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STIFF GROUPS AND WILD SOCLES

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All groups in this paper are assumed to be p-primary abelian groups for a fixed prime p. We follow the terminology and, with minor exceptions, the notation of [4]. We refer to $G[p]$ as the *socle* of G and by a *subsocle* of G we shall mean a subgroup of $G[p]$. All topological references are to the p-adic topology. By a *dense subsocle* of G we shall mean a subgroup of $G[p]$ that is dense in the topology on $G[p]$ induced by the p-adic of G. G^1 will denote the subgroup of elements of infinite height in G, that is, $G^1 = \bigcap_{n<\omega} p^n G$. If $G^1=0$, G is contained as a pure, dense subgroup of a closed p-group K. The purity of G in K implies that the p-adic topology of K induces that of G and G being dense in K means that K/G is divisible. For a given G, K is unique up to isomorphisms leaving the elements of G fixed and will be referred to as the *torsion-completion* of G.

We shall let $E(G)$ denote the endomorphism ring of G. Following Crawley [2], G is said to be *stiff* if for each $\phi \in E(G)$ there is an $n < \omega$ such that $\phi|(p^nG)[p]$ is multiplication by an integer. A dense subsocle S of G will be called *wild* if whenever $\phi \in E(G)$ and $\phi(S) \subseteq S$ there exists an $n < \omega$ such that $\phi|(p^nG)[p]$ is multiplication by an integer. If $G^1=0$ and $G[p]$ is a wild subsocle of the torsion completion K of G, then we shall say that *G has a wild socle.* Since endomorphisms of G extend uniquely to endomorphisms of K, a group G is stiff if it has a wild socle.

It is easily seen that stiff groups are *essentially indecomposable* (that is, if G is stiff and if $G=A\oplus B$, then one of the two groups A and B is bounded). Crawley [2] has also shown that stiff groups have the *exchange property.* The first construction of a stiff group was by Crawley in [1] where he found a wild subsocle of the torsion-completion of $\bigoplus_{n<\omega} C(p^n)$. This first construction of a group having a wild socle was in connection with finding an infinite reduced primary group isomorphic to no proper subgroup of itself. Indeed we have

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PROPOSITION. *If G is a stiff group and if $f_G(n)$ is finite for all $n < \omega$, then G is isomorphic to no proper subgroup of itself.*

$f_G(n)$, of couse, denotes the n-th Ulm invariant of G. The proof of the above proposition is not difficult and is contained in the proof of Theorem 6.3 in [8]. In fact, the proof of Theorem 6.3 in [8] actually establishes

THEOREM 1. *If K is a closed p-group with a countable basic subgroup and if S is a proper dense subsocle of K such that $|K[p]/S| < 2^{\aleph_0}$, then S supports a stiff pure subgroup of K.*

A subsocle S of G is said to *support* a subgroup H of G if $H[p] = S$. Recall from [7] that if S is a dense subsocle of G and if H is maximal among the subgroups supported by S, then H is a pure subgroup of G and G/H is divisible. We wish to exploit further the techniques used in the proof of Theorem 6.3 in [8]. For this purpose we cite the easily proved, but fundamental

LEMMA 1. *If ϕ is an endomorphism of a closed p-group K such that $|\phi(K[p])| < 2^{\aleph_0}$, then $(p^n K)[p] \subseteq \operatorname{Ker}\phi$ for some $n < \omega$.*

This lemma occurs as Lemma 6.1 in [8].

THEOREM 2. *Suppose $G^1 = 0$, G has a countable basic subgroup and $|K/G| < 2^{\aleph_0}$, where K is the torsion-completion of G. Then G contains a wild subsocle of K and, consequently, a pure, dense stiff subgroup.*

PROOF. Since G has a countable basic subgroup, $E(K)$ has cardinality of the continuum (assuming G to be unbounded). Let F be the family of all $\phi \in E(K)$ such that, for every $n < \omega$, $\phi|(p^n K)[p]$ is *not* multiplication by an integer. We need only find a dense subsocle S of G such that $\phi(S) \nsubseteq S$ for all $\phi \in F$. Such an S will be a wild subsocle of K, and if H is maximal among the subgroups of G supported by S, then H is a pure, dense subgroup of G. Since K is also the torsion-completion of any such H, H will be stiff.

In order to construct S, we first fix a well-ordering $\{\phi_\lambda\}_{\lambda<\Lambda}$ of F where Λ does not exceed the first ordinal having cardinality of the continuum. Let $S_0 = B[p]$ where B is a basic subgroup of G and let z be a fixed element of $G[p]$ not contained in S_0. We wish to find two families $\{x_\lambda\}_{\lambda<\Lambda}$ and $\{y_\lambda\}_{\lambda<\Lambda}$ of elements of $G[p]$ such that (1) $\phi_\lambda(x_\lambda) = y_\lambda + z$ for all $\lambda < \Lambda$ and (2) the subgroup S generated by S_0 and all the x_λ's and y_λ's has a direct decomposition $S = S_0 \oplus \bigoplus_{\lambda<\Lambda} (\langle x_\lambda\rangle \oplus \langle y_\lambda\rangle)$ and does not contain z. We proceed by induction. Suppose $\beta < \Lambda$ and that for each $\lambda < \beta$ we have an x_λ and y_λ

satisfying (1) and such that the subgroup T generated by S_0 and all the x_λ's and y_λ's with $\lambda<\beta$ has the direct decomposition $T=S_0\oplus\bigoplus_{\lambda<\beta}(<x_\lambda>\oplus<y_\lambda>)$ and $z\notin T$. We wish to find an $x_\beta\in G[p]$ such that $<T, x_\beta\phi_\beta(x_\beta)>$ does not contain z and has the direct decomposition $T\oplus<x_\beta>\oplus<\phi_\beta(x_\beta)>$. Assume that no such x_β exists and write $G[p]=T\oplus<z>\oplus V$. Then for each $x\in V$ there exists an $h\in T\oplus<z>$ and a positive integer $t<p$ such that $\phi_\beta(x)=h+tx$. It is easily seen that the integer t is independent of the choice of x. (Indeed suppose x and x_1 are linearly independent elements of V. Then we have $\phi_\beta(x_1)=h_1+t_1x_1$ and $\phi_\beta(x+x_1)=h_2+t_2(x+x_1)$ where $h_1, h_2\in T\oplus<z>$ and t_1 and t_2 are positive integers less than p. Therefore $t_2(x+x_1)=tx+t_1x_1=(t-t_1)x+t_1(x+x_1)$. Since x and $x+x_1$ are independent, $t-t_1=0$). Thus the endomorphism $\psi=t-\phi_\beta$ maps V into the subgroup $T\oplus<z>$ which has cardinality less than that of the continuum. Since $|K[p]/G[p]|<2^{\aleph_0}$, we then conclude that $|\psi(K)|<2^{\aleph_0}$. Therefore, by Lemma 1, there exists an $n<\omega$ such that $(p^nK)[p]\subseteqq\operatorname{Ker}\psi$, which contradicts the fact that $\phi_\beta\in F$. The desired x_β exists and we set $y_\beta=\phi_\beta(x_\beta)-z$.

We conclude then that there exists an $S=S_0\oplus\bigoplus_{\lambda<\Lambda}(<x_\lambda>\oplus<y_\lambda>)$ $G[p]$ such that $z\notin S$ and, for each λ, $\phi_\lambda(x_\lambda)=y_\lambda+z$. S is a dense subsocle of G (and, consequently, a dense subsocle of K) since S_0 is a dense subsocle of G. Since $\phi_\lambda(x_\lambda)=y_\lambda+z\notin S$ for each λ, we have that $\phi(S)\nsubseteqq S$ for all $\phi\in F$.

In the same vein as the preceding theorem, we mention the similarly proved

THEOREM 3. *If K is an unbounded closed p-group with a countable basic subgroup and if A is a countable subgroup of K, then K contains a wild subsocle S such that $S\cap A=0$.*

Stiff groups and groups with wild socles can be used to construct groups that are neither transitive nor fully transitive in the sense of Kaplansky [9]. We require the following lemma.

LEMMA 2. *Let G be such that either* (1) *G/G^1 is stiff or* (2) *G has a high subgroup having a wild socle. If $\phi\in E(G)$ and if $a\in G^1$, then there is an integer t such that $\phi(a)-ta\in pG^1$.*

PROOF. Let $\bar{\phi}\in E(G/G^1)$ be defined by $\bar{\phi}(g+G^1)=\phi(g)+G^1$. Suppose that (1) G/G^1 is stiff. Then choose t and n such that $\bar{\phi}\,|\,p^n(G/G^1)[p]$ is multiplication by t. Let $x\in p^nG-G^1$ be such that $px=a$. Then $x+G^1\in p^n(G/G^1)[p]$ and therefore $\phi(x)-tx\in G^1$. Thus $\phi(a)-ta=p(\phi(x)-tx)\in pG^1$, Recall that a *high subgroup* of G is one maximal among the subgroups of G that intersect G^1 trivially. Suppose that (2) G has a high subgroup H such that H has a wild socle. Let $\pi: G \to G/G^1$ be the canonical map. It is easily seen that $\bar{\phi}(\pi(H[p])) \subseteqq \pi(H[p])$. Since $\pi | H$ is an isomorphism of H onto a pure, dense subgroup of G/G^1, the torsion-completion of G/G^1 is also the torsion-completion of $\pi(H) \cong H$. Therefore $\pi(H[p])$ is a wild subsocle of G/G^1. Thus there is an $n < \omega$ such that $\bar{\phi} | p^n(G/G^1)[p]$ is multiplication by an integer. The proof is now completed as in the first case.

REMARK. If H and L are high subgroups of G, it is easily verified that $\pi(H[p]) = \pi(L[p])$. Consequently, every high subgroup of G has a wild socle if one does.

THEOREM 4. *Let G be a reduced group such that either* (1) G/G^1 *is stiff or* (2) *G has a high subgroup with a wild socle.*
Then

(i) *if G^1 is the direct sum of two or more cyclic groups of order p, then G is neither transitive nor fully tansitive*; *and*

(ii) *if G^1 is not cyclic, then G is not fully transitive.*

PROOF. (i) Suppose $G^1 = \bigoplus_{i \in I} \langle a_i \rangle$ where $O(a_i) = p$ for each i. Then $pG^1 = 0$ and each a_i has $(\omega, \infty, \infty, \cdots)$ as its Ulm sequence (see [9]). However, if $i \neq j$, there is no endomorphism of G mapping a_i to a_j. Indeed, Lemma 2 implies that each $\langle a_i \rangle$ is a fully-invariant subgroup of G.

(ii) Assume that G^1 is not cyclic. Then there exist elements a and b in G^1 such that $\langle a, b \rangle = \langle a \rangle \oplus \langle b \rangle$ is a pure subgroup of G^1 and $U(a) \leqq U(b)$ (see [9]). We shall show that $\phi(a) \neq b$ for all $\phi \in E(G)$. If $\phi \in E(G)$, we have, by Lemma 2, that $\phi(a) - ta \in pG^1$ and $b \notin a + pG^1$ because of the purity of $\langle a \rangle \oplus \langle b \rangle$ in G^1.

In [11] the first examples were given of primary groups that are not fully transitive. These examples depended strongly on the structure of the first Ulm factor of the groups. Our next theorem shows, however, that considerable freedom may be allowed in the first Ulm factor. For results in the opposite direction see [5] and [6].

THEOREM 5. *Let K be an unbounded closed p-group with a countable basic subgroup and suppose M is a pure subgroup such that K/M is a divisible group of cardinality less than $2^{\aleph_0}$. Then if A is a noncyclic reduced p-group with a countable basic subgroup, there exists a p-group G such that* (1) $G/G^1 \cong M$, (2) $G^1 = A$ *and* (3) *G is not fully transitive.*

PROOF. It is clear that the proof of Theorem 2 can be slightly modified so as to yield a wild subsocle S of K contained in M and such that $M[p]/S$ is countably infinite. Then, if D is the injective envelope of A and if H is a pure subgroup of M supported by S, $M/H \cong D/A$ – each is isomorphic to a direct sum of $\aleph_0$ copies of $C(p^\infty)$. Let G be a subdirect sum of M and D with kernels H and A. Then it follows from a remark in [10] that $G/G^1 \cong M$, $G^1 = A$ and H is a high subgroup of G. Since H has a wild socle, the desired conclusion follows from Theorem 4.

Lastly, we construct a counter example to a conjecture due to Cutler. Recall that G and H are said to be *quasi-isomorphic*, we write $G \dot{\cong} H$, if there exist subgroups V and W of G and H, respectively, such that $V \cong W$ and G/V and H/W are bounded. Cutler [3] has raised the following question: If $G \dot{\cong} H$ and if $G/G^1 \cong H/H^1$, does it follow that G and H are isomorphic? Although an affirmative answer can be given when G/G^1 is a direct sum of cylic groups, the answer is in the negative for the general case. Let K be the torsion-completion of $\bigoplus_{n<\omega} C(p^n)$. On the basis of techniques already used in this paper, we construct a group G with the following properties: $G/G^1 \cong K$, $G^1 \cong C(p)$ and G contains a high subgroup having a wild socle. An argument similar to that given in the proof of Theorem 6.3 in [8] shows that G is isomorphic to no proper subgroup of itself. We then have $G \dot{\cong} pG$ and $G/G^1 \cong K \cong pK \cong p(G/G^1) = pG/G^1 = pG/(pG)^1$, but $G \not\cong pG$.

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