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W^*-ALGEBRA WITH A NON-SEPARABLE CYCLIC REPRESENTATION

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1. Introduction. In this paper, we shall show some results on the non-separable cyclic representation of W^*-algebras. In [2], Feldman and Fell have raised the question whether any separable representation of a W^*-algebra M without the direct summand of finite type I is always σ-weakly continuous or not and they have showed that this is affirmative in the case of properly infinite and finite factor of type II_1. Furthermore, M. Takesaki [7] has showed that this is affirmative in the case of W^*-algebra of type II_1. From Theorem 5 in [5] and the above mentioned facts, we have a question whether a representation with singular part of a W^*-algebra is always non-separable or not, and we have to consider this question for the W^*-algebra with the direct summand of finite type I. We shall give a partial answer for this question [Theorem I].

Furthermore, in the representation theory of W^*-algebra, it has not been showed what abelian W^*-algebra admits a non-separable cyclic representation. We shall consider this problem more generally, and we shall show that every W^*-algebra, not finite dimensional, admits a non-separable cyclic representation [Theorem II].

Now we shall state two explained results in the following form:

THEOREM I. *Let M be a W^*-algebra such that $M=\sum_{n=1}^{\infty}\oplus Me_n$ and $Me_n \neq \{0\}$ for each n where e_n is an n-homogeneous central projection for each n. Let π is a non-trivial representation of M. If π satisfies the condition that $\pi^{-1}(0)$ contains e_n for all n where $\pi^{-1}(0)$ is the kernel of π, then π is a non-separable representation.*

THEOREM II. *Let M be an arbitrary W^*-algebra which is not finite dimensional, then M has a non-separable cyclic representation.*

Furthermore, we can show the following: Let M be a W^*-algebra which satisfies the assumption of Theorem I, then there exists a cyclic representation that satisfies the property in Theorem I.

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2. Notations and Preliminaries. Let A be a C^*-algebra and φ a positive linear functional on A. Putting

$$I_\varphi = \{a \in A\,;\ \varphi(a^*a) = 0\}$$

which is called the left kernel of φ, the quotient space A/I_φ becomes the pre-Hilbert space in the usual way canonically induced inner product by φ. We denote the element of A/I_φ corresponding to $a \in A$ by $\eta_\varphi(a)$. Then we get a Hilbert space H_φ as the completion of A/I_φ and a representation π_φ of A, as the left multiplication operators on H_φ, where π_φ is called cyclic representation. If H_φ is non-separable, then we shall call φ a non-separable positive linear functional.

Let M be a W^*-algebra, M_* the Banach space of all bounded normal functionals on M and M_*^+ the positive part of M_* (that is, the set of all functionals φ in M_* such that $\varphi(a^*a) \geqq 0$ for all $a \in M$). We may consider the s-topology defined by a family of semi-norms $\{\alpha_\varphi;\ \varphi \in M_*^+\}$ where $\alpha_\varphi(a) = \varphi(a^*a)^{1/2}$. In [4, p.1.64], S.Sakai has showed that whenever M is represented as a weakly closed algebra of operators on some Hilbert space, the s-topology coinsides with the strong operator topology on bounded sets of M.

3. Some lemmas. To prove theorems, we shall need some lemmas.

In the proof of Theorem I, we shall also use the following lemma which has played an essential role in [2] and [7].

LEMMA I. *Let S be the set of all sequences of integers $J = \{j_1, j_2, \cdots\}$ such that $1 \leqq j_n \leqq 2^n$ for each n. Then there exists a subset S_0 of S having the power of the continum, such that, for any two distinct sequences J, J' in S_0, the set of all n for which $j_n = j_n'$ is finite.*

LEMMA II. *Let M be a W^*-algebra which satisfies the assumption of Theorem* I. *If φ is a non-trivial positive linear functional on M such that I_φ contains e_n for each n, then φ is a non-separable positive linear functional.*

PROOF. Let Z be the center of M and Γ the spectrum of Z. Since M is finite, there exists the center valued trace $\natural$ of M. We define a numerical trace τ on M such that $\tau(a) = \varphi(a^\natural)$ for each $a \in M$. Furthermore, from the property of e_n for each n, there exists, in Me_n, a family $\{e_i^n\}_{i=1}^n$ of abelian projections which are mutually orthogonal and mutually equivalent, and satisfies the equality $e_n = \sum_{i=1}^n e_i^n$ for each n.

Next, let A be any fixed maximal abelian subalgebra that contains $\{e_i^n\}_{i=1,2,\dots,n}^{n=1,2,\dots}$ and μ_φ and μ_τ the Radon-measures on the spectrum Ω of A induced φ and τ, respectively. Furthermore let μ be the Radon-measure on Γ induced by $\varphi = \tau$.

We shall divide the proof into two cases according to the relation between the measures μ_φ and μ_τ.

(Case i). μ_φ is absolutely continuous with respect to μ_τ; In this case, there exists a compact subset K of Ω such that $\mu_\varphi(K) \neq 0$, and the restriction of μ_φ and μ_τ on K are equivalent each other.

Define $\alpha_{h,k}^i$, where $0 \leqq i \leqq k$, $0 \leqq h \leqq 2^k - 1$, as follows:

$$\alpha_{h,k}^i = (-1)^{[h/2^i]}$$

where $[r]$ denotes the largest integer $\leqq$ the real number r. For fixed k and $i < j$, we have:

$$\begin{aligned}\sum_{h=0}^{2^k-1} \alpha_{h,k}^i \alpha_{h,k}^j &= \sum_{h=0}^{2^k-1} (-1)^{[h/2^i]}(-1)^{[h/2^j]} \\ &= \sum_{l=0}^{2^{k-j}-1} \sum_{h=l\cdot 2^j}^{(l+1)2^j-1} (-1)^{[h/2^i]}(-1)^{[h/2^j]} \\ &= \sum_{l=0}^{2^{k-j}-1} (-1)^l \sum_{h=l\cdot 2^j}^{(l+1)2^j-1} (-1)^{[h/2^i]}.\end{aligned}$$

But $(-1)^{[h/2^i]}$ is positive and negative with equal frequency as h ranges from 2^j to $(l+1)2^j - 1$, so that

$$\sum_{h=0}^{2^k-1} \alpha_{h,k}^i \alpha_{h,k}^j = 0.$$

Let $k(s)$ be the largest integer k such that $2^k \leqq s$. We now define $a(i,n)$ in Ae_n, for any positive integer n, and $2^i \leqq n$:

$$a(i,n) = (n/2^{k(n)})^{1/2} \left(\sum_{h=0}^{2^{k(n)}-1} \alpha_{h,k(n)}^i e_h^n \right).$$

Then

$$(a(i,n)^*a(j,n))^\natural = (n/2^{k(n)})\frac{1}{n}\Big(\sum_{h=0}^{2^{k(n)}-1} \alpha^i_{h,k(n)} \times \alpha^j_{h,k(n)}\Big)e_n = \begin{cases} 0 & \text{if } i \neq j \\ e_n & \text{if } i = j. \end{cases}$$

Furthermore, $\|a(i,n)\| \leqq (n/2^{k(n)})^{1/2} \leqq 2^{1/2}$. Thus, given any sequence $i = \{i_1, i_2, \cdots\}$ with $2^{i_n} \leqq n$, the sequence $\{a(i_1, 1) + \cdots + a(i_n, n)\}_{n=1}^{\infty}$ is s-Cauchy and bounded, so that the above sequence converges to an element a^i of A with the s-topology. If i and j are two such sequences, and $i_n \neq j_n$ for all $n \geqq n_0$, then

$$(a(i_1,1)^*a(j_1,1) + \cdots + a(i_n,n)^*a(j_n,n))^\natural$$

$$= (a(i_1,1)^*a(j_1,1) + \cdots + a(i_{n_0},n_0)^*a(j_{n_0},n_0))^\natural.$$

Therefore we have:

$$(a^{i*}a^j) = (a(i_1,1)^*a(j_1,1) + \cdots + a(i_{n_0},n_0)^*a(j_{n_0},n_0))^\natural$$

and $$(a^{i*}a^i) = 1.$$

Furthermore we have:

$$\int_K a^i(\omega)\overline{a^j(\omega)}d\mu_\tau(\omega) = 0 \quad\text{and}\quad \int_K |a^i(\omega)|^2 d\mu_\tau(\omega) = \mu_\tau(K) > 0,$$

where $a^i(\cdot)$ is the element of $C(\Omega)$ corresponding to a^i and $\bar{\cdot}$ is the complex conjugate of $\cdot$. Therefore $\{a^i(\cdot);\ i \in S_0\}$, where S_0 appeared in Lemma I, is an orthogonal system in $L^2(K, \mu_\tau)$ and the cardinal number of S_0 is that of the continum. Therefore $L^2(K, \mu_\tau)$ is non-separable, so that $L^2(\Omega, \mu_\varphi)$ is non-separable.

Since $L^2(\Omega, \mu_\varphi)$ is imbedded in H_φ, H_φ is non-separable. Therefore φ is a non-separable positive linear functional.

(Case ii). μ_φ is not absolutely continuous with respect to μ_τ: In this case, there exists a compact subset K of Ω such that $\mu_\varphi(K) > 0$ and $\mu_\tau(K) = 0$. Furthermore there exists a sequence $\{P_n\}$ of open and closed sets in Ω such that

$$P_n \supset P_{n+1} \supset K \quad\text{and}\quad \lim_n \mu_\tau(P_n) = 0.$$

Let p_n be the projection of A corresponding to P_n, then we have

$$p_n \geqq p_{n+1} \geqq 0 \quad \text{and} \quad 1 \geqq p_n{}^\natural \geqq p_{n+1}{}^\natural \geqq 0\,.$$

It follows that the sequence $\{p_n^\natural(\gamma)\}$ of functions on Γ is convergent to zero μ-almost everywhere. Hence, by Egoroff's Theorem, $p_n^\natural(\gamma)$ is uniformly convergent to zero on some compact subset F of Γ with $\mu(F) > 1 - \varepsilon$ for any $\varepsilon > 0$. Therefore, considering a subsequence of $\{p_n\}$, we may assume $p_n^\natural(\gamma) < 1/4^{n+2}$ for all $\gamma \in F$. Put

$$G_n = \{\gamma \in \Gamma;\ p_n^\natural(\gamma) < 1/4^{n+2}\}.$$

Then G_n is open and contains F. We have $p_n^\natural(\gamma) \leqq 1/4^{n+2}$ on the closure $\overline{G}_n$ of G_n which is open and closed. Consider the projection g_n of Z corresponding to open and closed set $\overline{G}_1 \cap \cdots \cap \overline{G}_n$, and put $f_n = p_n g_n$, then we have

$$g_n \geqq g_{n+1}, f_n \geqq f_{n+1} \quad \text{and} \quad f_n^\natural \leqq 1/4^{n+2}\,,$$

so that f_n converges to zero σ-weakly. Let U_n be the open and closed subset of Ω corresponding to g_n and $U = \bigcap_{n=1}^{\infty} U_n$, we get

$$\begin{aligned}\mu_\varphi(U) &= \lim_{n\to\infty} \mu_\varphi(U_n) = \lim_{n\to\infty} \varphi(g_n) \\ &= \lim_{k\to\infty} \mu \bigcap_{n=1}^{k} \overline{G}_n \geqq \mu(F) > (1-\varepsilon)\,,\end{aligned}$$

which implies

$$\begin{aligned}\mu_\varphi(U \cap K) &= \mu_\varphi(U) + \mu_\varphi(K) - \mu_\varphi(U \cap K) \\ &> 1 - \varepsilon + \mu_\varphi(K) - \mu_\varphi(U \cap K) \\ &> \mu_\varphi(K) - \varepsilon > 0\end{aligned}$$

for sufficiently small $\varepsilon > 0$.

Now if we consider the space H_φ, then we have

$$\pi_\varphi(f_n) \geqq \pi_\varphi(f_{n+1})$$

and

$$\|\pi_\varphi(f_n)\eta_\varphi(1)\|_\varphi^2 = \varphi(f_n) = \varphi(p_n g_n) = \mu_\varphi(U_n \cap P_n) \geqq \mu_\varphi(U \cap K) > 0 \quad \text{for all } n.$$

It follows that $\pi_\varphi(f_n)\eta_\varphi(1)$ converges to a non-zero vector ξ of H_φ which belongs to $\bigcap_{n=1}^{\infty} \pi_\varphi(f_n)H_\varphi$.

Put $h_n \leqq f_n - f_{n+1}$, $e_{1,1} = h_1$ and suppose that orthogonal projections $\{e_{k,j}\}$ are constructed for $k = 1, \cdots, n-1$ and $1 \leqq j \leqq 2^k$ such as

$$h_k = e_{k,1} \sim e_{k,j} \quad \text{for} \quad j = 1, 2, \cdots, 2^k$$

and f_n is orthogonal to $e_{k,j}$. Let us put

$$q_n = \sum_{k=1}^{n-1} \sum_{j=1}^{2^k} e_{k,j} + f_n ,$$

then we have

$$q_n^\phi = \sum_{k=1}^{n-1} \sum_{j=1}^{2^k} e_{k,j}^\phi + f_n^\phi \leqq \sum_{k=1}^{n-1} 1/4^{k+2} + 1/4^{n+2}$$

$$= \sum_{k=1}^{n-1} (1/16)(1/2^k) + 1/4^{n+2} < 1/8 .$$

We get $(1 - q_n)^\phi \geqq 7/8$, so that there exist the orthogonal equivalent projections $e_{n,j}$ for $1 \leqq j \leqq 2^n$ such that

$$h_n = e_{n,1} \sim e_{n,j} \leqq 1 - q_n \quad \text{for} \quad 2 \leqq j \leqq 2^n .$$

By the mathematical induction, we conclude that there exists a family of orthogonal projections $\{e_{n,j}\}$ above mentioned.

Considering partial isometries $u_{n,j}$ such as

$$u_{n,j}^* u_{n,j} = e_{n,1} = h_n \quad \text{and} \quad u_{n,j} u_{n,j}^* = e_{n,j} ,$$

we have

$$u_{n,j}^* u_{n,j'} = u_{n,j}^* e_{n,j} e_{n,j'} u_{n,j'} = 0 \quad \text{for} \quad j \neq j'.$$

Hence, if we put $u(J) = \sum_{n=1}^{\infty} u_{n,j_n}$ for each sequence J of S_0, we have $u(J)^*u(J) = \sum_{n=1}^{\infty} h_n = f_1$ and $f_{n_0} u(J)^* u(J') f_{n_0} = 0$ if $j_n \neq j_n'$ for $n \geqq n_0$. It follows that

$$\begin{aligned}&(\pi_\varphi[u(J)]\xi, \pi_\varphi[u(J)]\xi)\\ &= (\pi_\varphi[u(J)^*u(J)]\xi, \xi)\\ &= (\pi_\varphi(f_1)\xi, \xi) = \|\xi\|^2 > 0\end{aligned}$$

and

$$\begin{aligned}(\pi_\varphi[u(J)]\xi, \pi_\varphi[u(J')]\xi) &= (\pi_\varphi[u(J')^*u(J)]\xi, \xi)\\ = \lim_n (\pi_\varphi(f_n)\pi_\varphi[u(J')^*u(J)]\pi_\varphi(f_n)\xi, \xi) &= 0\,.\end{aligned}$$

Therefore $\{\pi_\varphi[u(J)]\xi\}$ is an orthogonal system in H_φ, so that H_φ is non-separable by Lemma I. This completes the proof of Lemma II.

REMARK. In Case ii, we have used the method which has been used by M.Takesaki [7].

To prove Theorem II, we shall set the following lemma.

LEMMA III. *Let $l^\infty(Z)$ be the algebra of all bounded sequence where Z is the group of all integers. Then $l^\infty(Z)$ has a non-separable cyclic representation.*

PROOF. Let $\hat{Z}$ be the dual group of $\hat{Z}$, then $Z = T$ where T is the torus group. Define a function χ_t on Z where $t \in (0, 2\pi]$, as follows: $\chi_t(n) = \exp(itn)$. Then χ_t is a continuous character of Z, therefore χ_t is an element of T for each $t \in (0, 2\pi]$ and the family $\{\chi_t = \{\chi_t(n)\}_{n=-\infty}^{\infty};\ t \in (0, 2\pi]\}$ is contained in $l^\infty(Z)$.

For each $f \in l^\infty(Z)$ with $f = \{f(k)\}_{k=-\infty}^{\infty}$ and each positive integer n, we define a sequence $\{f_n\}_{n=1}^{\infty}$ as follows:

$$f_n = \frac{1}{2n} \sum_{|k| \leqq n} f(k)\,.$$

Furthermore let φ be the linear functional on $l^\infty(Z)$ such that

$$\varphi(f) = \operatorname*{Lim}_{n\to\infty} f_n$$

where $\operatorname*{Lim}_{n\to\infty}$ is a Banach-limit on $l^\infty(N)$ where N is the set of all positive integers. Then φ is a positive linear functional and $\varphi(1) = 1$. Furthermore, we have: for each $\chi_t = \{\chi_t(n)\}_n$,

$$(\chi_t \cdot \overline{\chi_t})_n = \frac{1}{2n} \sum_{|k| \leqq n} \chi_t(k) \cdot \overline{\chi_t(k)}$$

$$= \frac{1}{2n} \sum_{|k| \leqq n} \exp(itk)\exp(-itk)$$

$$= 1 + \frac{1}{2n} .$$

Let η_φ be the canonical mapping from $l^\infty(Z)/I_\varphi$ where I_φ is the left kernel induced by φ. Then we have: for each $t \in (0, 2\pi]$,

$$(\eta_\varphi(\chi_t) | \eta_\varphi(\chi_t))_\varphi = \varphi(\chi_t \cdot \overline{\chi_t}) = 1$$

by the properties of Banach-limit. If t is an element of $(0, 2\pi)$, then we have:

$$\frac{1}{2n} \sum_{|k| \leqq n} \exp(itk) = \frac{1}{n} \frac{\sin\left(n + \frac{1}{2}\right) t}{\sin\left(\frac{1}{2} t\right)} ,$$

and

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{|k| \leqq n} \exp(itk) = \lim_{n\to\infty} \frac{1}{n} \frac{\sin\left(n + \frac{1}{2} t\right)}{\sin\left(\frac{1}{2} t\right)} = 0 .$$

From the above argument and the properties of Banach-limit, if t and t' are two distinct elements of $(0, 2\pi]$, then we have:

$$(\eta_\varphi(\chi_t) | \eta_\varphi(\chi_{t'}))_\varphi = \varphi(\chi_{t-t'})$$

$$= \operatorname*{Lim}_{n\to\infty} (\chi_{(t-t')})_n$$

$$= \lim_{n\to\infty} \left(\frac{1}{2n} \sum_{|k| \leqq n} \exp(i(t-t')k) \right)$$

$$= 0 .$$

Therefore the family $\{\eta_\varphi(\chi_t);\ t \in (0, 2\pi]\}$ is a normalized orthogonal system in H_φ. Therefore H_φ is non-separable, so that $l^\infty(Z)$ has a non-separable cyclic representation. This completes the proof of Lemma III.

4. Proof of Theorems. In this section, we shall show the proof of Theorem I and Theorem II.

At first, we shall prove Theorem I by using Lemma I and Lemma II.

PROOF OF THEOREM I. Let H be the representing space of π and ξ an element of H with $\|\xi\| = 1$. Furthermore, define a positive linear functional φ in the following form:

$$\varphi(a) = (\pi(a)\xi \mid \xi) \quad \text{for all } a \in M.$$

Then $I_\varphi \supset \pi^{-1}(0) \in e_n$ for all n, and $I_\varphi = \{a \in M;\ \pi(a)\xi = 0\}$. Therefore, from Lemma II, H_φ is non-separable where H_φ is the Hilbert space canonically induced by φ.

Let δ be a mapping from the quotient space M/I_φ into H such that $\delta(\eta_\varphi(a)) = \pi(a)\xi$ where η_φ is the canonical mapping from M onto the quotient space M/I_φ. Then $\|\delta(\eta_\varphi(a))\| = \|\eta_\varphi(a)\|_\varphi$. Therefore δ is isometric from M/I_φ into H, so that we have the property that H_φ is imbedded in H. Therefore H is non-separable. This completes the proof of Theorem I.

REMARK. Let M be a W^*-algebra which satisfies the condition in Theorem I. Let π be a singular representation of M. If there exists an infinite subsequence $\{n_i\}_{i=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that $\pi^{-1}(0)$ contains e_{n_i} for all i and $\sum_{i=1}^{\infty} e_{n_i} \notin \pi^{-1}(0)$ where $\pi^{-1}(0)$ is the kernel of π, then π is a non-separable representation.

Next we shall show the proof of Theorem II by using Lemma III.

PROOF OF THEOREM II. Since M is not finite dimensional, there exists a countable family $\{e_n\}_{n=-\infty}^{\infty}$ of the orthogonal projections in M. Then a W^*-algebra N generated by $\{e_n\}_{n=-\infty}^{\infty}$ is abelian. Furthermore, by the function representation theory, N is $*$-isomorphic to $l^\infty(Z)$. Therefore, from Lemma III, there exists a positive linear functional φ such that the canonical cyclic representation π_φ of N induced by φ is non-separable. Let φ' be the positive linear functional on M which is the extension of φ by Hahn-Banach extension theorem. Furthermore, let $\{a_\alpha\}_{\alpha \in A}$ be a set in N such that $\{\eta_\varphi(a_\alpha)\}_{\alpha \in A}$ is a normalized orthogonal in H_φ where A is a index set with the continum cardinal number. Then, if α and β are two elements of A,

$$(\eta_{\varphi'}(a_\alpha) \mid \eta_{\varphi'}(a_\beta))_{\varphi'} = \varphi'(a_\beta^* a_\alpha) = \varphi(a_\beta^* a_\alpha)$$

$$= (\eta_\varphi(a_\alpha) \mid \eta_\varphi(a_\beta))_\varphi = \begin{cases} 1 \text{ if } \alpha = \beta \\ 0 \text{ if } \alpha \neq \beta . \end{cases}$$

Therefore $\{\eta_{\varphi'}(a_\alpha)\}_{\alpha \in A}$ is a normalized orthogonal system in $H_{\varphi'}$. Therefore $H_{\varphi'}$ is non-sparable, and M has a non-separable cyclic representation. This completes the proof of Theorem II.

By using the argument in proof of Theorem II, we can show the existence of a representation that satisfies the assumption of Theorem I.

PROPOSITION. *Let M be a W^*-algebra which satisfies the assumption of Theorem* I. *Then there exists a cyclic representation that satisfies the assumption of Theorem* I.

PROOF. We consider $\{e_n\}$ in the proof of Theorem II as the family $\{e_n\}$ of central projections in the assumption of Theorem I. Then, e_n corrsponds to an element $(\cdots, 0, \cdots, 0, \overset{\overset{n}{\vee}}{1}, 0, \cdots, 0, \cdots) \in l^\infty(Z)$ and, by the definition of $\pi_{\varphi'}$, e_n is contained in $\pi_{\varphi'}^{-1}(0)$. Therefore $\pi_{\varphi'}$ is a representation of M that satisfies the condition in Theorem I. This completes the proof of Proposition.

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