is seen that

$$q(m) = O(\log p(m+1)) = O(\log p(m)), \quad \text{as} \quad m \to +\infty.$$

Since $(x/\log x)^{1/2}$ is an increasing function, we have by (3. 6),

$$p(m-q(m)) = p(m) - \sum_{k=m-q(m)}^{m-1} \{p(k+1) - p(k)\}$$

$$> p(m) - q(m)\{p(m)/\log p(m)\}^{1/2} > p(m)/2, \quad \text{for} \quad m > m_0.$$

Therefore we have, by (3. 8) and the above relations,

$$\sum_{k=m-q(m)}^{m} \left(\sum_{j \in I(k)} a_j\right)^2 = O\left(\sum_{k=m-q(m)}^{m} B_k^2 \Big(\sum_{j \in I(k)} j^{-1/2}\Big)^2\right)$$

$$= O\left(B_m^2 \sum_{k=m-q(m)}^{m} \{p(k+1) - p(k)\}^2/p(k)\right)$$

$$= o\left(B_m^2 \sum_{k=m-q(m)}^{m} 1/\log p(k)\right) = o(B_m^2), \quad \text{as} \quad m \to +\infty.$$

Hence by Theorem 1 we can complete the proof.

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