

INEQUALITIES FOR POLYNOMIALS AND THEIR DERIVATIVES II

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Introduction. Many theorems of the theory of approximation depend upon the fact that a polynomial of degree n cannot change too rapidly; in other words its derivative cannot be too large. We know from A. A. Markov's inequality for polynomials that if $|P_n(x)| \leq M$ on $[-1, 1]$ then $|P_n'(x)| \leq Mn^2$ for $-1 \leq x \leq 1$ where $P_n(x)$ is an algebraic polynomial of degree n . But this is an extreme case. For special polynomials, one can sometimes improve the estimates of the derivative. For this we refer to papers of P. Erdős [4], G. Lorentz [8], and P. Turán [11].

1.1. In 1958 J. Balázs and P. Turán [3] obtained certain interesting inequalities which arise from their consideration of $(0, 2)$ interpolation on π -abscissas ($\pi_n(x) = (1-x^2)P_{n-1}(x)$, $P_n(x)$ being the Legendre polynomial of degree $= n$). By $(0, 2)$ interpolation they mean the problem of finding interpolatory polynomials $R_n(x)$ of degree $\leq 2n-1$ for which

$$(1.1.1) \quad R_n(x_k) = \alpha_k, \quad R_n''(x_k) = \beta_k, \quad k = 1, 2, \dots, n,$$

are prescribed. From this consideration they proved the following:

THEOREM 1.1.1 (J. Balázs and P. Turán). *Let n be even and further if we are given for a polynomial $Q_{2n-1}(x)$ of degree $\leq 2n-1$*

$$(1.1.2) \quad |Q_{2n-1}(x_k)| \leq A, \quad |Q_{2n-1}''(x_k)| \leq B, \quad k = 1, 2, \dots, n,$$

then for $-1 \leq x \leq 1$ we have

$$(1.1.3) \quad |Q_{2n-1}(x)| \leq \pi^6 n A + \frac{\pi^5 B}{n},$$

and

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$$(1.1.4) \quad |Q'_{2n-1}(x)| \leq \pi^8 n^{5/2} A + \pi^5 B n^{1/2}.$$

The appearance of the exponent $5/2$ in (1.1.4) is very significant. (1.1.3) gives the estimate of $Q_{2n-1}(x)$ uniformly in $[-1, 1]$, whereas (1.1.4) gives the estimate of $Q'_{2n-1}(x)$ uniformly in $[-1, 1]$ which is far better than what one can obtain from Markov's inequality applied on (1.1.3). Moreover (1.1.3) and (1.1.4) are best possible in a certain sense explained there in. We mention for the sake of orientation as remarked by P. Turán and J. Balázs [4] that if in $[-1, 1]$ $f'(x)$ exists everywhere and $|f(x)| \leq 1$, $|f''(x)| \leq 1$ then we have $|f'(x)| \leq 2$ for $[-1, 1]$. This has been proved after the first results of C. N. Moore and Hardy-Littlewood by E. Landau [7]. It would be of interest to point out that L. Fejér [6] in 1930 obtained the following result. Let $g(x)$ be an arbitrary polynomial of degree $\leq 2n-1$ satisfying

$$(1.1.5) \quad |g(x_i)| \leq A, \quad |g'(x_i)| \leq B, \quad i = 1, 2, \dots, n,$$

then uniformly in $[-1, 1]$ we have

$$(1.1.6) \quad |g(x)| \leq A + B.$$

Here x_i 's are the zeros of $T_n(x) = \cos n\theta$, $\cos \theta = x$ and $x_i = \cos \theta_i$, $\theta_i = \frac{(2i-1)\pi}{2n}$.

1.2. The object of this paper is to obtain analogous results when x_k 's are taken to be the zeros of $(1-x^2)u_n(x)$, $u_n(x)$ being the Tchebycheff polynomials of the second kind. We prove the following:

THEOREM 1.2.1. *Let $Q_{2n+1}(x)$ be an arbitrary polynomial of degree $\leq 2n+1$ (n even) satisfying*

$$(1.2.1) \quad |Q_{2n+1}(x_k)| \leq A, \quad k = 1, 2, \dots, n+2$$

and

$$(1.2.2) \quad |Q'_{2n+1}(x_k)| \leq \frac{B}{1-x_k^2}, \quad k = 2, 3, \dots, n+1.$$

Then for $-1 \leq x \leq 1$ we have

$$(1.2.3) \quad |Q_{2n+1}(x)| \leq c_1(A n^{3/2} + B n^{-1/2}), \quad (c_1 = 57)$$

and

$$(1.2.4) \quad |Q'_{2n+1}(x)| \leq c_2(A n^3 + B n), \quad (c_2 = 108).$$

First we remark that the result (1.2.3) and (1.2.4) are essentially best possible, i.e., we can find suitable polynomials $f_0(x)$ and $f_1(x)$ of degree $\leq 2n+1$ which satisfies (1.2.1) and (1.2.2) and for a numerical positive c_3 and c_4 , we have

$$(1.2.5) \quad |f_0(0)| > c_3(A n^{3/2} + B n^{-1/2}),$$

and

$$(1.2.6) \quad |f_1'(1)| > c_4(A n^3 + B n).$$

Here x_k 's mentioned in (1.2.1) and (1.2.2) are the zeros of $(1-x^2)u_n(x)$ given by

$$1 = x_1 > x_2 > \dots > x_{n+1} > x_{n+2} = -1.$$

Thus comparing the results on these two abscissas we find that (1.2.3) and (1.2.4) are not as good as (1.1.3) and (1.1.4) although (1.2.3) and (1.2.4) are proved to be best possible as explained above. But there is a common feature between these two theorems, namely the growth of their derivatives are of the order $n^{3/2}$. If we apply Markov's theorem to (1.2.3) we will get

$$(1.2.7) \quad |Q'_{2n+1}(x)| \leq c_1(A n^{7/2} + B n^{3/2}) \text{ in } -1 \leq x \leq 1.$$

From this we may conclude that the result stated in (1.2.4) is far better from (1.2.7) what we can obtain from Markov's theorem. For other interesting inequalities of the author when nodes of interpolation are taken as the zeros of $(1-x^2)T_n(x)$, $T_n(x)$ being the Tchebycheff polynomials of the first kind we refer to [12].

1.3. In an earlier work [14] we proved that for n even there exists a unique polynomial $R_n(x)$ of degree $\leq 2n+1$ for which

$$(1.3.1) \quad R_n(x_i) = a_i, \quad i = 1, 2, \dots, n+2,$$

$$(1.3.2) \quad R_n''(x_i) = b_i, \quad i = 2, 3, \dots, n+1,$$

are prescribed in advance. Here as mentioned earlier

$$(1.3.3) \quad 1 = x_1 > x_2 > \dots > x_{n+1} > x_{n+2} = -1$$

are the zeros of the polynomial $(1-x^2)u_n(x)$. We denote

$$(1.3.4) \quad R_n(x) = \sum_{k=1}^{n+2} a_k r_k(x) + \sum_{k=2}^{n+1} b_k \rho_k(x) \quad (n \text{ even})$$

where the fundamental polynomials $r_k(x)$ and $\rho_k(x)$ are stated in Theorem 2.1.1.

We now consider the sequence of points

$$(1.3.5) \quad 1 = x_{1,n} > x_{2,n} > \dots > x_{n+1,n} > x_{n+2,n} = -1$$

where $x_{k,n}$'s stand for the zeros of $(1-x^2)u_n(x)$. (The notation (1.3.3) of the zeros $(1-x^2)u_n(x)$ was more suitable when n was fixed). Then forming the interpolatory polynomials (1.3.4) satisfying (1.3.1) and (1.3.2) for each $n=2k$ we shall write the fundamental functions $r_k(x)$ and $\rho_k(x)$ as $r_{kn}(x)$ and $\rho_{kn}(x)$ respectively. Let now $f(x)$ be defined for $[-1, 1]$; we consider the sequence of polynomials

$$(1.3.6) \quad R_n[x, f] = \sum_{k=1}^{n+2} f(x_{k,n}) r_{kn}(x) + \sum_{k=2}^{n+1} b_{kn} \rho_{kn}(x) \quad (n \text{ even}).$$

We will also prove the convergence theorem in a similar way as obtained by J. Balázs and P. Turán [2].

THEOREM 1.3.1. *Let $f(x)$ have a continuous derivative in $[-1, 1]$ together with $f'(x) \in \text{Lip } \alpha$, $\alpha > 1/2$. Further let*

$$(1.3.7) \quad |b_{kn}| = o\left(\frac{n^{1/2}}{1-x_{k,n}^2}\right), \quad k = 2, 3, \dots, n+1,$$

then the sequence of interpolatory polynomials $R_n[x, f]$ as defined in (1.3.6) converges uniformly to $f(x)$ in $[-1, 1]$. The class $f'(x) \in \text{Lip } \alpha$, $\alpha > 1/2$ cannot be replaced by $f'(x) \in \text{Lip } 1/2$ even if all b_{kn} are zero.

2.1. The explicit forms of the fundamental polynomials were stated without proof in our earlier work [14]. Here we will state these fundamental polynomials with proof.

THEOREM 2.1.1. *The explicit forms of the fundamental polynomials are as follows.*

(a) For $k = 2, 3, \dots, n+1$ we have

$$(2.1.1) \quad \rho_k(x) = \frac{(1-x^2)u_n(x)q_{n-1}(x)}{2u'_n(x_k)}$$

where $q_{n-1}(x)$ is a polynomial in x of degree $\leq n-1$ which depends on k also and is given by

$$(2.1.2) \quad (1-x^2)^{1/4} q_{n-1}(x) = A_k \int_1^x u_n(t) w(t) dt + \int_1^x l_k(t) w(t) dt,$$

$$(2.1.3) \quad A_k \int_{-1}^1 u_n(t) w(t) dt = - \int_{-1}^1 l_k(t) w(t) dt,$$

$$(2.1.4) \quad l_k(t) = \frac{u_n(t)}{(t-x_k)u'_n(x_k)}, \quad w(t) = (1-t^2)^{-3/4}.$$

(b) For $k = 2, 3, \dots, n+1$ we have

$$(2.1.5) \quad r_k(x) = \frac{(1-x^2)l_k^2(x)}{1-x_k^2} + \frac{(1-x^2)u_n(x)s_{n-1}(x)}{2(1-x_k^2)u'_n(x_k)} + B_k \rho_k(x),$$

where $s_{n-1}(x)$ is a polynomial in x of degree $\leq n-1$ which depends on k also and is given by

$$(2.1.6) \quad (1-x^2)^{1/4} s_{n-1}(x) = A'_k \int_{-1}^x u_n(t) w(t) dt + \int_{-1}^x p_k(t) w(t) dt,$$

$$(2.1.7) \quad A'_k \int_{-1}^1 u_n(t) w(t) dt = - \int_{-1}^1 p_k(t) w(t) dt,$$

$$(2.1.8) \quad p_k(t) = (1-t^2)l_k''(t) - 3tl_k'(t) + n(n+2)l_k(t),$$

$$(2.1.9) \quad B_k = \frac{22x_k^2 - 7}{(1-x_k^2)^2}.$$

Lastly

$$(2.1.10) \quad r_1(x) = \frac{u_n(x)}{10(n+1)^2} [5(1+x)u_n(x) - 2(1-x^2)u'_n(x) - 5(1-x^2)t_{n-2}(x)]$$

$$(2.1.11) \quad r_{n+2}(x) = \frac{u_n(x)}{10(n+1)^2} [5(1-x)u_n(x) + 2(1-x^2)u'_n(x) - 5(1-x^2)t_{n-2}(x)]$$

where $t_{n-2}(x)$ is a polynomial in x of degree $\leq n-2$ and is given by

$$(2.1.12) \quad (1-x^2)^{1/4} t_{n-2}(x) = \int_1^x u'_n(t) w(t) dt.$$

2.2. In order to prove the above theorem we will require the following Lemmas.

LEMMA 2.2.1. For $x = \cos \theta$ we have

$$(2.2.1) \quad \int_{-1}^x u_{2r}(t) w(t) dt = c_r \int_{-1}^x w(t) dt + (1-x^2)^{1/4} h_{2r-1}(x), \quad r = 1, 2,$$

where

$$(2.2.2) \quad w(t) = (1-t^2)^{-3/4}, \quad c_r \int_{-1}^1 w(t) dt = \int_{-1}^1 u_{2r}(t) w(t) dt$$

and $h_{2r-1}(x)$ is a polynomial of degree $\leq 2r-1$ in x .

PROOF. This follows from the recurrence relation

$$(2.2.3) \quad \left(r - \frac{1}{4}\right) \int_{-1}^x u_{2r}(t) w(t) dt \\ = \left(r + \frac{1}{4}\right) \int_{-1}^x u_{2r-2}(t) w(t) dt + (1-x^2)^{1/4} u_{2r-1}(x), \quad r = 1, 2, \dots.$$

Here, as usual $u_{2r}(x)$ denote the Tchebycheff polynomials of the second kind.

LEMMA 2.2.2. For $x = \cos \theta$ we have

$$(2.2.4) \quad \int_{-1}^x u_{2r-1}(t) w(t) dt = (1-x^2)^{1/4} m_{2r-2}(x), \quad r = 1, 2, \dots$$

where $m_{2r-2}(x)$ is a polynomial in x of degree $\leq 2r-2$.

PROOF. This follows from the recurrence relation

$$(2.2.5) \quad \left(r - \frac{3}{4}\right) \int_{-1}^x u_{2r-1}(t) w(t) dt - \left(r - \frac{1}{4}\right) \int_{-1}^x u_{2r-3}(t) w(t) dt \\ = (1-x^2)^{1/4} u_{2r-2}(x), \quad r = 1, 2, \dots.$$

LEMMA 2.2.3. For $x = \cos \theta$, $k = 2, 3, \dots, n+1$ we have

$$(2.2.6) \quad \int_{-1}^x l_k(t) w(t) dt = d_k \int_{-1}^x w(t) dt + (1-x^2)^{1/4} e_{n-1}(x)$$

where

$$(2.2.7) \quad l_k(t) = \frac{u_n(t)}{(t-x_k)u'_n(x_k)}, \quad d_k \int_{-1}^1 w(t) dt = \int_{-1}^1 l_k(t) w(t) dt,$$

and $e_{n-1}(x)$ is a polynomial in x of degree $\leq n-1$.

PROOF. This is an easy consequence of formula (2.2.1), (2.2.3) and a well known result of L. Fejér [9]

$$(2.2.8) \quad l_k(t) = \frac{2(1-x_k^2)}{n+1} \sum_{r=0}^n u_r(x_k) u_r(t), \quad k = 2, 3, \dots, n+1.$$

For r even we use Lemma 2.2.1 and for r odd we use Lemma 2.2.2.

2.3. Here we will give the proof for the determination of the fundamental polynomials of the second kind. In view of uniqueness theorem stated on page 345 as theorem 4 in [14], it remains to verify that the fundamental functions are polynomials of degree $\leq 2n+1$ and satisfy the following conditions:

$$(2.3.1) \quad \rho_k(x_j) = 0, \quad j = 1, 2, \dots, n+2, \quad k = 2, 3, \dots, n+1$$

and

$$(2.3.2) \quad \rho'_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \quad j, k = 2, 3, \dots, n+1.$$

Consider the function

$$(2.3.3) \quad \chi_k(x) = \frac{(1-x^2)u_n(x)q_{n-1}(x)}{2u'_n(x_k)} \quad k = 2, 3, \dots, n+1$$

where for $k = 2, 3, \dots, n+1$

$$(2.3.4) \quad (1-x^2)^{1/4} q_{n-1}(x) = A_k \int_{-1}^x u_n(t) w(t) dt + \int_{-1}^x l_k(t) w(t) dt,$$

and

$$(2.3.5) \quad A_k \int_{-1}^1 u_n(t) w(t) dt + \int_{-1}^1 l_k(t) w(t) dt = 0, \quad k = 2, 3, \dots, n+1.$$

We observe from Lemma 2.2.1, Lemma 2.2.3 and formula (2.3.5) that $q_{n-1}(x)$ defined by (2.4.4) is indeed a polynomial in x of degree $\leq n-1$. It is obvious that

$$(2.3.6) \quad \mathcal{X}_k(x_j) = 0, \quad j = 1, 2, \dots, n+2, \quad k = 2, 3, \dots, n+1.$$

On differentiating once with respect to x equation (2.3.4) and putting $x=x_j$, $j=2, 3, \dots, n+1$ we have

$$(2.3.7) \quad 2(1-x_j^2) q'_{n-1}(x_j) - x_j q_{n-1}(x_j) = \begin{cases} 0 & j \neq k \\ 2 & j = k \end{cases}, \quad j, k = 2, 3, \dots, n+1.$$

A simple computation gives us

$$(2.3.8) \quad [(1-x^2)u_n(x)]'_{x=x_j} = -x_j u'_n(x_j), \quad j=2, 3, \dots, n+1$$

and

$$(2.3.9) \quad [(1-x^2)u_n(x)]'_{x=x_j} = (1-x_j^2) u'_n(x_j), \quad j = 2, 3, \dots, n+1.$$

Here dash indicates differentiation with respect to x . From (2.3.3), (2.3.7), (2.3.8) and (2.3.9) we get

$$(2.3.10) \quad \mathcal{X}_k''(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}, \quad j, k = 2, 3, \dots, n+1.$$

Thus we have shown that $\mathcal{X}_k(x)$ is a polynomial in x of degree $\leq 2n+1$ and satisfies conditions (2.3.6) and (2.3.10). Therefore we conclude in view of the remark stated earlier

$$\rho_k(x) \equiv \mathcal{X}_k(x).$$

2.4. Here we will give the determination of $r_k(x)$. Again, in view of the uniqueness theorem [14] it remains to verify that the fundamental functions $r_k(x)$ are polynomials of degree $\leq 2n+1$ and satisfy the following conditions, i.e.,

$$(2.4.1) \quad r_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}, \quad j, k = 1, 2, \dots, n+2,$$

$$(2.4.2) \quad r'_k(x_j) = 0 \quad j, k = 2, 3, \dots, n+1.$$

We need the following lemmas.

LEMMA 2.4.1. *Let $x = \cos \theta$, $k = 2, 3, \dots, n+1$, we have*

$$(2.4.3) \quad \int_{-1}^x p_k(t)w(t)dt = f_k \int_{-1}^x w(t)dt + (1-x^2)^{1/4} d_{n-2}(x)$$

where $d_{n-2}(x)$ is a polynomial in x of degree $\leq n-2$. Here

$$(2.4.4) \quad f_k \int_{-1}^1 w(t)dt = \int_{-1}^1 p_k(t)w(t)dt$$

and $p_k(t)$ is a polynomial in t of degree $\leq n-1$, given by

$$(2.4.5) \quad \begin{aligned} p_k(t) &= (1-t^2)l'_k(t) - 3tl'_k(t) + n(n+2)l_k(t) \\ &= \frac{2(1-x_k^2)}{n+1} \sum_{r=0}^{n-1} (n(n+2) - r(r+2)) u_r(x_k)u_r(t). \end{aligned}$$

PROOF. We use the above representation of $p_k(t)$ (which can be easily seen on using (2.2.8) and the differential equation of $u_r(t)$), Lemmas 2.2.1 and 2.2.2.

LEMMA 2.4.2. *Let $x = \cos \theta$, then $t_{n-2}(x)$ defined by the relation*

$$(2.4.6) \quad (1-x^2)^{1/4} t_{n-2}(x) = \int_{-1}^x u'_n(t)w(t)dt \quad (n \text{ even})$$

is a polynomial in x of degree $\leq n-2$.

PROOF. For n even we have

$$(2.4.7) \quad u_n(x) = 1 + 2 \sum_{r=1}^{n/2} T_{2r}(x), \quad T_{2r}(x) = \cos 2r\theta, \quad \cos \theta = x.$$

Differentiating (2.4.7) with respect to x and using Lemma 2.2.2, we get the above result.

LEMMA 2.4.3. *Let $x = \cos \theta$, $k=2, 3, \dots, n+1$ then $s_{n-1}(x)$ defined by*

$$(2.4.8) \quad (1-x^2)^{1/4} s_{n-1}(x) = A'_k \int_{-1}^x u_n(t) w(t) dt + \int_{-1}^x p_k(t) w(t) dt$$

with

$$(2.4.9) \quad A'_k \int_{-1}^1 u_n(t) w(t) dt + \int_{-1}^1 p_k(t) w(t) dt = 0$$

is a polynomial in x of degree $\leq n-1$.

PROOF. The proof of this Lemma follows at once if we use Lemmas 2.2.1, 2.2.2 and (2.4.9). Here we equated to zero the coefficient of $\int_{-1}^x w(t) dt$ obtained from the right-hand side of (2.4.8) which gives us the value of A'_k .

Now let us consider the function

$$(2.4.10) \quad M_k(x) = \frac{(1-x^2)l_k^2(x)}{1-x_k^2} + \frac{(1-x^2)u_n(x)s_{n-1}(x)}{2(1-x_k^2)u'_n(x_k)} + B_k \rho_k(x)$$

where $s_{n-1}(x)$ is defined in (2.4.8) and B_k is given by

$$(2.4.11) \quad B_k = \frac{22x_k^2 - 7}{(1-x_k^2)^2}.$$

In view of Lemma 2.4.3 and the fact that $\rho_k(x)$ is indeed a polynomial in x of degree $\leq 2n+1$, we have $M_k(x)$ also a polynomial in x of degree $\leq 2n+1$. It is also clear that

$$(2.4.12) \quad M_k(x_j) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \quad \begin{matrix} j = 1, 2, \dots, n+2 \\ k = 2, 3, \dots, n+1. \end{matrix}$$

A simple computation shows that

$$(2.4.13) \quad \left[\frac{(1-x^2)l_k^2(x)}{1-x_k^2} \right]''_{x=x_j} = 2 \frac{(1-x_j^2)}{(1-x_k^2)} [l'_k(x_j)]^2, \quad j \neq k.$$

From (2.4.5) we have

$$(2.4.14) \quad p_k(x_j) = (1-x_j^2)l'_k(x_j) - 3x_j l'_k(x_j) = \frac{-2(1-x_j^2)l'_k(x_j)}{(x_j-x_k)}, \quad j \neq k.$$

It can be easily verified that

$$(2.4.15) \quad [(1-x^2)^{3/4} u_n(x)]''_{x=x_j} = 0, \quad j = 2, 3, \dots, n+1.$$

On using (2.4.8), (2.4.15) and (2.4.14) we have

$$(2.4.16) \quad \begin{aligned} [(1-x^2)u_n(x)s_{n-1}(x)]''_{x=x_j} &= 2u'_n(x_j)p_k(x_j), \quad j \neq k \\ &= \frac{-4(1-x_j^2)l'_k(x_j)u'_k(x_j)}{(x_j-x_k)}, \quad j \neq k. \end{aligned}$$

Therefore, from (2.4.10), (2.4.13) and (2.4.16) we get

$$(2.4.17) \quad M''_k(x_j) = 0, \quad j \neq k \quad \text{since} \quad l'_k(x_j) = \frac{u'_n(x_j)}{(x_j-x_k)u'_n(x_k)}, \quad j \neq k.$$

Further B_k is determined by the fact that

$$(2.4.18) \quad M''_k(x_k) = 0, \quad k = 2, 3, \dots, n+1.$$

Thus $M_k(x)$ is a polynomial in x of degree $\leq 2n+1$ satisfying (2.4.12), (2.4.17) and (2.4.18). Therefore, by the uniqueness theorem

$$M_k(x) \equiv r_k(x) \quad k = 2, 3, \dots, n+1.$$

Proofs for $r_1(x)$, $r_{n+2}(x)$ are similar. Here one uses also Lemma 2.4.2.

3.1. Here we will mention those results concerning Tchebycheff polynomials of second kind which are required for our purpose.

$$(3.1.1) \quad |\sin(p+1)u| \leq (p+1)|\sin u|, \quad |u_p(x)| \leq (p+1), \quad -1 \leq x \leq 1,$$

$$(3.1.2) \quad \frac{2}{\pi}u \leq \sin u \leq u, \quad 0 \leq u \leq \frac{\pi}{2},$$

$$(3.1.3) \quad \frac{2}{\pi}(\pi-u) \leq \sin u \leq u, \quad -\frac{\pi}{2} \leq u \leq \pi.$$

For $0 \leq \theta \leq \pi/2$ and on using (3.1.2) we have

$$(3.1.4) \quad \int_0^\theta \frac{1}{(\sin u)^{1/2}} du \leq \pi(\sin \theta)^{1/2} = \pi(1-x^2)^{1/4}, \quad x = \cos \theta.$$

From a well known result of L. Fejér [9] we have

$$(3.1.5) \quad l_k(t) = \frac{2(1-x_k^2)}{n+1} \sum_{r=0}^n u_r(x_k) u_r(t), \quad k = 2, 3, \dots, n+1.$$

It follows from (3.1.5) on a simple computation (see [9]) that

$$(3.1.6) \quad |l_k(t)| \leq 2 \quad \text{for} \quad -1 \leq t \leq 1.$$

Following recurrence relations are easily verified.

$$(3.1.7) \quad \left(r - \frac{1}{4}\right) \int_{-1}^1 u_{2r}(t) w(t) dt = \left(r + \frac{1}{4}\right) \int_{-1}^1 u_{2r-2}(t) w(t) dt, \\ r = 1, 2, \dots,$$

$$(3.1.8) \quad \left(r + \frac{3}{4}\right) \int_{-1}^1 u_{2r}(t) (1-t^2)^{1/4} dt = \left(r - \frac{3}{4}\right) \int_{-1}^1 u_{2r-2}(t) (1-t^2)^{1/4} dt, \\ r = 1, 2, \dots.$$

3.2. Here we will prove certain lemmas which will lead us to the estimation of the fundamental polynomials.

LEMMA 3.2.1. *Following estimates are valid.*

$$(3.2.1) \quad \int_{-1}^1 u_{2r}(t) w(t) dt = 4\sqrt{\pi} \frac{\Gamma\left(r + \frac{5}{4}\right)}{\Gamma\left(r + \frac{3}{4}\right)},$$

$$(3.2.2) \quad \int_{-1}^1 u_{2r}(t) (1-t^2)^{1/4} dt = \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(r + \frac{1}{4}\right)}{\Gamma\left(r + \frac{7}{4}\right)},$$

$$(3.2.3) \quad \left| \int_1^x u_p(t) w(t) dt \right| \leq \pi(p+1)(1-x^2)^{1/4}, \quad 0 \leq x \leq 1,$$

$$(3.2.4) \quad \left| \int_1^x l_k(t) w(t) dt \right| \leq 2\pi(1-x^2)^{1/4}, \quad 0 \leq x \leq 1, \quad k = 2, 3, \dots, n+1.$$

PROOF. (3.2.1) and (3.2.2) follow from the recurrence relations (3.1.7) and (3.1.8) respectively. Using (3.1.1) and (3.1.4) we have (3.2.3). Similarly on using (3.1.4) and (3.1.6) we have (3.2.4) immediately. This proves the lemma. When $-1 \leq x \leq 0$ then on using (3.1.3) and (3.1.6) we have

$$(3.2.3a) \quad \left| \int_{-1}^x u_p(t) w(t) dt \right| \leq \pi(p+1)(1-x^2)^{1/4}, \quad -1 \leq x \leq 0$$

and

$$(3.2.4a) \quad \left| \int_{-1}^x l_k(t) w(t) dt \right| \leq 2\pi(1-x^2)^{1/4}, \quad -1 \leq x \leq 0.$$

LEMMA 3.2.2. *Following estimates are valid*

$$(3.2.5) \quad \left| \int_1^x u_p(t) w(t) dt \right| \leq 7(p+1)^{1/2}, \quad 0 \leq x \leq 1,$$

$$(3.2.6) \quad \left| \int_1^x l_k(t) w(t) dt \right| \leq \frac{8 \log n}{(n+1) \sin \theta_k} + \frac{14}{(n+1)^{1/2}},$$

for $k = 2, 3, \dots, n+1$ and $0 \leq x \leq 1$,

$$(3.2.7) \quad \left| \int_{-1}^1 l_k(t) w(t) dt \right| \leq \frac{8\sqrt{\pi}}{(n+1)} \frac{\Gamma\left(\frac{n}{2} + \frac{5}{4}\right)}{\Gamma\left(\frac{n}{2} + \frac{3}{4}\right)}, \quad k=2, 3, \dots, n+1.$$

PROOF. We observe that

$$(3.2.8) \quad \int_1^x u_p(t) w(t) dt = \int_0^\theta \frac{\sin(p+1)u}{(\sin u)^{3/2}} du, \quad x = \cos \theta, \quad t = \cos u.$$

If $0 \leq \theta \leq \pi/(p+1)$ then using (3.1.1) and (3.1.3) we have

$$(3.2.9) \quad \int_0^\theta \frac{\sin(p+1)u}{(\sin u)^{3/2}} du \leq (p+1) \int_0^\theta \frac{1}{(\sin u)^{1/2}} du \leq \sqrt{2} \pi(p+1)^{1/2}.$$

In case $\pi/(p+1) \leq \theta \leq \pi/2$ then we have

$$(3.2.10) \quad \int_0^\theta \frac{\sin(p+1)u}{(\sin u)^{3/2}} du = \int_0^{\pi/(p+1)} + \int_{\pi/(p+1)}^\theta.$$

But

$$(3.2.11) \quad \left| \int_{\pi/(p+1)}^\theta \frac{\sin(p+1)u}{(\sin u)^{3/2}} du \right| \leq \int_{\pi/(p+1)}^{\pi/2} \frac{1}{(\sin u)^{3/2}} du \leq \left(\frac{\pi}{2}\right)^{3/2} \int_{\pi/(p+1)}^{\pi/2} u^{-3/2} du$$

$$< \pi 2^{-1/2} (p+1)^{1/2}.$$

Therefore, on using (3.2.8)–(3.2.11) we have

$$\left| \int_1^x u_p(t)w(t) dt \right| \leq 2^{1/2} \pi(p+1)^{1/2} + \pi 2^{-1/2}(p+1)^{1/2} < 7(p+1)^{1/2},$$

from which (3.2.5) follows. In a similar way we can easily prove that

$$(3.2.12) \quad \left| \int_1^x u_p(t)(1-t^2)^{1/4} dt \right| \leq \frac{4}{p+1}.$$

As a consequence of (3.1.5) and (3.2.12) we have

$$(3.2.13) \quad \left| \int_1^x l_k(t)(1-t^2)^{1/4} dt \right| \leq \frac{8 \log n}{(n+1)} (1-x_k^2)^{1/2}.$$

But we know that

$$(3.2.14) \quad \int_1^x l_k(t)w(t) dt = (1-x_k^2)^{-1} \left[\int_1^x l_k(t)(1-t^2)^{1/4} dt + \frac{1}{u'_n(x_k)} \int_1^x \frac{(t+x_k)u_n(t)}{(1-t^2)^{3/4}} dt \right].$$

On using (3.2.5) and (3.2.13) we have

$$\left| \int_1^x l_k(t)w(t) dt \right| \leq \frac{8 \log n}{(n+1) \sin \theta_k} + 14(n+1)^{-1/2}$$

which proves (3.2.6). Proof of (3.2.7) follows from (3.1.5) and (3.2.1) and on using Abel’s lemma.

3.3. The above two lemmas lead us to formulate the estimation of the fundamental polynomials of the second kind. First we observe as a consequence of (2.1.3), (3.2.1) and (3.2.7) we have

$$(3.3.1) \quad |A_k| \leq \frac{2}{n+1}, \quad k = 2, 3, \dots, n+1.$$

On using (3.2.5), (3.2.6), (3.3.1) and (2.1.2) we have

$$(3.3.2) \quad |(1-x^2)^{1/4} q_{n-1}(x)| \leq \frac{28}{(n+1)^{1/2}} + \frac{8 \log n}{(n+1) \sin \theta_k}, \quad k = 2, 3, \dots, n+1.$$

Similarly from (2.1.2), (3.3.1), (3.2.3) and (3.2.4) we have

$$(3.3.3) \quad |q_{n-1}(x)| \leq \frac{2}{n+1} \pi(n+1) + 2\pi = 4\pi.$$

Now we prove

LEMMA 3.3.1. For $-1 \leq x \leq 1$ we have

$$(3.3.4) \quad \frac{|\rho_k(x)|}{1-x_k^2} \leq \frac{14}{(n+1)^{3/2}} + \frac{4 \log n}{(n+1)^2 \sin \theta_k}, \quad k = 2, 3, \dots, n+1,$$

$$(3.3.5) \quad \sum_{k=2}^{n+1} \frac{|\rho_k(x)|}{1-x_k^2} \leq 26(n+1)^{-1/2},$$

and

$$(3.3.6) \quad \frac{|\rho'_k(x)|}{1-x_k^2} \leq 12, \quad \sum_{k=2}^{n+1} \frac{|\rho'_k(x)|}{1-x_k^2} \leq 12n.$$

PROOF. We will obtain the estimates for $0 \leq x \leq 1$, similarly it holds for $-1 \leq x \leq 0$. For in view of (2.1.3) we can also write $\rho_k(x)$ in the form

$$\rho_k(x) = \frac{(1-x^2)^{1/2} u_n(x)}{2u'_n(x_k)} \left[A_k \int_{-1}^x u_n(t) w(t) dt + \int_{-1}^x l_k(t) w(t) dt \right].$$

From (2.1.1) and (3.3.2) we have immediately (3.3.4) and (3.3.5). From (2.1.1) and (2.1.2) we have

$$(3.3.7) \quad \rho'_k(x) = \frac{1}{2u'_n(x_k)} \left[- \left\{ (n+1) T_{n+1}(x) + \frac{x}{2} u_n(x) \right\} q_{n-1}(x) + A_k u_n^2(x) + u_n(x) l_k(x) \right],$$

on using (3.3.3), (3.1.1), (3.3.1) and (3.1.6) we have

$$\frac{|\rho'_k(x)|}{1-x_k^2} \leq \frac{1}{2(n+1)} \left[\frac{3(n+1)}{2} 4\pi + \frac{2}{(n+1)} (n+1)^2 + 2(n+1) \right] \leq 12,$$

from which (3.3.6) follow.

3.4. The object here is to prove that the estimation that we have obtained in Lemma 3.3.1 is best possible in a certain sense.

LEMMA 3.4.1. *There exists a positive number c_1 and c_2 such that for $n > n_0$*

$$(3.4.1) \quad \sum_{i=2}^{n+1} \frac{|\rho_i(0)|}{1-x_i^2} \geq c_1 n^{-1/2}$$

and

$$(3.4.2) \quad \sum_{i=2}^{n+1} \frac{|\rho_i'(1)|}{1-x_i^2} \geq c_2 n.$$

PROOF. From (2.1.1) and (2.1.2) we have

$$(3.4.3) \quad \frac{|\rho_k(0)|}{1-x_k^2} = \frac{1}{4(n+1)} \left| \int_0^1 \varphi_k(t) \omega(t) dt \right|$$

where

$$(3.4.4) \quad \varphi_k(t) = l_k(t) - l_k(-t) = \frac{4(1-x_k^2)}{n+1} \sum_{r=1}^{n/2} u_{2r-1}(x_k) u_{2r-1}(t).$$

First we note that owing to (3.4.4) and (3.2.12) we have immediately

$$(3.4.5) \quad \left| \int_0^1 \varphi_k(t) (1-t^2)^{1/4} dt \right| \leq \frac{16(1-x_k^2)}{n+1} \log n,$$

and a simple computation shows that

$$(3.4.6) \quad \int_0^1 u_{n-1}(t) \omega(t) dt = \frac{1\left(\frac{n}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{4}\right)} \left[\frac{2\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)} + \sum_{j=1}^{n/2} \frac{\Gamma\left(j - \frac{3}{4}\right)}{\Gamma\left(j + \frac{3}{4}\right)} (-1)^{j+1} \right] \geq c_3 n^{1/2},$$

on using (3.4.6) we have

$$(3.4.7) \quad \int_0^1 2t u_n(t) \omega(t) dt = \int_0^1 (u_{n+1}(t)' + u_{n-1}(t)) \omega(t) dt \geq c_4 n^{1/2}.$$

From the identity

$$1 - x_k^2 = 1 - t^2 + t^2 - x_k^2$$

we have

$$(3.4.8) \quad \int_0^1 \varphi_k(t) w(t) dt = (1-x_k^2)^{-1} \left[\int_0^1 \varphi_k(t) (1-t^2)^{1/4} dt + \frac{2}{u'_n(x_k)} \int_0^1 t u_n(t) w(t) dt \right].$$

Therefore, on using (3.4.7), (3.4.3) and (3.4.5) we have

$$\begin{aligned} \sum_{k=2}^{n+1} \frac{|\rho_k(0)|}{1-x_k^2} &\geq \sum_{k=\lceil n/8 \rceil}^{\lfloor n/4 \rfloor} \frac{|\rho_k(0)|}{1-x_k^2} \\ &\geq \frac{1}{4(n+1)} \left[\frac{c_4 n^{1/2}}{(n+1)} \frac{n}{4} - \frac{16}{(n+1)} \log n \sum_{k=\lceil n/8 \rceil}^{\lfloor n/4 \rfloor} \frac{1}{\sin \theta_k} \right] \\ &\geq c_5 n^{-1/2} - \frac{4 \log n}{(n+1)^2} \frac{3n}{4} \geq c_1 n^{-1/2} \quad \text{for } n \geq n_0. \end{aligned}$$

This completes the proof of (3.4.1). In order to prove (3.4.2) we use (3.3.7) and we get

$$(3.4.9) \quad \frac{|\rho_k(1)|}{1-x_k^2} = 2|(n+1)A_k + l_k(1)| = 2|(n+1)A_k + (-1)^{k-1}(1+x_k)|.$$

On using (2.1.3), (3.2.1) and (3.1.5) we have

$$(3.4.10) \quad (n+1)A_k + (-1)^{k+1} x_k = \frac{-\Gamma\left(\frac{n}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{n}{2} + \frac{5}{4}\right)} \left[\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} + \frac{1}{2} \sum_{r=1}^{n/2} \frac{\Gamma\left(r + \frac{1}{4}\right)}{\Gamma\left(r + \frac{3}{4}\right)} \cos 2r \theta_k \right].$$

Using Abel's Lemma we have

$$(3.4.11) \quad |(n+1)A_k + (-1)^{k+1} x_k| \leq c_6 n^{-1/2} (\sin \theta_k)^{-1}, \quad k=2, 3, \dots, n+1.$$

From (3.4.9) and (3.4.11) we have

$$\begin{aligned} \sum_{k=2}^{n+1} \frac{|\rho'_k(1)|}{1-x_k^2} &\cong \sum_{k=\lceil n/8 \rceil}^{\lceil n/4 \rceil} \frac{|\rho'_k(1)|}{1-x_k^2} \cong \frac{n}{4} - c_6 n^{-1/2} \sum_{k=\lceil n/8 \rceil}^{\lceil n/4 \rceil} (\sin \theta_k)^{-1} \\ &\cong \frac{n}{4} - c_7 n^{1/2} \cong c_2 n, \end{aligned}$$

which proves (3.4.2).

4.1. Here we shall prove lemmas which will lead to the estimates of the fundamental polynomials of the first kind.

LEMMA 4.1.1. *We have*

$$(4.1.1) \quad \sum_{k=2}^{n+1} \frac{(1-x^2)I_k^2(x)}{1-x_k^2} \leq 2,$$

$$(4.1.2) \quad \int_1^x u'_n(t) w(t) dt \leq 14n^{5/2}, \quad 0 \leq x \leq 1,$$

$$(4.1.3) \quad \left| \int_1^x p_k(t) w(t) dt \right| \leq 55(n+1)^{3/2}, \quad 0 \leq x \leq 1, \quad k=2, 3, \dots, n+1,$$

$$(4.1.4) \quad \left| \int_1^x p_k(t) w(t) dt \right| \leq 3\pi(n+1)^2(1-x^2)^{1/4}, \quad 0 \leq x \leq 1, \quad k=2, 3, \dots, n+1,$$

$$(4.1.5) \quad \left| \int_{-1}^1 p_k(t) w(t) dt \right| \leq 8\sqrt{\pi} (n+1) \frac{\Gamma\left(\frac{n}{2} + \frac{5}{4}\right)}{\Gamma\left(\frac{n}{2} + \frac{3}{4}\right)},$$

where $p_k(t)$ is defined by (2.1.8).

PROOF. (4.1.1) is due to P. Szasz [10]. (4.1.2) follows from (3.2.5) and

$$(4.1.6) \quad u'_n(t) = 4 \sum_{j=1}^{n/2} j u_{2j-1}(t) \quad (n \text{ even}).$$

Using (3.1.6) and Markov and Bernstein inequalities we have

$$(4.1.7) \quad |l'_k(x)| \leq 2n^2, \quad |l'_k(x)| \leq \frac{2n}{(1-x^2)^{1/2}}$$

respectively. Exactly in the way I proved (3.2.5) we have on using (4.1.7)

$$(4.1.8) \quad \left| \int_1^x \frac{t l'_k(t)}{(1-t^2)^{3/4}} dt \right| \leq 8n^{3/2}, \quad k=2, 3, \dots, n+1, \quad 0 \leq x \leq 1.$$

From (2.1.8) and integration by parts we have

$$\int_1^x p_k(t) w(t) dt = (1-x^2)^{1/4} l'_k(x) - \frac{7}{2} \int_1^x t l'_k(t) w(t) dt + n(n+2) \int_1^x l_k(t) w(t) dt.$$

Now using (4.1.7), (4.1.8) and (3.2.6) we have

$$\left| \int_1^x p_k(t) w(t) dt \right| \leq 2n^{3/2} + 28n^{3/2} + 25n^{3/2} = 55n^{3/2}.$$

This proves (4.1.3). A simple computation shows that

$$(4.1.9) \quad |p_k(t)| \leq 3(n+1)^2, \quad k = 2, 3, \dots, n+1.$$

Therefore on using (3.1.4) we have

$$(4.1.10) \quad \int_1^x p_k(t) w(t) dt \leq 3(n+1)^2 \int_1^x w(t) dt \leq 3\pi(n+1)^2(1-x^2)^{1/4}.$$

This proves (4.1.4). Proof of (4.1.5) follows from another representation of $p_k(t)$ and (3.2.1),

$$(4.1.11) \quad p_k(t) = \frac{2(1-x_k^2)}{n+1} \sum_{r=0}^n \alpha_r u_r(t) u_r(x_k),$$

with $\alpha_r = n(n+2) - r(r+2)$. This proves the lemma. As a consequence of the above lemma we have from (2.1.7), (3.2.1) and (4.1.5)

$$(4.1.12) \quad |A'_k| \leq 2(n+1).$$

Also on using (2.1.6), (4.1.12), (3.2.5) and (4.1.3) we get

$$(4.1.13) \quad |(1-x^2)^{1/4} s_{n-1}(x)| \leq 14(n+1)^{3/2} + 55(n+1)^{3/2} = 69(n+1)^{3/2}.$$

Similarly from (4.1.4), (2.1.6), (4.2.3) and (3.2.3) we have

$$(4.1.14) \quad |s_{n-1}(x)| \leq 2\pi(n+1)^2 + 3\pi(n+1)^2 = 5\pi(n+1)^2.$$

4.2. Here we shall give an estimate of the fundamental polynomials of the first kind.

LEMMA 4.2.1. *For $-1 \leq x \leq 1$ we have*

$$(4.2.1) \quad \sum_{k=2}^{n+1} |r_k(x)| \leq 48n^{3/2},$$

$$(4.2.2) \quad |r_1(x)| \leq 9n^{1/2}, \quad |r_{n+2}(x)| \leq 9n^{1/2},$$

$$(4.2.3) \quad |r'_1(x)| \leq 18n^{5/2}, \quad |r'_{n+2}(x)| \leq 18n^{5/2}.$$

PROOF. On using (2.1.10), (3.1.1) and (4.1.2) we have

$$|r_1(x)| \leq \frac{1}{10(n+1)^2} [10(n+1)^2 + 4(n+1)^2 + 70n^{5/2}] \leq 9n^{1/2}.$$

(4.2.3) follows from (4.2.2) on using Markov's inequality. (4.2.1) follows from (2.1.9), (3.3.4), (4.1.1) and (4.1.13). This proves the lemma.

LEMMA 4.2.2. *For $-1 \leq x \leq 1$ we have*

$$(4.2.4) \quad \sum_{k=2}^{n+1} |r'_k(x)| \leq 55n^3.$$

PROOF. From (4.1.1) and Schwarz inequality we have

$$(4.2.5) \quad \sum_{k=2}^{n+1} \frac{(1-x^2)^{1/2} |L_k(x)|}{1-x_k^2} \leq 4(n+1).$$

Therefore on using (4.1.7) and (4.2.5) we obtain

$$(4.2.6) \quad \sum_{k=2}^{n+1} \left| \frac{[(1-x^2)L'_k(x)]}{1-x_k^2} \right| \leq 12(n+1)^2.$$

From (2.1.9) and (3.3.6) we have

$$(4.2.7) \quad \sum_{k=2}^{n+1} |B_k \rho'_k(x)| \leq 180 \sum_{k=2}^{n+1} \frac{1}{1-x_k^2} \leq 90(n+1)^2.$$

On using (2.1.6), (4.1.14), (4.1.12) and (4.1.9) we have

$$[(1-x^2)u_n(x)s_{n-1}(x)]' = - [(n+1)T_{n+1}(x) + (x/2)u_n(x)]s_{n-1}(x) + u_n^2(x)A'_k + u_n(x)p_k(x).$$

So we get

$$(4.2.8) \quad |[[(1-x^2)u_n(x)s_{n-1}(x)]']| \leq 24(n+1)^3 + 5(n+1)^3 = 29(n+1)^3.$$

Now (4.2.4) follows from (4.2.6), (4.2.2), (2.1.5) and (4.2.8).

4.3. Here we will prove that the estimates obtained in Lemmas 4.2.1 and 4.2.2 are best possible in a certain sense.

LEMMA 4.3.1. *There exist positive constants c_6 and c_7 such that*

$$(4.3.1) \quad \sum_{k=2}^{n+1} |r_k(0)| \geq c_6 n^{3/2}$$

and

$$(4.3.2) \quad \sum_{k=2}^{n+1} |r'_k(1)| \geq c_7 n^3$$

for $n \geq n_1$.

PROOF. From (2.1.5), (4.1.1) and (3.3.4) we have

$$(4.3.3) \quad \sum_{k=2}^{n+1} |r_k(0)| \geq \sum_{k=[n/8]}^{[n/4]} |r_k(0)| \geq \frac{n+1}{2} \sum_{k=[n/8]}^{[n/4]} |s_{n-1}(0)| - 2 - c_8 n,$$

on using (2.1.6) and (2.1.7) we have

$$(4.3.4) \quad s_{n-1}(0) = -\frac{1}{2} \int_0^1 x_k(t) w(t) dt,$$

where

$$(4.3.5) \quad x_k(t) = p_k(t) - p_k(-t) = \frac{4(1-x_k^2)}{n+1} \sum_{r=1}^{n/2} \alpha_{2r-1} u_{2r-1}(x_k) u_{2r-1}(t).$$

Almost in a similar way as we proved (3.4.1) it follows that

$$(4.3.6) \quad \sum_{k=[n/8]}^{[n/4]} |s_{n-1}(0)| \geq c_9 n^{5/2},$$

using (4.3.3) and (4.3.6), (4.3.1) follows. The proof for (4.3.2) is similar as given for (3.4.2). Here we use (4.2.6), (4.2.7) and similar result as given in (3.4.10). This proves the lemma.

5.1. These lemmas proved above lead us easily to prove our main Theorems 1.2.1 and 1.3.1. First we shall prove Theorem 1.2.1. Owing to the uniqueness theorem we can write any arbitrary polynomial of degree $\leq 2n+1$ in the form

$$(5.1.1) \quad Q_{2n+1}(x) = \sum_{k=1}^{n+2} Q_{2n+1}(x_k) r_k(x) + \sum_{k=2}^{n+1} Q''_{2n+1}(x_k) \rho_k(x).$$

Now (1.2.3) follows from (4.2.1), (3.3.5), (1.2.1) and (1.2.2). That (1.2.3) is essentially best possible can be shown by the polynomial

$$(5.1.2) \quad f_0(x) = \sum_{k=1}^{n+2} A \operatorname{sign} r_k(0) r_k(x) + \sum_{k=2}^{n+1} \frac{B \operatorname{sign} \rho_k(0)}{1-x_k^2} \rho_k(x)$$

at $x = 0$. Here we use (3.4.1) and (4.3.1). Again (1.2.4) follows from (5.1.1), (1.2.1), (1.2.2), (3.3.6), (4.2.3) and (4.2.4). That (1.2.4) is essentially best possible is shown by the polynomial

$$(5.1.3) \quad f_1(x) = \sum_{k=1}^{n+2} A \operatorname{sign} r'_k(x) + \sum_{k=2}^{n+1} \frac{B \operatorname{sign} \rho'_k(1)}{1-x_k^2} \rho'_k(x)$$

at $x = 1$. This follows from (3.4.2) and (4.3.2). This completes the proof of Theorem 1.2.1.

PROOF OF THEOREM 1.3.1. Here we follow the notations given in section 1.3. From the uniqueness theorem it follows that any polynomial $\varphi_n(x)$ of degree $\leq 2n+1$ can be written in the form

$$(5.1.4) \quad \varphi_n(x) = \sum_{k=1}^{n+2} \varphi_n(x_{k,n}) r_{kn}(x) + \sum_{k=2}^{n+1} \varphi''_n(x_{k,n}) \rho_{kn}(x).$$

We have from (1.3.6)

$$(5.1.5) \quad R_n(x, f) = R_n(x) = \sum_{k=1}^{n+2} f(x_{k,n}) r_{kn}(x) + \sum_{k=2}^{n+1} b_{kn} \rho_{kn}(x).$$

From Lemma 8.2 of my earlier work [13] it follows that under the conditions

of Theorem 1.3.1 there exists a sequence of polynomials $\varphi_n(x)$ of degree $\leq n$ such that

$$(5.1.6) \quad |f(x) - \varphi_n(x)| = O(n^{-1-\alpha})[(1-x^2)^{\frac{1+\alpha}{2}} + n^{-1-\alpha}], \quad 0 < \alpha < 1,$$

and

$$(5.1.7) \quad |\varphi_n''(x_{k,n})| \leq c_8 n^{1-\alpha} (1-x_{k,n}^2)^{\frac{\alpha-1}{2}}.$$

Now writing

$$f(x) - R_n(x) = f(x) - \varphi_n(x) + \varphi_n(x) - R_n(x).$$

On using (5.1.4), (5.1.5) and the estimates of the fundamental polynomials we have at once

$$|\varphi_n(x) - R_n(x)| = o(1) \quad \text{for} \quad \alpha > 1/2.$$

From (5.1.6) of course we have

$$|f(x) - \varphi_n(x)| = o(1).$$

Therefore for $\alpha > 1/2$

$$|f(x) - R_n(x)| = o(1).$$

This proves the theorem. Theorem 1.3.1 is essentially best possible follows from (3.4.1), (4.3.1) and using the device given in the paper of P. Erdős and P. Turán [5] entitled on the role of Lebesgue function in the theory of Lagrange interpolation.

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