Tôhoku Math. Journ.
20(1968), 516-553.

ON THE GENERALIZED WHITEHEAD PRODUCTS AND THE GENERALIZED HOPH INVARIANT OF A COMPOSITION ELEMENT

HIDEO ANDO

(Received April 30, 1968)

Introduction. In [11] Hu generalized the Whitehead product to the relative product, moreover in [4] Blaker-Massey defined the mixed Whitehead product in the relative homotopy groups and the product in the triad homotopy groups. Recently, Arkowitz [2], Hilton [10] and Porter [12] have defined the generalized Whitehead product (GWP) $[\alpha,\beta]\in\pi(\Sigma(A\#B),X)$, for $\alpha\in\pi(\Sigma A,X)$ and $\beta\in\pi(\Sigma B,X)$, and studied various properties of the GWP. In the case when A and B are spheres $[\alpha,\beta]$ is essentially the Whitehead product.

The object of this paper is to investigate operations which are generalizations of the Whitehead products in the relative homotopy groups and the triad homotopy groups. We define the following six new products in this paper:

$$
\begin{aligned}
&[\alpha,\beta]_1\in\pi_1(\Sigma(A\#B),k) &&\text{for}\quad \alpha\in\pi_1(\Sigma A,k) &&\text{and}\quad \beta\in\pi(\Sigma B,X)\,,\\
&[\alpha,\beta]_2\in\pi_1(\Sigma(A\#B),k) &&\text{for}\quad \alpha\in\pi_1(\Sigma A,k) &&\text{and}\quad \beta\in\pi_1(\Sigma B,k)\,,\\
&[\alpha,\beta]_3\in\pi_2(\Sigma(A\#B),\Phi) &&\text{for}\quad \alpha\in\pi_1(\Sigma A,u) &&\text{and}\quad \beta\in\pi_1(\Sigma B,v)\,,\\
&[\alpha,\beta]_4\in\pi_2(\Sigma(A\#B),\Phi) &&\text{for}\quad \alpha\in\pi_2(\Sigma A,\Phi) &&\text{and}\quad \beta\in\pi(\Sigma B,W)\,,\\
&[\alpha,\beta]_5\in\pi_2(\Sigma(A\#B),\Phi) &&\text{for}\quad \alpha\in\pi_2(\Sigma A,\Phi) &&\text{and}\quad \beta\in\pi_1(\Sigma B,u)\,,\\
&[\alpha,\beta]_6\in\pi_2(\Sigma(A\#B),\Phi) &&\text{for}\quad \alpha\in\pi_2(\Sigma A,\Phi) &&\text{and}\quad \beta\in\pi_2(\Sigma B,\Phi)\,,
\end{aligned}
$$

where $k\colon X\to Y$ is a map and $\Phi=(v,v')\colon u\to u'$ is a pair-map in which $u\colon w\to X_1$, $v\colon W\to X_2$, $u'\colon X_2\to X$ and $v'\colon X_1\to X$.

If A, B are spheres and k is an inclusion map and $X_1,X_2\subset X$, $W=X_1\cap X_2$ then $[\alpha,\beta]_1$, $[\alpha,\beta]_2$ and $[\alpha,\beta]_3$ are essentially the mixed Whitehead product, the relative and triad Whitehead product, respectively. In section 2 we define generalized Whitehead elements and in section 3 we define the above GWP's by using the generalized Whitehead elements, and basic properties of these products are mentioned in section 4. The Jacobi identities of these products are obtained in section 5. In section 6 we define the generalized Hopf invariant H^* in the generalized homotopy groups by considering the Hopf invariant described in [1] and we generalize Theorem 1 in [8] .

Throughout this paper we shall assume that all spaces have base points and all maps (homotopies) are base point preserving and in section 1-5 spaces A, B and C are countable connected CW-complexes. In particular, in section 6 we shall assume that all spaces are finite CW-complexes.

1. Preliminaries. A map $q: X \to Y$ is called a cofibration if it has the homotopy lowering property for all spaces, i.e., if, for each space P and for all maps $f_0: Y \to P$ and homotopies $g_t: X \to P$ $(0 \leqq t \leqq 1)$ with $g_0 = f_0 \circ q$, there exists a homotopy $f_t: Y \to P$ with $g_t = f_t \circ q$. If q is an inclusion map, this is homotopy extension property. The quotient space $F = Y/q(X)$ is called the cofibre of q. Frequently the cofibration $q: X \to Y$ with cofibre F is denoted by the sequence $X \xrightarrow{q} Y \xrightarrow{p} F$, where p is the projection.

The set of all homotopy classes of maps $X \to Y$ is denoted by $\pi(X, Y)$, it contains the distinguished element 0, i.e., the homotopy class of the constant map $*: X \to Y$. A pair-map $(g_1, g_2): \iota_V \to f$ is by definition a map of maps such that the diagram

$$\begin{tikzcd} V \arrow[r, "g_1"] \arrow[d, "\iota_V"'] & X \arrow[d, "f"] \\ CV \arrow[r, "g_2"] & Y \end{tikzcd}$$

is commutative, where ι_V is the map $V \to CV$ which embeds V in the cone CV. Then $\pi_1(V, f)$ is defined as the set of homotopy classes of (g_1, g_2), and if V is a suspension space (a 2-fold suspension space) $\pi_1(V, f)$ is a group (an abelian group). Moreover, if f is an inclusion and $V = S^n$, $n \geqq 1$, we get the ordinary relative homotopy groups. The homotopy class of a pair-map (g_1, g_2): $\iota_V \to f$ is denoted by $\{(g_1, g_2)\}$. A map $\begin{pmatrix} h_1 h_2 \\ h_3 h_4 \end{pmatrix}: (\iota_V, C\iota_V) \to \Phi_Z$ is by definition a map of pair-maps such that the diagram

$$\begin{tikzcd} V \arrow[rr, "h_1"] \arrow[dr, "\iota_V"] \arrow[dd, "\iota_V"'] & & W \arrow[dr] \arrow[dd] & \\ & CV \arrow[rr, "h_2"] \arrow[dd, "\iota_{CV}"] & & X \arrow[dd] \\ CV \arrow[rr, "h_3"] \arrow[dr, "C\iota_V"'] & & Y \arrow[dr] \arrow[ur, phantom, "\Downarrow\Phi_Z"] & \\ & C^2V \arrow[rr, "h_4"] & & Z \end{tikzcd}$$

is commutative. Then $\pi_2(V, \Phi_Z)$ is defined as the set of homotopy classes of

$\begin{pmatrix} h_1 h_2 \\ h_3 h_4 \end{pmatrix}$, and if V is a suspension space (a 2-fold suspension space) $\pi_2(V, \Phi_Z)$ is a group (an abelian group) and a generalization of the triad homotopy groups. The homotopy class of $\begin{pmatrix} h_1 h_2 \\ h_3 h_4 \end{pmatrix}$ is denoted by $\left\{\begin{pmatrix} h_1 h_2 \\ h_3 h_4 \end{pmatrix}\right\}$.

The (reduced) suspension ΣX of X is the space obtained from $X \times I$ by identifying $X \times \dot{I} \cup * \times I$ to a point. The (reduced) cone CX of X is the space obtained from $X \times I$ by identifying $X \times 0 \cup * \times I$ to a point. We denote by $X \vee Y$ the subspace $X \times * \cup * \times Y$ of $X \times Y$. The smash product $X \# Y$ of X and Y is the space obtained from $X \times Y$ by identifying $X \vee Y$ to a point. For maps $f: X \to Y$ and $g: X' \to Y'$, we define a map $f \vee g: X \vee X' \to Y \vee Y'$ by $f \vee g = f \times g | X \vee X'$, and a map $f \# g: X \# X' \to Y \# Y'$ is defined by the following commutative diagram

$$\begin{array}{ccc} X \times X' & \xrightarrow{f \times g} & Y \times Y' \\ \Big\downarrow p_X & & \Big\downarrow p_Y \\ X \# X' & \xrightarrow{f \# g} & Y \# Y' \end{array} ,$$

where p's are identification maps.

The following properties are checked easily:

(1.1) (i) *If* $f \simeq f': X \to Y$ and $g \simeq g': X' \to Y'$ *then*

$$f \# g \simeq f' \# g': X \# X' \to Y \# Y'.$$

(ii) $$(f \# g) \# h = f \# (g \# h).$$

Let f and g be representatives of $\alpha \in \pi(X, Y)$ and $\beta \in \pi(X', Y')$ respectively. Then $f \# g$ is independent of the choice of f and g by (i) of (1.1) and the homotopy class is denoted by $\alpha \# \beta \in \pi(X \# X', Y \# Y')$. It follows from (1.1) that

$$(1.2) \qquad (\alpha \# \beta) \# \gamma = \alpha \# (\beta \# \gamma).$$

Next we consider two elements $\alpha \in \pi(X, Y)$ and $\beta \in \pi(Y, Z)$ and let $\alpha = \{f\}$ and $\beta = \{g\}$. Then the composition $g \circ f: X \to Z$ of f and g represents an element of $\pi(X, Z)$ which is independent of the choice of the representatives f and g, and the homotopy class of $g \circ f$ is denoted by $\beta \circ \alpha \in \pi(X, Z)$ and is called as the composition of α and β. The formula $\beta \circ \alpha = f^*(\beta) = g_*(\alpha)$ defines maps $f^*: \pi(Y, Z) \to \pi(X, Z)$ and $g_*: \pi(X, Y) \to \pi(X, Z)$ induced by f and g respectively. The join of A and B, $A * B$ is the quotient space obtained from $A\times B\times I$ by factoring out the relation: $(a, b_1, 0)\sim(a, b_2, 0)$ for all $b_1, b_2\in B$ and $(a_1, b, 1)\sim(a_2, b, 1)$ for all $a_1, a_2\in A$.

PROPOSITION (1.3) *$A*B$ is homotopy equivalent to $Q=CA\times B\cup A\times CB$* [12].

PROPOSITION (1.4) *$A*B$ is homotopy equivalent to $\Sigma(A\#B)$ and there exists a homotopy equivalence $\bar{h}_{A,B}\colon \Sigma(A\#B)\to Q$* ([10], [12]).

We denote by $\bar{p}_{A,B}\colon Q\to\Sigma A\vee\Sigma B$ the map which pinches $A\subset CA$ and $B\subset CB$ to $*$.

2. The generalized Whitehead elements. We consider the following pair-maps $\Psi=(1_{\Sigma A}\vee\iota_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})\colon \iota_{\Sigma A}\vee 1_{\Sigma B}\to\iota_{\Sigma A}\vee 1_{C\Sigma B}$, $\Psi_1=(\iota_{\Sigma A}\vee 1_{\Sigma B}, \iota_{C\Sigma A}\vee 1_{\Sigma B})\colon \iota_{\Sigma A}\vee 1_{\Sigma B}\to C\iota_{\Sigma A}\vee 1_{\Sigma B}$, $\Psi_2=(\iota_{\Sigma A}\vee 1_{\Sigma B}, \iota_{C\Sigma A}\vee 1_{C\Sigma B})\colon \iota_{\Sigma A}\vee\iota_{\Sigma B}\to C\iota_{\Sigma A}\vee\iota_{\Sigma B}$, $\Psi_3=(\iota_{\Sigma A}\vee\iota_{\Sigma B}, \iota_{C\Sigma A}\vee\iota_{C\Sigma B})\colon \iota_{\Sigma A}\vee\iota_{\Sigma B}\to C\iota_{\Sigma A}\vee C\iota_{\Sigma B}$:

$$\begin{tikzcd}
\Sigma A\vee\Sigma B \arrow[r, "\iota_{\Sigma A}\vee 1_{\Sigma B}"] \arrow[d, "1_{\Sigma A}\vee\iota_{\Sigma B}"'] \arrow[dr, phantom, "\Downarrow\Psi"] & C\Sigma A\vee\Sigma B \arrow[d, "1_{C\Sigma A}\vee\iota_{\Sigma B}"] \\
\Sigma A\vee C\Sigma B \arrow[r, "\iota_{\Sigma A}\vee 1_{C\Sigma B}"] & C\Sigma A\vee C\Sigma B
\end{tikzcd}\quad,\qquad
\begin{tikzcd}
\Sigma A\vee\Sigma B \arrow[r, "\iota_{\Sigma A}\vee 1_{\Sigma B}"] \arrow[d, "\iota_{\Sigma A}\vee 1_{\Sigma B}"'] \arrow[dr, phantom, "\Downarrow\Psi_1"] & C\Sigma A\vee\Sigma B \arrow[d, "\iota_{C\Sigma A}\vee 1_{\Sigma B}"] \\
C\Sigma A\vee\Sigma B \arrow[r, "C\iota_{\Sigma A}\vee 1_{\Sigma B}"'] & C^2\Sigma A\vee\Sigma B
\end{tikzcd}\quad,$$

(2. 1)

$$\begin{tikzcd}
\Sigma A\vee\Sigma B \arrow[r, "\iota_{\Sigma A}\vee\iota_{\Sigma B}"] \arrow[d, "\iota_{\Sigma A}\vee 1_{\Sigma B}"'] \arrow[dr, phantom, "\Downarrow\Psi_2"] & C\Sigma A\vee C\Sigma B \arrow[d, "\iota_{C\Sigma A}\vee 1_{C\Sigma B}"] \\
C\Sigma A\vee\Sigma B \arrow[r, "C\iota_{\Sigma A}\vee\iota_{\Sigma B}"] & C^2\Sigma A\vee C\Sigma B
\end{tikzcd}\quad,\qquad
\begin{tikzcd}
\Sigma A\vee\Sigma B \arrow[r, "\iota_{\Sigma A}\vee\iota_{\Sigma B}"] \arrow[d, "\iota_{\Sigma A}\vee\iota_{\Sigma B}"'] \arrow[dr, phantom, "\Downarrow\Psi_3"] & C\Sigma A\vee C\Sigma B \arrow[d, "\iota_{C\Sigma A}\vee\iota_{C\Sigma B}"] \\
C\Sigma A\vee C\Sigma B \arrow[r, "C\iota_{\Sigma A}\vee C\iota_{\Sigma B}"'] & C^2\Sigma A\vee C^2\Sigma B,
\end{tikzcd}$$

where $1_{\Sigma B}(1_{\Sigma B})\colon \Sigma A(\Sigma B)\to\Sigma A(\Sigma B)$, $1_{C\Sigma A}(1_{C\Sigma B})\colon C\Sigma A(C\Sigma B)\to C\Sigma A(C\Sigma B)$ and $1_{C^2\Sigma A}\colon C^2\Sigma A\to C^2\Sigma B$ are identity maps, and $\iota_{\Sigma A}(\iota_{\Sigma B})\colon \Sigma A(\Sigma B)\subset C\Sigma A(C\Sigma B)$, $\iota_{C\Sigma A}(\iota_{C\Sigma B})\colon C\Sigma A(C\Sigma B)\subset C^2\Sigma A(C^2\Sigma B)$, $C\iota_{\Sigma A}(C\iota_{\Sigma B})\colon C\Sigma A(C\Sigma B)\subset C^2\Sigma A(C^2\Sigma B)$.

Then we have the following commutative diagrams

(2. 1. 1)

$$\begin{tikzcd}
\pi_2(\Sigma\Lambda, \Psi) \arrow[r, "\partial_\Psi"] \arrow[d, "\partial_{\Psi^T}"'] & \pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee 1_{\Sigma B}) \arrow[r, "(1_{\Sigma A}\vee\iota_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})_*"] \arrow[d, "\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}"] & \pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee 1_{C\Sigma B}) \arrow[d, "\partial_{\iota_{\Sigma A}\vee 1_{C\Sigma B}}"] \\
\pi_1(\Sigma\Lambda, 1_{\Sigma A}\vee\iota_{\Sigma B}) \arrow[r, "\partial_{1_{\Sigma A}\vee\iota_{\Sigma B}}"] & \pi(\Sigma\Lambda, \Sigma A\vee\Sigma B) \arrow[r, "(1_{\Sigma A}\vee\iota_{\Sigma B})_*"] & \pi(\Sigma\Lambda, \Sigma A\vee C\Sigma B)\,,
\end{tikzcd}$$

(2.1.2)

$$\begin{tikzcd}
\pi_2(\Sigma\Lambda, \Psi_3) \arrow[r, "\partial_{\Psi_3}"] \arrow[d, "\partial_{\Psi_3^T}"] & \pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee\iota_{\Sigma B}) \arrow[r, "(\iota_{\Sigma A}\vee\iota_{\Sigma B},\, \iota_{C\Sigma A}\vee\iota_{C\Sigma B})_*"] \arrow[d, "\partial_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}"] & \pi_1(\Sigma\Lambda, C\iota_{\Sigma A}\vee C\iota_{\Sigma B}) \arrow[d, "\partial_{C\iota_{\Sigma A}\vee C\iota_{\Sigma B}}"] \\
\pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee\iota_{\Sigma B}) \arrow[r, "\partial_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}"] & \pi(\Sigma\Lambda, \Sigma A\vee\Sigma B) \arrow[r, "(\iota_{\Sigma A}\vee\iota_{\Sigma B})_*"] & \pi(\Sigma\Lambda, C\Sigma A\vee C\Sigma B),
\end{tikzcd}$$

(2.1.3)

$$\begin{tikzcd}
\pi_2(\Sigma\Lambda, \Psi_2) \arrow[r, "\partial_{\Psi_2}"] \arrow[d, "\partial_{\Psi_2^T}"] & \pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee\iota_{\Sigma B}) \arrow[r, "(\iota_{\Sigma A}\vee 1_{\Sigma B},\, \iota_{C\Sigma A}\vee 1_{C\Sigma B})_*"] \arrow[d, "\partial_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}"] & \pi_1(\Sigma\Lambda, C\iota_{\Sigma A}\vee \iota_{\Sigma B}) \arrow[d, "\partial_{C\iota_{\Sigma A}\vee \iota_{\Sigma B}}"] \\
\pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee 1_{\Sigma B}) \arrow[r, "\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}"] & \pi(\Sigma\Lambda, \Sigma A\vee\Sigma B) \arrow[r, "(\iota_{\Sigma A}\vee\iota_{\Sigma B})_*"] & \pi(\Sigma\Lambda, C\Sigma A\vee \Sigma B),
\end{tikzcd}$$

(2.1.4)

$$\begin{tikzcd}
\pi_2(\Sigma\Lambda, \Psi_1) \arrow[r, "\partial_{\Psi_1}"] \arrow[d, "\partial_{\Psi_1^T}"] & \pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee 1_{\Sigma B}) \arrow[r, "(\iota_{\Sigma A}\vee 1_{\Sigma B},\, \iota_{C\Sigma A}\vee 1_{\Sigma B})_*"] \arrow[d, "\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}"] & \pi_1(\Sigma\Lambda, C\iota_{\Sigma A}\vee 1_{\Sigma B}) \arrow[d, "\partial_{C\iota_{\Sigma A}\vee 1_{\Sigma B}}"] \\
\pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee 1_{\Sigma B}) \arrow[r, "\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}"] & \pi(\Sigma\Lambda, \Sigma A\vee\Sigma B) \arrow[r, "(\iota_{\Sigma A}\vee 1_{\Sigma B})_*"] & \pi(\Sigma\Lambda, C\Sigma A\vee \Sigma B),
\end{tikzcd}$$

in which ∂'s are the boundary homomorphisms (c.f. [5], [10]) and $\Sigma\Lambda=\Sigma(A\# B)$, and $\pi_2(\Sigma\Lambda, \Psi)$ and $\pi_2(\Sigma\Lambda, \Psi_i)$ $(i = 1, 2, 3)$ are identified with $\pi_2(\Sigma\Lambda, \Psi^T)$ and $\pi_2(\Sigma\Lambda, \Psi_i^T)$ respectively, under the isomorphism τ defined in [5, p. 291]. Then we see

PROPOSITION (2.2) (i) $\partial_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}$, $\partial_{\iota_{\Sigma A}\vee 1_{C\Sigma B}}$, $\partial_{C\iota_{\Sigma A}\vee\iota_{\Sigma B}}$, $\partial_{C\iota_{\Sigma A}\vee 1_{\Sigma B}}$, ∂_{Ψ_3} *and* $\partial_{\Psi_3^T}$ *are isomorphisms,* (ii) $(1_{\Sigma A}\vee\iota_{\Sigma B})_*$, $(\iota_{\Sigma A}\vee 1_{\Sigma B})_*$, $(1_{\Sigma A}\vee\iota_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})_*$, $(\iota_{\Sigma A}\vee 1_{\Sigma B}, \iota_{C\Sigma A}\vee 1_{C\Sigma B})_*$ *and* $(\iota_{\Sigma A}\vee 1_{\Sigma B}, \iota_{C\Sigma A}\vee 1_{\Sigma B})_*$ *are epimorphisms,* (iii) $\partial_{1_{\Sigma A}\vee\iota_{\Sigma B}}$, $\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}$, ∂_{Ψ}, ∂_{Ψ_1} *and* ∂_{Ψ_2} *are monomorphisms.*

PROOF. (i) Since $\pi(\Sigma\Lambda, C\Sigma A\vee C\Sigma B) = 0$ in the homotopy exact sequence of the map $\iota_{\Sigma A}\vee\iota_{\Sigma B}$, $\partial_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}$ is an isomorphism. Similarly, the other boundary homomorphisms are isomorphisms. (ii) Let $(i_{\Sigma A}, i_{C\Sigma A})$ be a pair-map: $\iota_{\Sigma A}\to\iota_{\Sigma A}\vee 1_{\Sigma B}$, where $i_{\Sigma A}$: $\Sigma A\subset\Sigma A\vee\Sigma B$ and $i_{C\Sigma A}$: $C\Sigma A\subset C\Sigma A\vee\Sigma B$ and $\iota_{\Sigma A}$: $\Sigma A\subset C\Sigma A$. Then we consider the following commutative diagram

$$\begin{tikzcd}
\pi_1(\Sigma\Lambda, \iota_{\Sigma A}) \arrow[r, "(i_{\Sigma A}, i_{C\Sigma A})_*"] \arrow[d, "\partial_{\iota_{\Sigma A}}", "\cong"'] & \pi_1(\Sigma\Lambda, \iota_{\Sigma A} \vee 1_{\Sigma B}) \arrow[r, "(1_{\Sigma A} \vee \iota_{\Sigma B}, 1_{C\Sigma A} \vee \iota_{\Sigma B})_*"] \arrow[d, "\partial_{\iota_{\Sigma A} \vee 1_{\Sigma B}}"] & \pi(\Sigma\Lambda, \iota_{\Sigma A} \vee 1_{C\Sigma B}) \arrow[d, "\partial_{\iota_{\Sigma A} \vee 1_{C\Sigma B}}", "\cong"'] \\
\pi(\Sigma\Lambda, \Sigma A) \arrow[r, "i_{\Sigma A *}"] & \pi(\Sigma\Lambda, \Sigma A \vee \Sigma B) \arrow[r, "(1_{\Sigma A} \vee \iota_{\Sigma B})_*"] & \pi(\Sigma\Lambda, \Sigma A \vee C\Sigma B),
\end{tikzcd}$$

$((1_{\Sigma A} \vee \iota_{\Sigma B}) \circ i_{\Sigma A})_* = (1_{\Sigma A} \vee \iota_{\Sigma B})_* \circ i_{\Sigma A *}$ is an epimorphism if and only if $(1_{\Sigma A} \vee \iota_{\Sigma B}, 1_{C\Sigma A} \vee \iota_{\Sigma B})_* \circ (i_{\Sigma A}, i_{C\Sigma A})_*$ is an epimorphism, and hence $(1_{\Sigma A} \vee \iota_{\Sigma B}, 1_{C\Sigma A} \vee \iota_{\Sigma B})_*$ is an epimorphism. Also since $(1_{\Sigma A} \vee \iota_{\Sigma B})_* i_{\Sigma A *}$ is an epimorphism we deduce that $(1_{\Sigma A} \vee \iota_{\Sigma B})_*$ is an epimorphism. Similarly, the other homomorphisms are epimorphisms. (iii) is obvious.

Next we define the generalized Whitehead elements.

(a) Let $\overline{h}_{A,B}: \Sigma(A \# B) \to Q_{A,B} = CA \times B \cup A \times CB$ and $\overline{p}_{A,B}: Q_{A,B} \to \Sigma A \vee \Sigma B$ be the maps defined in section 1 (c.f. [12]). Then we define θ by

$$(2.3) \qquad \theta = \{\overline{p}_{A,B}\overline{h}_{A,B}\} \in \pi(\Sigma(A \# B), \Sigma B \vee \Sigma B),$$

where $\overline{h}_{A,B}$ is the homotopy equivalence described in (1. 4).

(b) Consider a commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & \pi_1(\Sigma\Lambda, \iota_{\Sigma A} \vee 1_{\Sigma B}) \arrow[r, "\partial_{\iota_{\Sigma A} \vee 1_{\Sigma B}}"] & \pi(\Sigma\Lambda, \Sigma A \vee \Sigma B) \arrow[r, "(\iota_{\Sigma A} \vee 1_{\Sigma B})_*"] \arrow[d, "j_{\Sigma A \vee \Sigma B *}"] & \pi(\Sigma\Lambda, C\Sigma A \vee \Sigma B) \arrow[d, "j_{C\Sigma A \vee \Sigma B *}"] \\
& & \pi(\Sigma\Lambda, \Sigma A \times \Sigma B) \arrow[r, "(\iota_{\Sigma A} \times 1_{\Sigma B})_*"] & \pi(\Sigma\Lambda, C\Sigma A \times \Sigma B),
\end{tikzcd}$$

where the top row sequence is exact, and $j_{\Sigma A \vee \Sigma B}: \Sigma A \vee \Sigma B \subset \Sigma A \times \Sigma B$, $j_{C\Sigma A \vee \Sigma B}: C\Sigma A \vee \Sigma B \subset C\Sigma A \times \Sigma B$. Then we have $j_{C\Sigma A \vee \Sigma B *} \circ (\iota_{\Sigma A} \vee 1_{\Sigma B})_*(\theta) = (\iota_{\Sigma A} \times 1_{\Sigma B})_* j_{\Sigma A \vee \Sigma B *}(\theta) = 0$ (c.f. [12]) and since $j_{C\Sigma A \vee \Sigma B *}$ is isomorphic we deduce that $(\iota_{\Sigma A} \vee 1_{\Sigma B})_* = 0$. So there is $\theta_1 \in \pi_1(\Sigma\Lambda, \iota_{\Sigma A} \vee 1_{\Sigma B})$ such that $\partial_{\iota_{\Sigma A} \vee 1_{\Sigma B}}(\theta_1) = \theta$. Since $\partial_{\iota_{\Sigma A} \vee 1_{\Sigma B}}$ is a monomorphism, the element θ_1 is determined uniquely. Hence a representative of θ_1 has the form $(\overline{p}_{A,B}\overline{h}_{A,B}, a):$ $\iota \to \iota_{\Sigma A} \vee 1_{\Sigma B}$, where $\iota: \Sigma(A \# B) \to C\Sigma(A \# B)$, that is,

$$(2.4) \qquad \theta_1 = \{(\overline{p}_{A,B}\overline{h}_{A,B}, a)\} \in \pi_1(\Sigma(A \# B), \iota_{\Sigma A} \vee 1_{\Sigma B}).$$

Similarly we define

(2.4′) $$\theta_1' = \{(\bar{p}_{B,A}\bar{h}_{B,A}, a')\} \in \pi_1(\Sigma(B\#A), 1_{\Sigma B} \vee \iota_{\Sigma A}).$$

(c) Since $\partial_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}: \pi_1(\Sigma(A\#B), \iota_{\Sigma B}\vee\iota_{\Sigma B}) \cong \pi_1(\Sigma(A\#B), \Sigma A\vee\Sigma B)$ there exists $\theta_2 \in \pi_1(\Sigma(A\#B), \iota_{\Sigma A}\vee\iota_{\Sigma B})$ such that $\partial_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}(\theta_2) = \theta$.

On the other hand we have the commutative diagram

$$\begin{array}{ccc} \Sigma(A\#B) & \xrightarrow{\bar{p}_{A,B}\bar{h}_{A,B}} & \Sigma A\vee\Sigma B \\ \Big\downarrow \iota & & \Big\downarrow \iota_{\Sigma A}\vee 1_{\Sigma B} \\ C\Sigma(A\#B) & \xrightarrow{C(\bar{p}_{A,B}\bar{h}_{A,B})} & C(\Sigma A\vee\Sigma B) = C\Sigma A\vee C\Sigma B. \end{array}$$

So a representative of θ_2 is $(\bar{p}_{A,B}\bar{h}_{A,B}, C(\bar{p}_{A,B}\bar{h}_{A,B}))$, that is,

(2.5) $$\theta_2 = \{(\bar{p}_{A,B}\bar{h}_{A,B}, C(\bar{p}_{A,B}\bar{h}_{A,B})\} \in \pi_1(\Sigma(A\#B), \iota_{\Sigma A}\vee\iota_{\Sigma B}).$$

Similarly we define

(2.5′) $$\theta_2' = \{(\bar{p}_{B,A}\bar{h}_{B,A}, C(\bar{p}_{B,A}\bar{h}_{B,A})\} \in \pi_1(\Sigma(B\#A), \iota_{\Sigma B}\vee\iota_{\Sigma A}).$$

The homomorphism $(1_{\Sigma A}\vee 1_{\Sigma B}, 1_{C\Sigma B}\vee\iota_{\Sigma B})_*: \pi_1(\Sigma(A\#B), \iota_{\Sigma A}\vee 1_{\Sigma B}) \to \pi_1(\Sigma(A\#B), \iota_{\Sigma A}\vee\iota_{\Sigma B})$ induced by $(1_{\Sigma A}\vee 1_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})$ is a monomorphism. Then we have

(2.6) $$(1_{\Sigma A}\vee 1_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})_*(\theta_1) = \theta_2.$$

For, since $\theta = \partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}(\theta_1) = \partial_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}(1_{\Sigma A}\vee 1_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})_*(\theta_1)$ we have $(1_{\Sigma A}\vee 1_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})_*(\theta_1) = \partial^{-1}_{\iota_{\Sigma A}\vee\iota_{\Sigma B}}(\theta) = \theta_2$.

(d) In (2.1.1) we saw that $0 = (1_{\Sigma A}\vee\iota_{\Sigma B})_*(\theta) = (1_{\Sigma A}\vee\iota_{\Sigma B})_* \circ \partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}(\theta_1) = \partial_{\iota_{\Sigma A}\vee 1_{C\Sigma B}}(1_{\Sigma A}\vee\iota_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})_*(\theta_1)$, and since $\partial_{\iota_{\Sigma A}\vee 1_{C\Sigma B}}$ is an isomorphism we have $(1_{\Sigma A}\vee\iota_{\Sigma B}, 1_{C\Sigma A}\vee\iota_{\Sigma B})_*(\theta_1)=0$. Hence there is an element $\theta_3 \in \pi_2(\Sigma(A\#B), \Psi)$ such that $\partial_\psi(\theta_3)=\theta_1$. Then since ∂_Ψ is a monomorphism, θ_3 is determined uniquely. And θ_3 has a representative of the form $\begin{pmatrix} \bar{p}_{A,B}\bar{h}_{A,B} & a \\ b & c \end{pmatrix}: \Pi \to \Psi$; that is,

(2.7) $$\theta_3 = \left\{\begin{pmatrix} \bar{p}_{A,B}\bar{h}_{A,B} & a \\ b & c \end{pmatrix}\right\} \in \pi_2(\Sigma(A\#B), \Psi),$$

where $a: C\Sigma(A\#B) \to C\Sigma A\vee\Sigma B$, $b: C\Sigma(A\#B) \to \Sigma A\vee C\Sigma B$, $c: C^2\Sigma(A\#B) \longrightarrow C\Sigma A\vee C\Sigma B$ and Π is the pair-map

$$\begin{tikzcd} \Sigma(A\#B) \arrow[r, "\iota"] \arrow[d, "\iota"] \arrow[dr, phantom, "\Downarrow\Pi"] & C\Sigma(A\#B) \arrow[d, "\iota_{C\Sigma}"] \\ C\Sigma(A\#B) \arrow[r, "C\iota_{\Sigma}"'] & C^2\Sigma(A\#B). \end{tikzcd}$$

Similarly we define

$$\theta_3' = \left\{\begin{pmatrix} \bar{\rho}_{B,A}\bar{h}_{B,A} & a' \\ b' & c' \end{pmatrix}\right\} \in \pi_2(\Sigma(B\#A), \Psi'), \tag{2.7}$$

where $\Psi' = (\iota_{\Sigma B} \vee 1_{\Sigma A}, \iota_{\Sigma B} \vee 1_{C\Sigma A}) : 1_{\Sigma B} \vee \iota_{\Sigma A} \to 1_{C\Sigma B} \vee \iota_{\Sigma A}$.

(e) In (2.1.4), we had $0 = (\iota_{\Sigma A} \vee 1_{\Sigma B})_*(\theta) = (\iota_{\Sigma A} \vee 1_{\Sigma B})_* \circ \partial_{\iota_{\Sigma A} \vee 1_{\Sigma B}}(\theta_1)$ $= \partial_{C\iota_{\Sigma A} \vee 1_{\Sigma B}}(\iota_{\Sigma A} \vee 1_{\Sigma B}, 1_{C\Sigma A} \vee 1_{\Sigma B})_*(\theta_1)$, and since $\partial_{C\iota_{\Sigma A} \vee 1_{\Sigma B}}$ is an isomorphism we have $(\iota_{\Sigma A} \vee 1_{\Sigma B}, 1_{C\Sigma A} \vee 1_{\Sigma B})_*(\theta_1) = 0$. Hence there is an element $\theta_4 \in \pi_2(\Sigma(A\#B), \Psi_1)$ such that $\partial_{\Psi_1}(\theta_4) = \theta_1$. And since ∂_{Ψ_1} is a monomorphism θ_4 is determined uniquely. Then the element θ_4 has a representative of the form $\begin{pmatrix} \bar{\rho}_{A,B}\bar{h}_{A,B} & a \\ \bar{b} & \bar{c} \end{pmatrix}$: $\Pi \to \Psi_1$; that is,

$$\theta_4 = \left\{\begin{pmatrix} \bar{\rho}_{A,B}\bar{h}_{A,B} & a \\ \bar{b} & \bar{c} \end{pmatrix}\right\} \in \pi_2(\Sigma(A\#B), \Psi_1), \tag{2.8}$$

where $\bar{b}: C\Sigma(A\#B) \to C\Sigma A \vee \Sigma B$, $\bar{c}: C^2\Sigma(A\#B) \to C^2\Sigma A \vee \Sigma B$.

Similarly we define

$$\theta_4' = \left\{\begin{pmatrix} \bar{\rho}_{B,A}\bar{h}_{B,A} & a' \\ \bar{b}' & \bar{c}' \end{pmatrix}\right\} \in \pi_2(\Sigma(B\#A), \Psi_1'), \tag{2.8'}$$

where $\Psi_1' = (1_{\Sigma B} \vee \iota_{\Sigma A}, 1_{\Sigma B} \vee \iota_{C\Sigma A}) : 1_{\Sigma B} \vee \iota_{\Sigma A} \to 1_{\Sigma B} \vee C\iota_{\Sigma A}$.

(f) From (2.1.3) we see easily that $\theta_2 \in \mathrm{Ker.}(\iota_{\Sigma A} \vee 1_{\Sigma B}, \iota_{C\Sigma A} \vee 1_{C\Sigma B})_*$ $= \mathrm{Im.}\partial_{\Psi_2}$. Hence there is an element $\theta_5 \in \pi_2(\Sigma(A\#B), \Psi_2)$ such that $\partial_{\Psi_2}(\theta_5)$ $= \theta_2$, and since ∂_{Ψ_2} is a monomorphism θ_5 is determined uniquely. Then the element θ_5 has a representative of the form $\begin{pmatrix} \bar{\rho}_{A,B}\bar{h}_{A,B} & C(\bar{\rho}_{A,B}\bar{h}_{A,B}) \\ \bar{\bar{b}} & \bar{\bar{c}} \end{pmatrix}$: $\Pi \to \Psi_2$; that is,

$$\theta_5 = \left\{\begin{pmatrix} \bar{\rho}_{A,B}\bar{h}_{A,B} & C(\bar{\rho}_{A,B}\bar{h}_{A,B}) \\ \bar{\bar{b}} & \bar{\bar{c}} \end{pmatrix}\right\} \in \pi_2\Sigma(A\#B), \Psi_2), \tag{2.9}$$

where $\overline{\overline{b}}\colon C\Sigma(A\#B)\to C\Sigma A\vee\Sigma B$, $\overline{\overline{c}}\colon C^2\Sigma(A\#B)\to C^2\Sigma A\vee C\Sigma B$.

Similarly we define

$$(2.9')\qquad \theta'_5=\left\{\begin{pmatrix}\bar{\rho}_{B,A}\bar{h}_{B,A} & C(\bar{\rho}_{B,A}\bar{h}_{B,A})\\ \overline{\overline{b'}} & \overline{\overline{c'}}\end{pmatrix}\right\}\in\pi_2(\Sigma(B\#A),\Psi'_2)\,,$$

where $\Psi'_2=(1_{\Sigma B}\vee\iota_{\Sigma A},1_{C\Sigma B}\vee\iota_{C\Sigma A})\colon\iota_{\Sigma B}\vee\iota_{\Sigma A}\to\iota_{\Sigma B}\vee C\iota_{\Sigma A}$.

(g) As we have seen in (2.1.2) $\partial_{\Psi_3}\colon\pi_2(\Sigma(A\#B),\Psi_3)\cong\pi_1(\Sigma(A\#B),\iota_{\Sigma A}\vee\iota_{\Sigma B})$, so we obtain $\theta_6\in\pi_2(\Sigma(A\#B),\Psi_3)$ such that $\partial_{\Psi_3}(\theta_6)=\theta_1$, and θ_6 is determined uniquely. And we have the map $\begin{pmatrix}\bar{\rho}_{A,B}\bar{h}_{A,B} & C(\bar{\rho}_{A,B}\bar{h}_{A,B})\\ C(\bar{\rho}_{A,B}\bar{h}_{A,B}) & C^2(\bar{\rho}_{A,B}\bar{h}_{A,B})\end{pmatrix}\colon\Pi\to\Psi_3$.

Hence a representative of θ_6 is $\begin{pmatrix}\bar{\rho}_{A,B}\bar{h}_{A,B} & C(\bar{\rho}_{A,B}\bar{h}_{A,B})\\ C(\bar{\rho}_{A,B}\bar{h}_{A,B}) & C^2(\bar{\rho}_{A,B}\bar{h}_{A,B})\end{pmatrix}$, that is,

$$(2.10)\qquad \theta_6=\left\{\begin{pmatrix}\bar{\rho}_{A,B}\bar{h}_{A,B} & C(\bar{\rho}_{A,B}\bar{h}_{A,B})\\ C(\bar{\rho}_{A,B}\bar{h}_{A,B}) & C^2(\bar{\rho}_{A,B}\bar{h}_{A,B})\end{pmatrix}\right\}\in\pi_2(\Sigma(A\#B),\Psi_3)\,.$$

Similarly we define

$$(2.10')\qquad \theta'_6=\left\{\begin{pmatrix}\bar{\rho}_{B,A}\bar{h}_{B,A} & C(\bar{\rho}_{B,A}\bar{h}_{B,A})\\ C(\bar{\rho}_{B,A}\bar{h}_{B,A}) & C^2(\bar{\rho}_{B,A}\bar{h}_{B,A})\end{pmatrix}\right\}\in\pi_2(\Sigma(B\#A),\Psi'_3)\,,$$

where $\Psi'_3=(\iota_{\Sigma B}\vee\iota_{\Sigma A},\iota_{C\Sigma B}\vee\iota_{C\Sigma A})\colon\iota_{\Sigma B}\vee\iota_{\Sigma A}\to C\iota_{\Sigma B}\vee C\iota_{\Sigma A}$.

Let $\varphi_*\colon\pi_2(\Sigma(A\#B),\Psi)\to\pi_2(\Sigma(A\#B),\Psi_3)$, $\varphi_{2,3*}\colon\pi_2(\Sigma(A\#B),\Psi_2)\to\pi_2(\Sigma(A\#B),\Psi_3)$ and $\varphi_{1,2*}\colon\pi_2(\Sigma(A\#B),\Psi_1)\to\pi_2(\Sigma(A\#B),\Psi_2)$ be natural homomorphisms induced by $\varphi,\varphi_{2,3}$ and $\varphi_{1,2}$, respectively, where $\varphi=\begin{pmatrix}1_{\Sigma A}\vee 1_{\Sigma B} & 1_{C\Sigma A}\vee\iota_{\Sigma B}\\ \iota_{\Sigma A}\vee 1_{C\Sigma B} & \iota_{C\Sigma A}\vee C\iota_{\Sigma B}\end{pmatrix}\colon\Psi\to\Psi_3$, $\varphi_{2,3}=\begin{pmatrix}1_{\Sigma A}\vee 1_{\Sigma B} & 1_{C\Sigma A}\vee 1_{C\Sigma B}\\ 1_{C\Sigma A}\vee\iota_{C\Sigma B} & 1_{C^2\Sigma A}\vee\iota_{C\Sigma B}\end{pmatrix}\colon\Psi_2\to\Psi_3$, and $\varphi_{1,2}=\begin{pmatrix}1_{\Sigma A}\vee 1_{\Sigma B} & 1_{C\Sigma A}\vee\iota_{\Sigma B}\\ 1_{C\Sigma A}\vee 1_{\Sigma B} & 1_{C^2\Sigma A}\vee\iota_{\Sigma B}\end{pmatrix}\colon\Psi_1\to\Psi_2$. Then we see easily

$$(2.11)\qquad \varphi_*(\theta_3)=\varphi_{2,3*}(\theta_5)=\theta_6\quad\text{and}\quad\varphi_{1,2*}(\theta_4)=\theta_5\,.$$

3. The generalized Whitehead products. By using the generelized Whitehead elements defined in section 2, we may define the various products in the generalized homotopy groups.

(A) The generalized Whitehead product (GWP) $[\alpha,\beta]\in\pi(\Sigma(A\#B),X)$ of $\{f\}=\alpha\in\pi(\Sigma A,X)$ and $\{g\}=\beta\in\pi(\Sigma B,X)$ is defined as follows [2], [12]:

(3.1) $$[\alpha, \beta] = (\alpha, \beta)_*(\theta), \quad \text{where} \quad (\alpha, \beta) = \nabla \circ (f \vee g),$$

that is,

$$[\alpha, \beta] = \{\nabla \circ (f \vee g) \circ \bar{\rho}_{A,B} \bar{h}_{A,B}\} \in \pi(\Sigma(A \# B), X).$$

(B) The GWP $[\alpha, \beta]_1 \in \pi_1(\Sigma(A \# B), k)$ of $\{(f, f')\} = \alpha \in \pi_1(\Sigma A, k)$ and $\{g\} = \beta \in \pi(\Sigma B, X)$, $k: X \to Y$, is defined by

(3.2) $$[\alpha, \beta]_1 = (\alpha, \beta)_{1*}(\theta_1), \text{ where } (\alpha, \beta)_1 = (\nabla \circ (f \vee g), \ \nabla \circ (f' \vee kg)),$$

that is,

$$[\alpha, \beta]_1 = \{(\nabla \circ (f \vee g) \circ \bar{\rho}_{A,B} \bar{h}_{A,B}, \ \nabla \circ (f' \vee kg) \circ a)\} \in \pi_1(\Sigma(A \# B), k).$$

Then $[\alpha, \beta]_1$ is independent of choice of (f, f') and g. Similarly we define

(3.2') $$[\beta, \alpha]_1 = (\beta, \alpha)_{1*}(\theta_1'), \text{ where } (\beta, \alpha)_1 = (\nabla \circ (g \vee f), \nabla \circ (kg \vee f')),$$

that is,

$$[\beta, \alpha]_1 = \{(\nabla \circ (g \vee f) \circ \bar{\rho}_{B,A} \bar{h}_{B,A}, \ \nabla \circ (kg \vee f') \circ a')\} \in \pi_1(\Sigma(B \# A), k).$$

REMARK. This GWP is a generalization of the relative product in the sense of M. G. Baratt [9; p. 164].

(C) The GWP $[\alpha, \beta]_2 \in \pi_1(\Sigma(A \# B), k)$ of $\{(f, f')\} = \alpha \in \pi_1(\Sigma A, k)$ and $\{(g, g')\} = \beta \in \pi_1(\Sigma B, k)$, $k: X \to Y$, is defined by

(3.3) $$[\alpha, \beta]_2 = (\alpha, \beta)_{2*}(\theta_2), \text{ where } (\alpha, \beta)_2 = (\nabla \circ (f \vee g), \nabla \circ (f' \vee g')),$$

that is,

$$[\alpha, \beta]_2 = \{(\nabla \circ (f \vee g) \circ \bar{\rho}_{A,B} \bar{h}_{A,B}, \ \nabla \circ (f' \vee g') \circ C(\bar{\rho}_{A,B} \bar{h}_{A,B})\} \in \pi_1(\Sigma(A \# B), k).$$

Then $[\alpha, \beta]_2$ is independent of the choice of (f, f') and (g, g').
Similarly we define

(3.3') $$[\beta, \alpha]_2 = (\beta, \alpha)_{2*}(\theta_2'), \text{ where } (\beta, \alpha)_2 = (\nabla \circ (g \vee f), \nabla \circ (g' \vee f')),$$

that is,

$$[\beta, \alpha]_2 = \{(\nabla \circ (g \vee f) \circ \bar{\rho}_{A,B} \bar{h}_{B,A}), \ \nabla \circ (g' \vee f') \circ C(\bar{\rho}_{B,A} \bar{h}_{B,A}))\} \in \pi_1(\Sigma(B \# A), k).$$

(D) We consider a pair-map $\Phi = (v, v'): u \to u'$;

$$\begin{tikzcd} W \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\Downarrow\Phi"] & X_1 \arrow[d, "v'"] \\ X_2 \arrow[r, "u'"] & X \end{tikzcd},$$

Hereafter, in this paper Φ denotes the above map. Then the GWP $[\alpha, \beta]_3 \in \pi_2(\Sigma(A\#B), \Phi)$ of $\{(f, f')\} = \alpha \in \pi_1(\Sigma A, u)$ and $\{(g, g')\} = \beta \in \pi_1(\Sigma B, v)$ is given by

$$(3.4)\quad [\alpha, \beta]_3 = (\alpha, \beta)_{3*}(\theta)_3, \text{ where } (\alpha, \beta)_3 = \begin{pmatrix} \nabla\circ(f\vee g) & \nabla\circ(f'\vee ug) \\ \nabla\circ(vf\vee g') & \nabla\circ(v'f\vee u'g') \end{pmatrix},$$

that is,

$$[\alpha, \beta]_3 = \left\{\begin{pmatrix} \nabla\circ(f\vee g)\circ \bar{p}_{A,B}\bar{h}_{A,B} & \nabla\circ(f'\vee ug)\circ a \\ \nabla\circ(vf\vee g')\circ b & \nabla\circ(v'f\vee u'g')\circ c \end{pmatrix}\right\} \in \pi_2(\Sigma(A\#B), \Phi).$$

Then $[\alpha, \beta]_3$ is independent of the choice of (f, f') and (g, g').
Similarly we define

$$(3.4')\quad [\beta, \alpha]_3 = (\beta, \alpha)_{3*}(\theta_3'), \text{ where } (\beta, \alpha)_3 = \begin{pmatrix} \nabla\circ(g\vee f) & \nabla\circ(ug\vee f') \\ \nabla\circ(g'\vee vf) & \nabla\circ(u'g'\vee v'f) \end{pmatrix},$$

that is,

$$[\beta, \alpha]_3 = \left\{\begin{pmatrix} \nabla\circ(g\vee f)\circ\bar{p}_{A,B}\bar{h}_{A,B} & \nabla\circ(ug\vee f')\circ a' \\ \nabla\circ(g'\vee vf)\circ b' & \nabla\circ(u'g'\vee v'f)\circ c' \end{pmatrix}\right\} \in \pi_2(\Sigma(B\#A), \Phi).$$

(E) The GWP $[\alpha, \beta]_4 \in \pi_2(\Sigma(A\#B), \Phi)$ of $\left\{\begin{pmatrix} f & f' \\ f'' & f''' \end{pmatrix}\right\} = \alpha \in \pi_2(\Sigma A, \Phi)$ and $\{g\} = \beta \in \pi(\Sigma B, W)$ is defined by

$$(3.5)\quad [\alpha, \beta]_4 = (\alpha, \beta)_{4*}(\theta_4), \text{ where } (\alpha, \beta)_4 = \begin{pmatrix} \nabla\circ(f\vee g) & \nabla\circ(f'\vee ug) \\ \nabla\circ(f''\vee vg) & \nabla\circ(f'''\vee v'ug) \end{pmatrix},$$

that is,

$$[\alpha, \beta]_4 = \left\{\begin{pmatrix} \nabla\circ(f\vee g)\circ\bar{p}_{A,B}\bar{h}_{A,B} & \nabla\circ(f'\vee ug)\circ \bar{a} \\ \nabla\circ(f''\vee vg)\circ\bar{b} & \nabla\circ(f'''\vee v'ug)\circ\bar{c} \end{pmatrix}\right\} \in \pi_2(\Sigma(A\#B), \Phi).$$

Then $[\alpha, \beta]_4$ is independent of the choice of $\begin{pmatrix} f & f' \\ f'' & f''' \end{pmatrix}$ and g.

Similarly we define

$$(3.5') \quad [\beta, \alpha]_4 = (\beta, \alpha)_{4*}(\theta_4'), \text{ where } (\beta, \alpha)_4 = \begin{pmatrix} \nabla\circ(g\vee f) & \nabla\circ(ug\vee f') \\ \nabla\circ(vg\vee f'') & \nabla\circ(v'ug\vee f''') \end{pmatrix},$$

that is,

$$[\beta, \alpha]_4 = \left\{\begin{pmatrix} \nabla\circ(g\vee f)\circ\bar{\rho}_{B,A}\bar{h}_{B,A} & \nabla\circ(ug\vee f')\circ\bar{\bar{a}}' \\ \nabla\circ(vg\vee f'')\circ\bar{\bar{b}}' & \nabla\circ(v'ug\vee f''')\circ\bar{\bar{c}}' \end{pmatrix}\right\} \in \pi_2(\Sigma(B\#A), \Phi).$$

(F) The GWP $[\alpha, \beta]_5 \in \pi_2(\Sigma(A\#B), \Phi)$ of $\left\{\begin{pmatrix} f & f' \\ f'' & f''' \end{pmatrix}\right\} = \alpha \in \pi_2(\Sigma A, \Phi)$ and $\{(g, g')\} = \beta \in \pi_1(\Sigma B, u)$ is defined by

$$(3.6) \quad [\alpha, \beta]_5 = (\alpha, \beta)_{5*}(\theta_5), \text{ where } (\alpha, \beta)_5 = \begin{pmatrix} \nabla\circ(f\vee g) & \nabla\circ(f'\vee g') \\ \nabla\circ(f''\vee vg) & \nabla\circ(f'''\vee v'g') \end{pmatrix},$$

that is,

$$[\alpha, \beta]_5 = \left\{\begin{pmatrix} \nabla\circ(f\vee g)\circ\bar{\rho}_{A,B}\bar{h}_{A,B} & \nabla\circ(f'\vee g')\circ C(\bar{\rho}_{A,B}\bar{h}_{A,B}) \\ \nabla\circ(f''\vee vg)\circ\bar{\bar{b}} & \nabla\circ(f'''\vee v'g')\circ\bar{\bar{c}} \end{pmatrix}\right\} \in \pi_2(\Sigma(A\#B), \Phi).$$

Then $[\alpha, \beta]_5$ is independent of the choice of $\begin{pmatrix} f & f' \\ f'' & f''' \end{pmatrix}$ and (g, g').

Similarly we define

$$(3.6') \quad [\beta, \alpha]_5 = (\beta, \alpha)_{5*}(\theta_5'), \text{ where } (\beta, \alpha)_5 = \begin{pmatrix} \nabla\circ(g\vee f) & \nabla\circ(g'\vee f') \\ \nabla\circ(vg\vee f'') & \nabla\circ(v'g'\vee f''') \end{pmatrix},$$

that is,

$$[\beta, \alpha]_5 = \left\{\begin{pmatrix} \nabla\circ(g\vee f)\circ\bar{\rho}_{B,A}\bar{h}_{B,A} & \nabla\circ(g'\vee f')\circ C(\bar{\rho}_{B,A}\bar{h}_{B,A}) \\ \nabla\circ(vg\vee f'')\circ\bar{\bar{b}}' & \nabla\circ(v'g'\vee f''')\circ\bar{\bar{c}}' \end{pmatrix}\right\} \in \pi_2(\Sigma(B\#A), \Phi),$$

(G) The GWP $[\alpha, \beta]_6 \in \pi_2(\Sigma(A\#B), \Phi)$ of $\left\{\begin{pmatrix} f & f' \\ f'' & f''' \end{pmatrix}\right\} = \alpha \in \pi_2(\Sigma A, \Phi)$ and $\left\{\begin{pmatrix} g & g' \\ g'' & g''' \end{pmatrix}\right\} = \beta \in \pi_2(\Sigma B, \Phi)$ is defined by

$$(3.7) \quad [\alpha, \beta]_6 = (\alpha, \beta)_{6*}(\theta_6), \text{ where } (\alpha, \beta)_6 = \begin{pmatrix} \nabla\circ(f\vee g) & \nabla\circ(f'\vee g') \\ \nabla\circ(f''\vee g'') & \nabla\circ(f'''\vee g''') \end{pmatrix},$$

that is,

$$[\alpha,\beta]_6=\left\{\begin{pmatrix}\nabla\circ(f\vee g)\circ\bar{\rho}_{A,B}\bar{h}_{A,B} & \nabla\circ(f'\vee g')\circ C(\bar{\rho}_{A,B}\bar{h}_{A,B}) \\ \nabla\circ(f''\vee g'')\circ C(\bar{\rho}_{A,B}\bar{h}_{A,B}) & \nabla\circ(f'''\vee g''')\circ C^2(\bar{\rho}_{A,B}\bar{h}_{A,B})\end{pmatrix}\right\}$$
$$\in \pi_2(\Sigma(A\# B),\Phi)\,.$$

Then $[\alpha,\beta]_6$ is independent of the choice of $\begin{pmatrix} f & f' \\ f'' & f''' \end{pmatrix}$ and $\begin{pmatrix} g & g' \\ g'' & g''' \end{pmatrix}$. Similarly we define

(3.7′) $[\beta,\alpha]_6=(\beta,\alpha)_{6*}(\theta_6')$, where $(\beta,\alpha)_6=\begin{pmatrix}\nabla\circ(g\vee f) & \nabla\circ(g'\vee f') \\ \nabla\circ(g''\vee f'') & \nabla\circ(g'''\vee f''')\end{pmatrix}$,

that is,

$$[\beta,\alpha]_6=\left\{\begin{pmatrix}\nabla\circ(g\vee f)\circ\bar{\rho}_{B,A}\bar{h}_{B,A} & \nabla\circ(g'\vee f')\circ C(\bar{\rho}_{B,A}\bar{h}_{B,A}) \\ \nabla\circ(g''\vee f'')\circ C(\bar{\rho}_{B,A}\bar{h}_{B,A}) & \nabla\circ(g'''\vee f''')\circ C(\bar{\rho}_{B,A}\bar{h}_{B,A})\end{pmatrix}\right\}$$
$$\in \pi_2(\Sigma(B\# A),\Phi)\,.$$

4. Properties of the generalized Whitehead products. We consider properties of the generalized Whitehead products defined in section 3.

Let $k\colon X\to Y$ be a map and let $(m,n)\colon k\to k'$ be a pair-map:

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "k"] & X' \arrow[d, "k'"] \\ Y \arrow[r, "n"] & Y' \end{tikzcd},$$

and let $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\colon \Phi\to\bar{\Phi}$ be a map

$$\begin{tikzcd} W \arrow[rr, "a_1"] \arrow[dr] \arrow[dd] & & \bar{W} \arrow[dr] \arrow[dd] & \\ & X_1 \arrow[rr, "a_2"] \arrow[dd] & & \bar{X}_1 \arrow[dd] \\ X_2 \arrow[rr, "b_1"] \arrow[dr] \arrow[ur, phantom, "\Downarrow\Phi"] & & \bar{X}_2 \arrow[dr] \arrow[ur, phantom, "\Downarrow\bar{\Phi}"] & \\ & X \arrow[rr, "b_2"] & & \bar{X} \end{tikzcd}.$$

Then we obtain easily the following formulas:

PROPOSITION (4.1)

$$k_*[\alpha,\beta]=[k_*\alpha,k_*\beta] \qquad \textit{for}\ \ \alpha\in\pi(\Sigma A,X),\ \beta\in\pi(\Sigma B,X),$$

$$(m,n)_* [\alpha,\beta]_1 = [(m,n)_*\alpha, m_*\beta]_1 \qquad \textit{for} \quad \alpha\in\pi_1(\Sigma A,k),\ \beta\in\pi(\Sigma B,X)\,,$$

$$(m,n)_* [\alpha,\beta]_2 = [(m,n)_*\alpha, (m,n)_*\beta]_2 \qquad \textit{for} \quad \alpha\in\pi_1(\Sigma A,k),\ \beta\in\pi_1(\Sigma B,k)\,,$$

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}_* [\alpha,\beta]_3 = [(a_1,a_2)_*\alpha, (a_1,b_1)_*\beta]_3 \qquad \textit{for} \quad \alpha\in\pi_1(\Sigma A,u),\ \beta\in\pi_1(\Sigma B,v)\,,$$

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}_* [\alpha,\beta]_4 = \left[\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}_*\alpha, a_{1*}\beta\right]_4 \qquad \textit{for} \quad \alpha\in\pi_2(\Sigma A,\Phi),\ \beta\in(\Sigma B,W)\,,$$

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}_* [\alpha,\beta]_5 = \left[\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}_*\alpha, (a_1,a_2)_*\beta\right]_5 \qquad \textit{for} \quad \alpha\in\pi_2(\Sigma A,\Phi),\ \beta\in(\Sigma B,u)\,,$$

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}_* [\alpha,\beta]_6 = \left[\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}_*\alpha, \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}_*\beta\right]_6 \qquad \textit{for} \quad \alpha\in\pi_2(\Sigma A,\Phi),\ \beta\in\pi_2(\Sigma B,\Phi)\,.$$

Let $\partial_k:\ \pi_1(\Sigma D,k)\longrightarrow\pi(\Sigma D,X)$, $\quad\partial_k:\ \pi_1(\Sigma\Lambda,k)\longrightarrow\pi(\Sigma\Lambda,X)$,

$\partial_v:\ \pi_1(\Sigma B,v)\longrightarrow\pi(\Sigma B,W)$, $\quad\partial_\Phi:\ \pi_2(\Sigma\Lambda,\Phi)\longrightarrow\pi_1(\Sigma\Lambda,u)$

denote boundary homomorphisms, where $D=A$ or B and $\Sigma\Lambda=\Sigma(A\#B)$ or $\Sigma(B\#A)$.

Then we have

PROPOSITION (4.2)

$$\partial_k[\alpha,\beta]_1 = [\partial_k\alpha,\beta],\ \partial_k[\beta,\alpha]_1 = [\beta,\partial_k\alpha] \qquad \textit{for}\ \alpha\in\pi_1(\Sigma A,k),\ \beta\in\pi(\Sigma B,X),$$

$$\partial_k[\alpha,\beta]_2 = [\partial_k\alpha,\partial_k\beta],\ \partial_k[\beta,\alpha]_2 = [\partial_k\beta,\partial_k\alpha] \qquad \textit{for}\ \alpha\in\pi_1(\Sigma A,k),\ \beta\in\pi_1(\Sigma B,k),$$

$$\partial_\Phi[\alpha,\beta]_3 = [\alpha,\partial_v\beta]_1,\ \partial_\Phi[\beta,\alpha]_3 = [\partial_v\beta,\alpha]_1 \qquad \textit{for}\ \alpha\in\pi_1(\Sigma A,u),\ \beta\in\pi_1(\Sigma B,v),$$

$$\partial_\Phi[\alpha,\beta]_4 = [\partial_\Phi\alpha,\beta]_1,\ \partial_\Phi[\beta,\alpha]_4 = [\beta,\partial_\Phi\alpha]_1 \qquad \textit{for}\ \alpha\in\pi_2(\Sigma A,\Phi),\ \beta\in\pi(\Sigma B,W),$$

$$\partial_\Phi[\alpha,\beta]_5 = [\partial_\Phi\alpha,\beta]_2,\ \partial_\Phi[\beta,\alpha]_5 = [\beta,\partial_\Phi\alpha]_2 \qquad \textit{for}\ \alpha\in\pi_2(\Sigma A,\Phi),\ \beta\in\pi_1(\Sigma B,u),$$

$$\partial_\Phi[\alpha,\beta]_6 = [\partial_\Phi\alpha,\partial_\Phi\beta]_2,\ \partial_\Phi[\beta,\alpha]_6 = [\partial_\Phi\beta,\partial_\Phi\alpha]_2 \qquad \textit{for}\ \alpha\in\pi_2(\Sigma A,\Phi),\ \beta\in\pi_2(\Sigma B,\Phi).$$

PROOF. We shall prove $\partial_k[\alpha,\beta]_1=[\partial_k\alpha,\beta]$. Proofs of the other five formulas are similar to it. We consider the following commutative diagram

$$\begin{array}{ccc}
\pi_1(\Sigma(A\#B),\iota_{\Sigma A}\vee 1_{\Sigma B}) & \xrightarrow{\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}} & \pi_1(\Sigma(A\#B),\Sigma A\vee\Sigma B) \\
\Big\downarrow (\alpha,\beta)_* & & \Big\downarrow (\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}\alpha,\beta)_* \\
\pi_1(\Sigma(A\#B),k) & \xrightarrow{\partial_k} & \pi(\Sigma(A\#B),X)
\end{array}$$

Then we have $\partial_k[\alpha,\beta]_1=\partial_k(\alpha,\beta)_{1*}(\theta_1)=(\partial_k\alpha,\beta)_*\circ\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}(\theta_1)=(\partial_k\alpha,\beta)_*(\theta)$ $=[\partial_k\alpha,\beta]$.

We have the following properties for the relations of GWP's defined in section 3.

PROPOSITION (4.3)

$$\begin{aligned}
&[\alpha,\beta]_2 = [\alpha,\partial_k\beta]_1\,, \\
&[\beta,\alpha]_2 = [\partial_k\beta,\alpha]_1\,,
\end{aligned}\qquad \textit{for}\quad \alpha\in\pi_1(\Sigma A,k),\ \beta\in\pi_1(\Sigma B,k)\,,$$

$$\begin{aligned}
&[\alpha,\beta]_5 = [\alpha,\partial_u\beta]_4\,, \\
&[\beta,\alpha]_5 = [\partial_u\beta,\alpha]_4\,,
\end{aligned}\qquad \textit{for}\quad \alpha\in\pi_2(\Sigma A,\Phi),\ \beta\in\pi_1(\Sigma B,u)\,,$$

$$\begin{aligned}
&[\alpha,\beta]_6 = [\partial_\Phi\alpha,\partial_{\Phi^T}\circ\tau(\beta)]_3\,, \\
&[\beta,\alpha]_6 = [\partial_{\Phi^T}\circ\tau(\beta),\partial_\Phi\alpha]_3\,,
\end{aligned}\qquad \textit{for}\quad \alpha\in\pi_2(\Sigma A,\Phi),\ \beta\in\pi_2(\Sigma B,\Phi)\,,$$

$$\begin{aligned}
&[\alpha,\beta]_6 = [\alpha,\partial_\Phi\beta]_5\,, \\
&[\beta,\alpha]_6 = [\partial_\Phi\beta,\alpha]_5\,,
\end{aligned}\qquad \textit{for}\quad \alpha\in\pi_2(\Sigma A,\Phi)\,,\ \beta\in\pi_2(\Sigma B,\Phi)\,,$$

where $\tau:\ \pi_2(\Sigma B,\Phi)\cong\pi_2(\Sigma B,\Phi^T)$ *is the natural isomorphism defined in* [5; p. 291] *and* $\partial_{\Phi^T}:\ \pi_2(\Sigma B,\Phi^T)\to\pi_1(\Sigma B,v)$ *is the boundary homomorphism.*

PROOF. To prove $[\alpha,\beta]_2=[\alpha,\partial_k\beta]_1$, let $\alpha=\{(f,f')\}$ and $\beta=\{(g,g')\}$. By (2.6) we have $(\alpha,\beta)_{2*}(\theta_2)=(\alpha,\beta)_{2*}(1_{\Sigma A}\vee 1_{\Sigma B},1_{C\Sigma A}\vee\iota_{\Sigma B})_*(\theta_1)$, where $(\alpha,\beta)_2=(\nabla\circ(f\vee g),\nabla\circ(f'\vee g'))$. Hence we obtain

$$\begin{aligned}
(\alpha,\beta)_2\circ(\bar\rho_{A,B}\bar h_{A,B},C(\bar\rho_{A,B}\bar h_{A,B}))&\simeq(\alpha,\beta)_2(1_{\Sigma A}\vee 1_{\Sigma B},1_{C\Sigma A}\vee\iota_{\Sigma B})(\bar\rho_{A,B}\bar h_{A,B},a)\\
&=(\nabla\circ(f\vee g)\circ\bar\rho_{A,B}\bar h_{A,B},\nabla\circ(f'\vee kg)\circ a)\\
&=(\alpha,\partial_k\beta)_1\circ(\bar\rho_{A,B}\bar h_{A,B},a)\,.
\end{aligned}$$

Thus $[\alpha,\beta]_2=(\alpha,\beta)_{2*}(\theta_2)=(\alpha,\partial_k\beta)_{1*}(\theta_1)=[\alpha,\partial_k\beta]_1$. By using the formula analogous to (2.6) we have similarly $[\beta,\alpha]_2=[\partial_k\beta,\alpha]_1$.

Next we shall prove that $[\alpha,\beta]_6=[\partial_\Phi\alpha,\partial_{\Phi^T}\circ\tau(\beta)]_3$. Let $\alpha=\left\{\begin{pmatrix}f & f'\\ f'' & f'''\end{pmatrix}\right\}$ and $\beta=\left\{\begin{pmatrix}g & g'\\ g'' & g'''\end{pmatrix}\right\}$, then $\begin{pmatrix}g & g'\\ g' & g'''u\end{pmatrix}$, (g,g'') and (f,f') are representatives of $\tau(\beta)$, $\partial_{\Phi^T}\circ\tau(\beta)$ and $\partial_\Phi(\alpha)$, respectively, where $u:C^2B\to C^2B$ is the homeomorphism defined by $u(b,s,t)=(b,t,s)$ [5]. By (2.11) $(\alpha,\beta)_{6*}\varphi_*(\theta_3)=(\alpha,\beta)_{6*}(\theta_6)$. Hence we obtain

$$(\alpha,\beta)_6\circ\begin{pmatrix}\bar\rho_{A,B}\bar h_{A,B} & C(\bar\rho_{A,B}\bar h_{A,B})\\ C(\bar\rho_{A,B}\bar h_{A,B}) & C^2(\bar\rho_{A,B}\bar h_{A,B})\end{pmatrix}\simeq(\alpha,\beta)_6\circ\varphi\begin{pmatrix}\rho_{A,B}h_{A,B} & a\\ b & c\end{pmatrix}$$

$$=\begin{pmatrix}\nabla\circ(f\vee g) & \nabla\circ(f'\vee g')\\ \nabla\circ(f''\vee g'') & \nabla\circ(f'''\vee g''')\end{pmatrix}\circ\begin{pmatrix}1_{\Sigma A}\vee 1_{\Sigma B} & 1_{C\Sigma A}\vee\iota_{\Sigma B}\\ \iota_{\Sigma A}\vee 1_{C\Sigma B} & \iota_{C\Sigma A}\vee C\iota_{\Sigma B}\end{pmatrix}\circ\begin{pmatrix}\bar\rho_{A,B}\bar h_{A,B} & a\\ b & c\end{pmatrix}$$

$$=\begin{pmatrix}\nabla\circ(f\vee g) & \nabla\circ(f'\vee ug)\\ \nabla\circ vf\vee g'') & \nabla\circ(v'f'\vee u'g'')\end{pmatrix}\circ\begin{pmatrix}\bar\rho_{A,B}\bar h_{A,B} & a\\ b & c\end{pmatrix}$$

$$=(\partial_\Phi\alpha,\partial_{\Phi^T}\circ\tau(\beta))_3\begin{pmatrix}\bar\rho_{A,B}\bar h_{A,B} & a\\ b & c\end{pmatrix}.$$

Thus the desired formula is proved.

The other formulas are proved by using (2. 11) and the formulas analogous to (2. 11).

PROPOSITION (4.4) [2; Proposition 3.4]. *If A and B are suspensions then*

$$[\alpha,\beta_1+\beta_2]=[\alpha,\beta_1]+[\alpha,\beta_2],$$

$$[\alpha_1+\alpha_2,\beta]=[\alpha_1,\beta]+[\alpha_2,\beta],$$

where $\alpha,\alpha_i\in\pi(\Sigma A,X)$ and $\beta,\beta_i\in\pi(\Sigma B,X)$, $i=1,2$.

PROPOSITION (4.5) *If A and B are suspensions, then*

$$[\alpha,\beta_1+\beta_2]_1=[\alpha,\beta_1]_1+[\alpha,\beta_2]_1 \quad \textit{for}\ \alpha,\alpha_i\in\pi_1(\Sigma A,k)\ \textit{and}$$
$$[\alpha_1+\alpha_2,\beta]_1=[\alpha_1,\beta]_1+[\alpha_2,\beta]_1 \quad \beta,\beta_i\in\pi(\Sigma B,X),\ i=1,2,$$
$$[\alpha,\beta_1+\beta_2]_2=[\alpha,\beta_1]_2+[\alpha,\beta_2]_2 \quad \textit{for}\ \alpha,\alpha_i\in\pi_1(\Sigma A,k)\ \textit{and}$$
$$[\alpha_1+\alpha_2,\beta]_2=[\alpha_1,\beta]_2+[\alpha_2,\beta]_2 \quad \beta,\beta_i\in\pi_1(\Sigma B,k),\ i=1,2,$$
$$[\alpha,\beta_1+\beta_2]_3=[\alpha,\beta_1]_3+[\alpha,\beta_2]_3 \quad \textit{for}\ \alpha,\alpha_i\in\pi_1(\Sigma A,u)\ \textit{and}$$
$$[\alpha_1+\alpha_2,\beta]_3=[\alpha_1,\beta]_3+[\alpha_2,\beta]_3 \quad \beta,\beta_i\in\pi_1(\Sigma B,v),\ i=1,2,$$
$$[\alpha,\beta_1+\beta_2]_4=[\alpha,\beta_1]_4+[\alpha,\beta_2]_4 \quad \textit{for}\ \alpha,\alpha_i\in\pi_2(\Sigma A,\Phi)\ \textit{and}$$
$$[\alpha_1+\alpha_2,\beta]_4=[\alpha_1,\beta]_4+[\alpha_2,\beta]_4 \quad \beta,\beta_i\in\pi(\Sigma B,W),\ i=1,2,$$
$$[\alpha,\beta_1+\beta_2]_5=[\alpha,\beta_1]_5+[\alpha,\beta_2]_5 \quad \textit{for}\ \alpha,\alpha_i\in\pi_2(\Sigma A,\Phi)\ \textit{and}$$
$$[\alpha_1+\alpha_2,\beta]_5=[\alpha_1,\beta]_5+[\alpha_2,\beta]_5 \quad \beta,\beta_i\in\pi_1(\Sigma B,u),\ i=1,2,$$
$$[\alpha,\beta_1+\beta_2]_6=[\alpha,\beta_1]_6+[\alpha,\beta_2]_6 \quad \textit{for}\ \alpha,\alpha_i\in\pi_2(\Sigma A,\Phi)\ \textit{and}$$
$$[\alpha_1+\alpha_2,\beta]_6=[\alpha_1,\beta]_6+[\alpha_2,\beta]_6 \quad \beta,\beta_i\in\pi_2(\Sigma B,\Phi),\ i=1,2.$$

PROOF. We shall prove $[\alpha, \beta_1+\beta_2]_1=[\alpha, \beta_1]_1+[\alpha, \beta_2]_1$. Let $\alpha=\{(f_1, f_2)\}$, $\beta_1=\{g_1\}$ and $\beta_2=\{g_2\}$, and let $i_{\Sigma A}: \Sigma A\subset\Sigma A\vee\Sigma B_1\vee\Sigma B_2$, $i_{\Sigma B_1}: \Sigma B_1\subset\Sigma A\vee\Sigma B_1\vee\Sigma B_2$ and $i_{\Sigma B_2}: \Sigma B_2\subset\Sigma A\vee\Sigma B_1\vee\Sigma B_2$, $B=B_1=B_2$. Now we consider the commutative diagram

$$\begin{array}{ccc}\Sigma A\vee\Sigma B_1\vee\Sigma B_2 & \xrightarrow{q} & X\\ \Big\downarrow{\scriptstyle \iota_{\Sigma A}\vee 1_{\Sigma B_1}\vee 1_{\Sigma B_2}} & & \Big\downarrow{\scriptstyle k}\\ C\Sigma A\vee\Sigma B_1\vee\Sigma B_2 & \xrightarrow{r} & Y\end{array},$$

where $q=\nabla\circ(1\vee\nabla)\circ(f_1\vee g_1\vee g_2)$ and $r=\nabla\circ(1\vee\nabla)\circ(f_2\vee kg_1\vee kg_2)$. If $\gamma\in\pi_1(\Sigma A, \iota_{\Sigma A}\vee 1_{\Sigma B_1}\vee 1_{\Sigma B_2})$ is an element represented by a map $(i_{\Sigma A}, i_{C\Sigma A})$: $\iota_{\Sigma A}\rightarrow\iota_{\Sigma A}\vee 1_{\Sigma B_1}\vee 1_{\Sigma B_2}$, $i_{C\Sigma A}: C\Sigma A\subset C\Sigma A\vee\Sigma B_1\vee\Sigma B_2$, then we obtain the following formulas:

$$(4.5.1)\ \begin{cases}(q, r)_*([\gamma, \{i_{\Sigma B_1}\}+\{i_{\Sigma B_2}\}]_1)=[(q, r)_*\gamma, \beta_1+\beta_2]_1,\\ (q, r)_*([\gamma, \{i_{\Sigma B_1}\}]_1+[\gamma, \{i_{\Sigma B_2}\}]_1)=[(q, r)_*\gamma, \beta_1]_1+[(q, r)_*\gamma, \beta_2]_1,\\ [\gamma, \{i_{\Sigma B_1}\}+\{i_{\Sigma B_2}\}]_1=[\gamma, \{i_{\Sigma B}\}]_1+[\gamma, \{i_{\Sigma B_2}\}]_1.\end{cases}$$

For, the first and second formulas are obtained by (4.1) and the third formula is proved as follows: In the homotopy exact sequence

$$\longrightarrow\pi_1(\Sigma\Lambda, \iota_{\Sigma A}\vee 1_{\Sigma B_1}\vee 1_{\Sigma B_2})\xrightarrow{\partial}\pi(\Sigma\Lambda, \Sigma A\vee\Sigma B_1\vee\Sigma B_2)$$
$$\xrightarrow{(\iota_{\Sigma A}\vee 1_{\Sigma B_1}\vee 1_{\Sigma B_2})_*}\pi(\Sigma\Lambda, C\Sigma A\vee\Sigma B_1\vee\Sigma B_2),$$

∂ is a monomorphism since $(\iota_{\Sigma A}\vee 1_{\Sigma B_1}\vee 1_{\Sigma B_2})$ is onto, and we have

$$\partial[\gamma, \{i_{\Sigma B_1}\}+\{i_{\Sigma B_2}\}]_1=[\partial\gamma, \{i_{\Sigma B_1}\}+\{i_{\Sigma B_2}\}]\quad(\text{by }(4.2)),$$
$$\partial([\gamma, \{i_{\Sigma B_1}\}]_1+[\gamma, \{i_{\Sigma B_2}\}]_1)=[\partial\gamma, \{i_{\Sigma B_1}\}]+[\partial\gamma, \{i_{\Sigma B_2}\}]\quad(\text{by }(4.2)).$$

Hence, by using (4.4) and the fact that ∂ is a monomorphism, we get desired formula.

Next, by using (4.5.1) and $\alpha=(q, r)_*\gamma$, we have

$$\begin{aligned}[\alpha, \beta_1+\beta_2]_1 &= [(q, r)_*\gamma, \beta_1+\beta_2]_1\\ &= (q, r)_*([\gamma, \{i_{\Sigma B_1}\}+\{i_{\Sigma B_2}\}]_1)\\ &= (q, r)_*([\gamma, \{i_{\Sigma B_1}\}]_1+(q, r)_*[\gamma, \{i_{\Sigma B_2}\}]_1\end{aligned}$$

$$= [(q,r)_*\gamma, \beta_1]_1 + [(q,r)_*\gamma, \beta_2]_1$$
$$= [\alpha, \beta_1]_1 + [\alpha, \beta_2]_1 .$$

Similarly we have

$$[\alpha_1+\alpha_2, \beta]_1 = [\alpha_1, \beta]_1 + [\alpha_2, \beta]_1 .$$

Next we shall prove $[\alpha, \beta_1+\beta_2]_3 = [\alpha, \beta_1]_3 + [\alpha, \beta_2]_3$. Let $\alpha = \{(f_1, f_2)\}$, $\beta_1 = \{(g_1, g_1')\}$ and $\beta_2 = \{(g_2, g_2')\}$, and let $i_{C\Sigma B_1} : C\Sigma B_1 \subset \Sigma A \vee C\Sigma B_1 \vee C\Sigma B_2$ and $i_{C\Sigma B_2} : C\Sigma B_2 \subset \Sigma A \vee C\Sigma B_1 \vee C\Sigma B_2$, and let Λ be a map such that

$$\begin{array}{ccc} \Sigma A \vee \Sigma B_1 \vee \Sigma B_2 & \xrightarrow{\iota_{\Sigma A} \vee 1_{\Sigma B_1} \vee 1_{\Sigma B_2}} & C\Sigma A \vee \Sigma B_1 \vee \Sigma B_2 \\ \Big\downarrow 1_{\Sigma A} \vee \iota_{\Sigma B_1} \vee \iota_{\Sigma B_2} & \Downarrow \Lambda & \Big\downarrow 1_{C\Sigma A} \vee \iota_{\Sigma B_1} \vee \iota_{\Sigma B_2} \\ \Sigma A \vee C\Sigma B_1 \vee C\Sigma B_2 & \xrightarrow{\iota_{\Sigma A} \vee 1_{C\Sigma B_1} \vee 1_{C\Sigma B_2}} & C\Sigma A \vee C\Sigma B_1 \vee C\Sigma B_2 \end{array} ,$$

Then we consider the map $\begin{pmatrix} q_1 & q_2 \\ r_1 & r_2 \end{pmatrix} : \Lambda \to \Phi$, where $q_1 = \nabla \circ (1 \vee \nabla) \circ (f_1 \vee g_1 \vee g_2)$, $q_2 = \nabla \circ (1 \vee \nabla) \circ (f_2 \vee u g_1 \vee u g_2)$, $r_1 = \nabla \circ (1 \vee \nabla) \circ (v f_1 \vee g_1' \vee g_2')$ and $r_2 = \nabla \circ (1 \vee \nabla)(v' f_2 \vee u' g_1' \vee u' g_2')$. Now if $\gamma_1, \gamma_2 \in \pi_1(\Sigma B,\ 1_{\Sigma A} \vee \iota_{\Sigma B_1} \vee \iota_{\Sigma B_2})$ and $\delta \in \pi_1(\Sigma A, \iota_{\Sigma A} \vee 1_{\Sigma B_1} \vee 1_{\Sigma B})$ are the elements represented by maps $(i_{\Sigma B_1}, i_{C\Sigma B_1}) : \iota_{\Sigma B_1} \to 1_{\Sigma A} \vee \iota_{\Sigma B_1} \vee \iota_{\Sigma B_2}$, $(i_{\Sigma B_2}, i_{C\Sigma B_2}) : \iota_{\Sigma B_2} \to 1_{\Sigma A} \vee \iota_{\Sigma B_1} \vee \iota_{\Sigma B_2}$ and $(i_{\Sigma A}, i_{C\Sigma A}) : \iota_{\Sigma A} \to \iota_{\Sigma A} \vee 1_{\Sigma B_1} \vee 1_{\Sigma B_2}$, respectively then we have the following formulas:

$$(4.5.2)\quad \begin{cases} \begin{pmatrix} q_1 & q_2 \\ r_1 & r_2 \end{pmatrix}_* [\delta, \gamma_1+\gamma_2]_3 = [(q_1, q_2)_*\delta, \beta_1, \beta_2]_3 , \\ \begin{pmatrix} q_1 & q_2 \\ r_1 & r_2 \end{pmatrix}_* ([\delta, \gamma_1]_3 + [\delta, \gamma_2]_3) = [(q_1, q_2)_*\delta, \beta_1]_3 + [(q_1, q_2)_*\delta, \beta_2]_3 , \\ [\delta, \gamma_1+\gamma_2]_3 = [\delta, \gamma_1]_3 + [\delta, \gamma_2]_3 . \end{cases}$$

We shall prove only the third formula, the other fomulas are obvious.

The boundary homomorphism $\partial_\Lambda : \pi_2(\Sigma(A \# B), \Lambda) \to \pi_1(\Sigma(A \# B), \iota_{\Sigma A} \vee 1_{\Sigma B_1} \vee 1_{\Sigma B_2})$ is a monomorphism, and we have

$$\partial_\Lambda[\delta, \gamma_1+\gamma_2]_3 = [\delta, \partial\gamma_1+\partial\gamma_2]_1 ,$$

$$\partial_\Lambda([\delta, \gamma_1]_3 + [\delta, \gamma_2]_3) = [\delta, \partial\gamma_1]_1 + [\delta, \partial\gamma_2]_1 .$$

Hence we get $[\delta, \gamma_1+\gamma_2]_3 = [\delta, \gamma_1]_3 + [\delta, \gamma_2]_3$.

By using (4.5.2) we obtain the desired formula $[\alpha, \beta_1+\beta_2]_3 = [\alpha, \beta_1]_3 + [\alpha, \beta_2]_3$.

If we choose appropriate universal examples, the case $[\ ,\ ]_4$ is proved similarly, and the other formulas are proved easily by (4.3).

Next by using Hardie's method [6] we may define the GWP $[\alpha, \beta]_1$ as follows: Let $f: \Sigma A \to X$ and $g: \Sigma B \to X$ be maps, if we use Hardie's notation $w(f, g)$ is the composition map $\nabla \circ (f \vee g) \circ \bar{\rho}_{A,B} \bar{h}_{A,B}: \Sigma(A \# B) \to X$.

Consider the commutative diagram

$$\begin{array}{ccccccccc}
\Sigma(A\#B) & \xrightarrow{\bar{h}_{A,B}} & CA\times B\cup A\times CB & \xrightarrow{\bar{\rho}_{A,B}} & \Sigma A\vee\Sigma B & \xrightarrow{*\vee g} & X\vee X & \xrightarrow{\nabla} & X \\
\downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow & & & & \| \\
C\Sigma(A\#B) & \xrightarrow{h_{A,B}} & CA\times CB & \xrightarrow{\rho_{A,B}} & \Sigma A\times\Sigma B & \xrightarrow{p_B} & B & \xrightarrow{g} & X,
\end{array}$$

where $p_B: \Sigma A \times \Sigma B \to \Sigma B$ is the projection, and $\rho_{A,B}$, $h_{A,B}$ are maps defined in [12], and $\lambda_A(g) = g \circ p_B \circ \rho_{A,B} h_{A,B}$. Let $l: C\Sigma A \to X$ be a map. Then we define $\bar{w}(l, g): C\Sigma(A\#B) \to X$ by

$$\bar{w}(l, g)(y, t) = \begin{cases} \lambda_A(g)(y, 2t), & 0 \leqq t \leqq 1/2, \\ & \qquad\qquad y \in \Sigma(A\#B), \\ w(l\sigma'_{2t-1}, g)y, & 1/2 \leqq t \leqq 1, \end{cases}$$

where $\sigma'_{2t-1}: \Sigma A \to C\Sigma A$ is defined by $\sigma'_{2t-1}(x) = (x, 2t-1)$ for $x \in \Sigma A$.

Let $k: X \to Y$ be a map. And if $\{(f, f')\} = \alpha \in \pi_1(\Sigma A, k)$ and $\{g\} = \beta \in \pi(\Sigma B, X)$, then $\bar{w}(f', kg): C\Sigma(A\#B) \to Y$ is given by

$$\bar{w}(f', kg)(y, t) = \begin{cases} \lambda_A(kg)(y, 2t), & 0 \leqq t \leqq 1/2, \\ & \qquad\qquad y \in \Sigma(A\#B), \\ w(f'\sigma'_{2t-1}, kg)y, & 1/2 \leqq t \leqq 1. \end{cases}$$

And the diagram

$$\begin{array}{ccc}
\Sigma(A\#B) & \xrightarrow{w(f, g)} & X \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle k} \\
C\Sigma(A\#B) & \xrightarrow{\bar{w}(f', kg)} & Y
\end{array}$$

is commutative.

THEOREM (4.6) $\{w(f, g), \bar{w}(f', kg)\} = [\alpha, \beta]_1 \in \pi_1(\Sigma(A\#B), k)$.

PROOF. Let $i_{\Sigma A}: \Sigma A \subset \Sigma A \vee \Sigma B$, $i_{\Sigma B} : \Sigma B \subset \Sigma A \vee \Sigma B$ and $i_{C\Sigma A} : C\Sigma A \subset C\Sigma A \vee \Sigma B$. If we choose $\{(w(i_{\Sigma A}, i_{\Sigma B}), \bar{w}(i_{C\Sigma A}, (\iota_{\Sigma A} \vee 1_{\Sigma A}) i_{\Sigma B})\} \in \pi_1(\Sigma(A\#B), \iota_{\Sigma A} \vee 1_{\Sigma B})$ we have $\partial_{\iota_{\Sigma A} \vee 1_{\Sigma B}} \{(w(i_{\Sigma A}, i_{\Sigma B}), \bar{w}(i_{C\Sigma A}, (\iota_{\Sigma A} \vee 1_{\Sigma B}) i_{\Sigma B})\} = \{w(i_{\Sigma A}, i_{\Sigma B})\} = \theta$. Since $\partial_{\iota_{\Sigma A} \vee 1_{\Sigma B}}$ is a monomorphism we obtain $\{(w(i_{\Sigma A}, i_{\Sigma B}), \bar{w}(i_{C\Sigma A}, (\iota_{\Sigma A} \vee 1_{\Sigma B}) i_{\Sigma B})\} = \partial^{-1}_{\iota_{\Sigma A} \vee 1_{\Sigma B}}(\theta) = \theta_1 = \{(\bar{\rho}_{A,B} \bar{h}_{A,B}, a)\}$, thus we deduce $(\alpha, \beta)_{1*}(w(i_{\Sigma A} \vee i_{\Sigma B}), \bar{w}(i_{C\Sigma A}, (\iota_{\Sigma A} \vee 1_{\Sigma B}) i_{\Sigma B})\} = (\alpha, \beta)_{1*}(\theta_1) = [\alpha, \beta]_1$.

Hence we now proceed to prove

$$(\alpha, \beta)_{1*}\{(w(i_{\Sigma A} \vee i_{\Sigma B}), \bar{w}(i_{C\Sigma A}, (\iota_{\Sigma A} \vee 1_{\Sigma A}) i_{\Sigma B})\} = \{w(f, g), \bar{w}(f', kg)\}\,.$$

The left hand side of the above formula is $\{(\nabla \circ (f \vee g) \circ \bar{\rho}_{A,B} \bar{h}_{A,B}, \nabla \circ (f' \vee kg) \circ \bar{w}(i_{C\Sigma A}, (\iota_{\Sigma A} \vee 1_{\Sigma B}) i_{\Sigma B})$ and

$$\begin{aligned}
&\nabla \circ (f' \vee kg)\, \bar{w}(i_{C\Sigma A}, (\iota_{\Sigma A} \vee 1_{\Sigma B})\, i_{\Sigma B}) \\
&\qquad = \begin{cases} \nabla(f' \vee kg)\, \lambda_A(\iota_{\Sigma A} \vee 1_{\Sigma B}) i_{\Sigma B})(y, 2t), & 0 \leqq t \leqq 1/2\,, \\ \nabla(f' \vee kg)\, w(i_{C\Sigma A}\, \sigma'_{2t-1}, (\iota_{\Sigma A} \vee 1_{\Sigma B}) i_{\Sigma B}) y\,, & 1/2 \leqq t \leqq 1\,, \end{cases} \\
&\qquad = \begin{cases} \lambda_A(kg)(y, 2t)\,, & 0 \leqq t \leqq 1/2\,, \\ w(f' \sigma'_{2t-1}, kg)\, y\,, & 1/2 \leqq t \leqq 1\,, \end{cases} \\
&\qquad = \bar{w}(f', kg)(y, t)\,.
\end{aligned}$$

Therefore we have the desired result.

LEMMA (4.7). [2; Proposition 3.3].

$$[\beta, \alpha] = -(\Sigma\sigma)^*[\alpha, \beta], \quad \textit{for} \quad \alpha \in \pi(\Sigma A, X) \quad \textit{and} \quad \beta \in \pi(\Sigma B, X)\,,$$

where $\sigma: B\#A \to A\#B$ *is induced by a map* $B \times A \to A \times B$ *defined by* $(y, x) \to (x, y)$ *for* $x \in A$ *and* $y \in B$.

Consider maps $(\Sigma\sigma, C\Sigma\sigma) : \iota_{B,A} \to \iota_{A,B}$, $\begin{pmatrix} \Sigma\sigma & C\Sigma\sigma \\ C\Sigma\sigma & C^2\Sigma\sigma \end{pmatrix} : \Pi_{B,A} = (\iota_{B,A}, \iota_{C_{B,A}}) \to \Pi_{A,B} = (\iota_{A,B}, \iota_{C_{A,B}})$:

$$\begin{tikzcd}
\Sigma(B\#A) \arrow[r, "\Sigma\sigma"] \arrow[d, "\iota_{B,A}"] & \Sigma(A\#B) \arrow[d, "\iota_{A,B}"] \\
C\Sigma(B\#A) \arrow[r, "C\Sigma\sigma"] & C\Sigma(A\#B)
\end{tikzcd},
\quad
\begin{tikzcd}
\Sigma(B\#A) \arrow[rr] \arrow[dr] \arrow[dd] & & \Sigma(A\#B) \arrow[dr] \arrow[dd] & \\
& C\Sigma(B\#A) \arrow[rr] \arrow[dd] & & C\Sigma(A\#B) \arrow[dd] \\
C\Sigma(B\#A) \arrow[rr] \arrow[dr] \arrow[ur, phantom, "\Downarrow\Pi_{B,A}"] & & C\Sigma(A\#B) \arrow[dr] \arrow[ur, phantom, "\Downarrow\Pi_{A,B}"] & \\
& C^2\Sigma(B\#A) \arrow[rr] & & C^2\Sigma(A\#B)
\end{tikzcd}.$$

Then we have

THEOREM (4.8).

$$[\beta,\alpha]_1 = -(\Sigma\sigma, C\Sigma\sigma)^*[\alpha,\beta]_1 \qquad \textit{for} \quad \alpha\in\pi_1(\Sigma A,k) \quad \textit{and} \quad B\in\pi_1(\Sigma B,X),$$

$$[\beta,\alpha]_2 = -(\Sigma\sigma, C\Sigma\sigma)^*[\alpha,\beta]_2 \qquad \textit{for} \quad \alpha\in\pi_1(\Sigma A,k) \quad \textit{and} \quad B\in\pi_1(\Sigma B,k),$$

$$[\beta,\alpha]_3 = -\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[\alpha,\beta]_3 \qquad \textit{for} \quad \alpha\in\pi_1(\Sigma A,u) \quad \textit{and} \quad \beta\in\pi_1(\Sigma B,v),$$

$$[\beta,\alpha]_4 = -\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[\alpha,\beta]_4 \qquad \textit{for} \quad \alpha\in\pi_2(\Sigma A,\Phi) \quad \textit{and} \quad \beta\in\pi_2(\Sigma B,W),$$

$$[\beta,\alpha]_5 = -\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[\alpha,\beta]_5 \qquad \textit{for} \quad \alpha\in\pi_2(\Sigma A,\Phi) \quad \textit{and} \quad \beta\in\pi_1(\Sigma B,u),$$

$$[\beta,\alpha]_6 = -\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[\alpha,\beta]_6 \qquad \textit{for} \quad \alpha\in\pi_2(\Sigma A,\Phi) \quad \textit{and} \quad B\in\pi_2(\Sigma B,\Phi).$$

PROOF. We shall prove the first formula. Let $\alpha=\{(f,f')\}$ and $\beta=\{g\}$ and let

$$i_{1,\Sigma A}: \Sigma A\subset\Sigma A\vee\Sigma B, \qquad i_{1,\Sigma B}: \Sigma B\subset\Sigma A\vee\Sigma B, \qquad i_{1,C\Sigma A}: C\Sigma A\subset C\Sigma A\vee\Sigma B,$$

$$i_{2,\Sigma A}: \Sigma A\subset\Sigma B\vee\Sigma A, \qquad i_{2,\Sigma B}: \Sigma B\subset\Sigma B\vee\Sigma A, \qquad i_{2,C\Sigma A}: C\Sigma A\subset\Sigma B\vee C\Sigma A.$$

Consider maps

$$p_1 = \nabla\circ(f\vee g): \Sigma A\vee\Sigma B\longrightarrow X, \qquad q_1 = \nabla\circ(f'\vee kg): C\Sigma A\vee\Sigma B\longrightarrow Y,$$

$$p_2 = \nabla\circ(g\vee f): \Sigma B\vee\Sigma A\longrightarrow X, \qquad q_2 = \nabla\circ(kg\vee f'): \Sigma B\vee C\Sigma A\longrightarrow Y.$$

Let $\sigma_{U\times V}: V\times U\to U\times V$ be the map by $(v,u)\to(u,v)$ and let $\sigma_{U\vee V} = \sigma_{U\times V}|V\vee U: V\vee U\to U\vee V$, where U and V are any spaces. Then $(\Sigma\sigma_{A\vee B},\Sigma\sigma_{CA\vee B}): 1_{\Sigma B}\vee\iota_{\Sigma A}\to\iota_{\Sigma A}\vee 1_{\Sigma B}$ is a map of pair as follows:

$$\begin{array}{ccc}
\Sigma B\vee\Sigma A=\Sigma(B\vee A) & \xrightarrow{\Sigma\sigma_{A\vee B}} & \Sigma(A\vee B)=\Sigma A\vee\Sigma B\\
\big\downarrow{\scriptstyle 1_{\Sigma B}\vee\iota_{\Sigma A}} & & \big\downarrow{\scriptstyle \iota_{\Sigma A}\vee 1_{\Sigma B}}\\
\Sigma B\vee C\Sigma A=\Sigma(B\vee CA) & \xrightarrow{\Sigma\sigma_{CA\vee B}} & \Sigma(CA\vee B)=C\Sigma A\vee\Sigma B.
\end{array}$$

Let $\gamma_1=\{(i_{1,\Sigma A},i_{1,C\Sigma A})\}\in\pi_1(\Sigma A,\iota_{\Sigma A}\vee 1_{\Sigma B})$ and $\gamma_2=\{(i_{2,\Sigma A},i_{2,C\Sigma A})\}\in\pi_1(\Sigma A,1_{\Sigma B}\vee\iota_{\Sigma A})$. Then we have

(4. 8. 1) $(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*[\{i_{2,\Sigma B}\}, \gamma_2]_1 + (\Sigma\sigma, C\Sigma\sigma)^*[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2), \{i_{1,\Sigma B}\}]_1 = 0.$

This is proved as follows:

$$\begin{aligned}
&\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}\{(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*[\{i_{2,\Sigma B}\}, \gamma_2]_1 + (\Sigma\sigma, C\Sigma\sigma)^*[\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2), \{i_{1,\Sigma B}\}]_1\\
&\quad = [(\Sigma\sigma_{A\vee B})_*\{i_{2,\Sigma B}\}, (\Sigma\sigma_{A\vee B})_*\circ\partial_{1_{\Sigma B}\vee\iota_{\Sigma A}}(\gamma_2)]\\
&\qquad\qquad + (\Sigma\sigma)^*\circ\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2), \{i_{1,\Sigma B}\}]_1\\
&\quad = [\{i_{1,\Sigma B}\}, (\Sigma\sigma_{A\vee B})_*\circ\partial_{1_{\Sigma B}\vee\iota_{\Sigma A}}(\gamma_2)] + (\Sigma\sigma)^*[(\Sigma\sigma_{A\vee B})_*\circ\partial_{1_{\Sigma B}\vee\iota_{\Sigma A}}(\gamma_2), \{i_{1,\Sigma B}\}]\\
&\quad = 0 \quad (\text{by } (4.7)).
\end{aligned}$$

Since $\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}$ is monomorphic we obtain the desired formula.
Then we have

$$\begin{aligned}
(p_1, q_1)_*(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*[\{i_{2,\Sigma B}\}, \gamma_2]_1 &= (p_2, q_2)_*[\{i_{2,\Sigma B}\}, \gamma_2]_1\\
&= [p_2\{i_{2,\Sigma B}\}, (p_2, q_2)_*\gamma_2]_1 \quad (\text{by } (4.1))\\
&= [\beta, \alpha]_1 .
\end{aligned}$$

On the other hand

$$\begin{aligned}
&(p_1, q_1)_*(\Sigma\sigma, C\Sigma\sigma)^*[\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2), \{i_{1,\Sigma B}\}]_1\\
&\qquad = (\Sigma\sigma, C\Sigma\sigma)^*(p_1, q_1)_*[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2), \{i_{1,\Sigma B}\}]_1\\
&\qquad = (\Sigma\sigma, C\Sigma\sigma)^*[(p_1, q_1)_*(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2), p_{1*}\{i_{1,\Sigma B}\}]_1\\
&\qquad = (\Sigma\sigma, C\Sigma\sigma)^*[(p_1, q_1)_*\gamma_2, \beta]_1\\
&\qquad = (\Sigma\sigma, C\Sigma\sigma)^*[\alpha, \beta]_1 .
\end{aligned}$$

Hence, by (4.8.1) we have $[\beta, \alpha]_1 = -(\Sigma\sigma, C\Sigma\sigma)^*[\alpha, \beta]_1$.

Next we shall prove the third formula. Let $\alpha = \{(f, f')\}$ and $\beta = \{(g, g')$, and let $i_{1,C\Sigma B} : C\Sigma B \subset \Sigma A \vee C\Sigma B$ and $i_{2,C\Sigma B} : C\Sigma B \subset C\Sigma B \vee \Sigma A$. We consider the map $\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} : \Psi_{A,B} \to \Phi$, where $p_1 = \nabla\circ(f\vee g)$, $q_1 = \nabla\circ(f'\vee ug)$, $r_1 = \nabla\circ(vf\vee g')$ and $s_1 = \nabla\circ(v'f\vee u'g')$, $\Psi_{A,B} = \Psi$. Similarly we set $p_2 = \nabla\circ(g\vee f)$, $q_2 = \nabla\circ(ug\vee f')$, $r_2 = \nabla\circ(g'\vee vf)$ and $s_2 = \nabla\circ(u'g'\vee v'f)$.

Let $\Theta_{A,B}$, $\Theta_{B,A}$ be the maps of pair $(\Sigma(1_A\vee\iota_B), \Sigma(\iota_{CA}\vee\iota_B)) : \Sigma(\iota_A\vee 1_B) \to \Sigma(\iota_A \vee 1_{CB})$, $(\Sigma(\iota_B\vee 1_A), (\Sigma(\iota_B\vee 1_{CA})) : \Sigma(1_B\vee\iota_A) \to \Sigma(1_{CB}\vee\iota_B)$:

$$\begin{tikzcd}
\Sigma(A\vee B) \arrow[r, "\Sigma(\iota_A\vee 1_B)"] \arrow[d, "\Sigma(1_A\vee\iota_B)"'] \arrow[dr, phantom, "\Downarrow\Theta_{A,B}"] & \Sigma(CA\vee B) \arrow[d, "\Sigma(\iota_{CA}\vee\iota_B)"] \\
\Sigma(A\vee CB) \arrow[r, "\Sigma(\iota_A\vee 1_{CB})"] & \Sigma(CA\vee CB)
\end{tikzcd}, \qquad
\begin{tikzcd}
\Sigma(B\vee A) \arrow[r, "\Sigma(1_B\vee\iota_A)"] \arrow[d, "\Sigma(\iota_B\vee 1_A)"'] \arrow[dr, phantom, "\Downarrow\Theta_{B,A}"] & \Sigma(B\vee CA) \arrow[d, "\Sigma(\iota_B\vee 1_{CA})"] \\
\Sigma(CB\vee A) \arrow[r, "\Sigma(1_{CB}\vee\iota_B)"] & \Sigma(CB\vee CA)
\end{tikzcd}.$$

Then $\Theta_{A,B}$ and $\Theta_{B,A}$ are identified with $\Psi_{A,B}$ and $\Psi_{B,A}$ respectively, in the obvious way, where $\Psi_{B,A}=\Psi'$. Consider a map of pair of pair $\begin{pmatrix}\Sigma\sigma_{A\vee B} & \Sigma\sigma_{CA\vee B}\\ \Sigma\sigma_{A\vee CB} & \Sigma\sigma_{CA\vee CB}\end{pmatrix}$: $\Psi_{B,A}=\Theta_{B,A}\to\Theta_{A,B}=\Psi_{A,B}$, and let $\gamma_1=\{(i_{1,\Sigma A}, i_{1,C\Sigma A})\}\in\pi_1(\Sigma A, \iota_{\Sigma A}\vee 1_{\Sigma B})$, $\gamma_2=\{(i_{1,\Sigma B}, i_{1,C\Sigma B})\}\in\pi_1(\Sigma B, 1_{\Sigma A}\vee\iota_{\Sigma B})$, $\gamma_1'=\{(i_{2,\Sigma B}, i_{2,C\Sigma B})\}\in\pi_1(\Sigma B, \iota_{\Sigma B}\vee 1_{\Sigma A})$ and $\gamma_2'=\{(i_{2,\Sigma A}, i_{2,C\Sigma A})\}\in\pi_1(\Sigma A, 1_{\Sigma B}\vee\iota_{\Sigma A})$. Then we have

$$(4.8.2)\quad \begin{pmatrix}\Sigma\sigma_{A\vee B} & \Sigma\sigma_{CA\vee B}\\ \Sigma\sigma_{A\vee CB} & \Sigma\sigma_{CA\vee CB}\end{pmatrix}_*[\gamma_1', \gamma_2']_3+\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2'), \gamma_2]_3=0.$$

This is proved as follows:

$$\begin{aligned}
&\partial_{\Psi_{A,B}}\text{ (the left hand side of (4.8.2))}\\
&=\partial_{\Psi_{A,B}}[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{A\vee CB})_*(\gamma_1'), (\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2')]_3\\
&\qquad\qquad+(\Sigma\sigma, C\Sigma\sigma)^*\partial_{\Psi_{A,B}}[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2'), \gamma_2]_3\\
&=[\partial_{1_{\Sigma A}\vee\iota_{\Sigma B}}(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{A\vee CB})_*(\gamma_1'), (\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2')]_1\\
&\qquad\qquad+(\Sigma\sigma, C\Sigma\sigma)^*[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2'), \partial_{1_{\Sigma A}\vee\iota_{\Sigma B}}(\gamma_2)]_1\\
&=[(\Sigma\sigma_{A\vee B})_*\{i_{2,\Sigma B}\}, (\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2')]_1\\
&\qquad\qquad+(\Sigma\sigma, C\Sigma\sigma)^*[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})^*(\gamma_2'), \{i_{1,\Sigma B}\}]_1\\
&=[\{i_{1,\Sigma B}\}, (\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2')]_1+(\Sigma\sigma, C\Sigma\sigma)^*[(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2'), \{i_{1,\Sigma B}\}]_1\\
&=0.
\end{aligned}$$

Since $\partial_{\Psi_{A,B}}: \pi_2(\Sigma(B\#A), \Psi_{A,B})\to\pi_1(\Sigma(B\#A), \iota_{\Sigma A}\vee 1_{\Sigma B})$ is monomorphic we obtain the desired formula.

Next we have

$$\begin{aligned}
&\begin{pmatrix}p_1 & q_1\\ r_1 & s_1\end{pmatrix}_*\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[(\Sigma\sigma_{A,B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2'), \gamma_2]_3\\
&\qquad=\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*\circ\begin{pmatrix}p_1 & q_1\\ r_1 & s_1\end{pmatrix}_*[(\Sigma\sigma_{A,B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2'), \gamma_2]_3\\
&\qquad=\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[(p_1, q_1)_*(\Sigma\sigma_{A\vee B}, \Sigma\sigma_{CA\vee B})_*(\gamma_2'), \beta]_3\\
&\qquad=\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[(p_2, q_2)_*(\gamma_2'), \beta]_3
\end{aligned}$$

$$= \begin{pmatrix} \Sigma\sigma & C\Sigma\sigma \\ C\Sigma\sigma & C^2\Sigma\sigma \end{pmatrix}^* [\alpha, \beta]_3,$$

while we deduce

$$\begin{pmatrix} p_1 & q_1 \\ r_1 & s_2 \end{pmatrix}_* \begin{pmatrix} \Sigma\sigma_{A\vee B} & \Sigma\sigma_{CA\vee B} \\ \Sigma\sigma_{CA\vee B} & \Sigma\sigma_{CA\vee CB} \end{pmatrix}_* [\gamma_1', \gamma_2']_3 = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}_* [\gamma_1', \gamma_2']_3$$
$$= [(p_2, r_2)_*(\gamma_1'), (p_2, q_2)_*(\gamma_2')]_3$$
$$= [\beta, \alpha]_3 .$$

Hence we have $[\beta, \alpha]_3 = -\begin{pmatrix} \Sigma\sigma & C\Sigma\sigma \\ C\Sigma\sigma & C^2\Sigma\sigma \end{pmatrix}^* [\alpha, \beta]_3$ by (4. 8. 2).

If we choose appropriate universal examples, then other formulas are proved in the same way.

LEMMA (4.9) [10; Lemma 16. 5′]. *Let $f\colon X \to A\vee B$ be a map, and let $p_A(p_B)\colon A\vee B \to A(B)$ be projection. If $p_A \circ f \simeq *$ and $p_B f \simeq *$, then $\Sigma f \simeq *$.*

Let $\{f\} \in \pi(\Sigma(A\#B), X)$, $\{(g, g')\} \in \pi_1(\Sigma(A\#B), k\ )$ and $\left\{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right\}$ $\in \pi_2(\Sigma(A\#B), \Phi)$. Then suspension homomorphisms

$$\Sigma_X\colon \pi(\Sigma(A\#B), X) \longrightarrow \pi(\Sigma^2(A\#B), \Sigma X),$$
$$\Sigma_k\colon \pi_1(\Sigma(A\#B), k) \longrightarrow \pi_1(\Sigma^2(A\#B), \Sigma k),$$
$$\Sigma_\Phi\colon \pi_2(\Sigma(A\#B), \Phi) \longrightarrow \pi_2(\Sigma^2(A\#B), \Sigma\Phi)$$

are defined by $\Sigma_X\{f\} = \{\Sigma f\}$, $\Sigma_k\{(g, g')\} = \{(\Sigma g, \Sigma g')\}$ and $\Sigma_\Phi\left\{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\right\}$ $= \left\{\begin{pmatrix} \Sigma a_1 & \Sigma a_2 \\ \Sigma b_1 & \Sigma b_2 \end{pmatrix}\right\}$, respectively.

THEOREM (4.10). *All GWP's $[\alpha, \beta]$ and $[\alpha, \beta]_i$, $i = 1, 2, 3, 4, 5, 6$ are annihilated by suspension homomorphisms.*

PROOF. We now shall prove $\Sigma_X[\alpha, \beta] = 0$ for $\alpha \in \pi(\Sigma A, X)$ and $\beta \in \pi(\Sigma B, X)$.

We consider the commutative diagram

$$\begin{array}{ccccc} \Sigma(A\#B) & \xrightarrow{\bar\rho_{A,B}\bar h_{A,B}} & \Sigma A \vee \Sigma B & \xrightarrow{p_A} & \Sigma A \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle j} & & \| \\ C\Sigma(A\#B) & \xrightarrow{\rho_{A,B} h_{A,B}} & \Sigma A \times \Sigma B & \xrightarrow{\pi_A} & \Sigma A , \end{array}$$

where j is the inclusion map, and p_A and π_A are projections. Since $C\Sigma(A\#B)$ is contractible we have $\pi_A\circ(\rho_{A,B}h_{A,B})\circ\iota = p_A\circ(\bar\rho_{A,B}\bar h_{A,B})\simeq *$. Similarly we have $p_B\circ(\bar\rho_{A,B}\bar h_{A,B})\simeq *$. Thus we obtain $\Sigma(\bar\rho_{A,B}\bar h_{A,B})\simeq *$ by Lemma (4.9).

Let $\alpha=\{f\}$ and $\beta=\{g\}$. We consider the commutative diagram

$$\begin{array}{ccc}
\pi(\Sigma\Lambda.\ \Sigma A\vee\Sigma B) & \xrightarrow{(\alpha,\beta)_*} & \pi(\Sigma\Lambda, X)\\
\downarrow{\scriptstyle\Sigma} & & \downarrow{\scriptstyle\Sigma_X}\\
\pi(\Sigma^2\Lambda, \Sigma^2 A\vee\Sigma^2 B) & \xrightarrow{(\Sigma(\alpha,\beta))_*} & \pi(\Sigma^2\Lambda, \Sigma X)\,,
\end{array}$$

where $(\alpha,\beta)=\nabla\circ(f\vee g)$ and $\Sigma(\alpha,\beta)=\nabla\circ(\Sigma f\vee\Sigma g)$, and $\Sigma\Lambda=\Sigma(A\#B)$. Then $\Sigma_X[\alpha,\beta]=\Sigma_X(\alpha,\beta)_*(\theta)=(\Sigma(\alpha,\beta))_*\Sigma(\theta)=0$.

Next we shall prove $\Sigma_k[\alpha,\beta]_1=0$ for $\alpha=\{(f,f')\}\in\pi_1(\Sigma A,k)$ and $\beta=\{g\}\in\pi(\Sigma B,X)$ and $\Sigma_\Phi[\alpha,\beta]_3=0$ for $\alpha=\{(f,f')\}\in\pi_1(\Sigma A,u)$ and $\beta=\{(g,g')\}\in\pi_1(\Sigma B,v)$. We consider the commutative diagram

$$\begin{array}{ccccc}
\pi_2(\Sigma\Lambda,\Psi) & \xrightarrow{\partial_\Psi} & \pi_1(\Sigma\Lambda,\iota_{\Sigma A}\vee 1_{\Sigma B}) & \xrightarrow{\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}} & \pi(\Sigma\Lambda,\Sigma A\vee\Sigma B)\\
\downarrow{\scriptstyle\Sigma_\Psi} & & \downarrow{\scriptstyle\Sigma_{\iota_{\Sigma A}\vee 1_{\Sigma B}}} & & \downarrow{\scriptstyle\Sigma}\\
\pi_2(\Sigma^2\Lambda,\Sigma\Psi) & \xrightarrow{\partial_{\Sigma\Psi}} & \pi_1(\Sigma^2\Lambda,\Sigma(\iota_{\Sigma A}\vee 1_{\Sigma B})) & \xrightarrow{\partial_{\Sigma(\iota_{\Sigma A}\vee 1_{\Sigma B})}} & \pi(\Sigma^2\Lambda,\Sigma^2 A\vee\Sigma^2 B)\,.
\end{array}$$

Then $\partial_{\iota_{\Sigma A}\vee 1_{\Sigma B}}$, $\partial_{\Sigma(\iota_{\Sigma A}\vee 1_{\Sigma B})}$, ∂_Ψ and $\partial_{\Sigma\Psi}$ are monomorphisms by ((2.2) (iii)). From the commutativity of the above diagram and $\Sigma(\theta)=0$ we deduce $\Sigma_{\iota_{\Sigma A}\vee 1_{\Sigma B}}(\theta)_1=0$ and $\Sigma_\Psi(\theta_3)=0$, and we have $\Sigma_k[\alpha,\beta]_1=\Sigma_k\circ(\alpha,\beta)_{1*}(\theta_1)=(\Sigma(\alpha,\beta)_1)_*(\Sigma_{\iota_{\Sigma A}\vee 1_{\Sigma B}}(\theta_1))=0$, $\Sigma_\Phi[\alpha,\beta]_3=\Sigma_\Phi\circ(\alpha,\beta)_{3*}(\theta_3)=((\Sigma(\alpha,\beta)_3)_*(\Sigma_\Psi\theta_3)=0$, where $\Sigma(\alpha,\beta)_1=(\nabla\circ(\Sigma f\vee\Sigma g),\ \nabla\circ(\Sigma f'\vee\Sigma k\circ\Sigma g))$ and $\Sigma(\alpha,\beta)_3=\begin{pmatrix}\nabla(\Sigma f\vee g) & \nabla(\Sigma f'\vee\Sigma u\circ\Sigma g)\\ \nabla(\Sigma v\circ\Sigma f\vee\Sigma g') & \nabla(\Sigma v'\circ\Sigma f\vee\Sigma u'\circ\Sigma g')\end{pmatrix}$.

The other cases are proved similarly.

5. The Jacobi identity. In [2], Arkowitz described that it is possible to prove the appropriate Jacobi identity for GWP of elements $\alpha\in\pi(\Sigma A,X)$, $\beta\in\pi(\Sigma B,X)$ and $\gamma\in\pi(\Sigma C,X)$, when A,B and C are suspensions.

In this section we represent the Jacobi identity in the explicit form and we consider the appropriate Jacobi identities for other GWP's.

First, we recall the concept mentioned in [3]. Let $p_A\colon \Sigma A\times\Sigma B\to\Sigma A$ and $p_B\colon \Sigma A\times\Sigma B\to\Sigma B$ be projections, and let $p_A^*(p_B^*)\colon \pi(\Sigma A,\Omega X)(\pi(\Sigma B,\Omega X))\to\pi(\Sigma A\times\Sigma B,\Omega X)$ be the homomorphism induced by $p_A(p_B)$. And if we choose an $\alpha \in \pi(\Sigma A, \Omega X)$ and a $\beta \in \pi(\Sigma B, \Omega X)$ then these determine $p_A^*(\alpha)$, $p_B^*(\beta)$ $\in \pi(\Sigma A \times \Sigma B, \Omega X)$. We consider the commutator $(p_A^*(\alpha), p_B^*(\beta)) = (p_A^*(\alpha)^{-1} \cdot p_B^*(\beta)^{-1}) \cdot (p_A^*(\alpha) \cdot p_B^*(\beta))$ in $\pi(\Sigma A \times \Sigma B, \Omega X)$, and for the cofibration $\Sigma A \vee \Sigma B \to \Sigma A \times \Sigma B \to \Sigma A \# \Sigma B$ there exists an exact sequence

$$\pi(\Sigma A \# \Sigma B, \Omega X) \xrightarrow{q^*} \pi(\Sigma A \times \Sigma B, \Omega X) \xrightarrow{j^*} \pi(\Sigma A \vee \Sigma B, \Omega X),$$

where q is the projection. Then $j^*(p_A^*(\alpha), p_B^*(\beta)) = 0$ and q^* is a monomorphism [3], and the commutator product of $\alpha \in \pi(\Sigma A, \Omega X)$ and $\beta \in \pi(\Sigma B, \Omega X)$ is the element $<\alpha, \beta> \in \pi(\Sigma A \# \Sigma B, \Omega X)$ uniquely defined by

$$<\alpha, \beta> = q^{*-1}(p_A^*(\alpha), p_B^*(\beta)).$$

Let $p_1: \Sigma A \times \Sigma B \times \Sigma C \to \Sigma A$, $p_2: \Sigma A \times \Sigma B \times \Sigma C \to \Sigma B$ and $p_3: \Sigma A \times \Sigma B \times \Sigma C \to \Sigma C$ be projections. Now let $\{f\} = \alpha \in \pi(\Sigma A, \Omega X)$, $\{g\} = \beta \in \pi(\Sigma B, \Omega X)$ and $\{h\} = \gamma \in \pi(\Sigma C, \Omega X)$. Then we obtain maps $f \circ p_1 = f'$, $g \circ p_2 = g'$, $h \circ p_3 = h'$: $\Sigma A \times \Sigma B \times \Sigma C \to \Omega X$. We consider the map $a = (f', (g', h'))$: $\Sigma A \times \Sigma B \times \Sigma C \to \Omega X$, where $(g', h') = g'^{-1} \cdot h'^{-1} \cdot g' \cdot h'$ and the products and inverse in the commutator come from the loop space structure of ΩX. Then we see $a | T \simeq *$: $T \to \Omega X$ since $a | \Sigma A \times \Sigma B \times * \simeq *$, $a | \Sigma A \times * \times \Sigma C \simeq *$ and $a | * \times \Sigma B \times \Sigma C \simeq *$, where $T = \Sigma A \times \Sigma B \times * \cup \Sigma A \times * \times \Sigma C \cup * \times \Sigma B \times \Sigma C$. For the cofibration $T \xrightarrow{j_T} \Sigma A \times \Sigma B \times \Sigma C \xrightarrow{p_{1,2,3}} \Sigma A \# \Sigma B \# \Sigma C$ there exists an exact sequence

$$\pi(\Sigma A \# \Sigma B \# \Sigma C, \Omega X) \xrightarrow{p_{1,2,3}^*} \pi(\Sigma A \times \Sigma B \times \Sigma C, \Omega X) \xrightarrow{j_T^*} \pi(T, \Omega X),$$

where j_T is the inclusion map and $p_{1,2,3}$ is the projection. Then $p_{1,2,3}^*$ is a monomorphism [3: Proposition 8] and $j_T^*\{a\} = 0$. By using the method analogous to that in [14], we have

PROPOSITION (5.1). $<\alpha, <\beta, \gamma>> = p_{1,2,3}^{*-1}(p_1^* \alpha, (p_2^* \beta, p_3^* \gamma))$,

where $\alpha \in \pi(\Sigma A, \Omega X)$, $\beta \in \pi(\Sigma B, \Omega X)$ *and* $\gamma \in \pi(\Sigma C, \Omega X)$.

Let $\rho_{B,C,A}: \Sigma A \times \Sigma B \times \Sigma C \to \Sigma B \times \Sigma C \times \Sigma A$ and $\rho_{C,A,B}: \Sigma A \times \Sigma B \times \Sigma C \to \Sigma C \times \Sigma A \times \Sigma B$ be the maps given by $\rho_{B,C,A}(x, y, z) = (y, z, x)$ and $\rho_{C,A,B}(x, y, z) = (z, x, y)$ for $x \in \Sigma A$, $y \in \Sigma B$ and $z \in \Sigma C$, respectively. Then $\rho_{B,C,A}$ and $\rho_{C,A,B}$ induce

$$\rho'_{B,C,A}: \Sigma A \# \Sigma B \# \Sigma C \to \Sigma B \# \Sigma C \# \Sigma A, \quad \rho'_{C,A,B}: \Sigma A \# \Sigma B \# \Sigma C \to \Sigma C \# \Sigma A \# \Sigma B.$$

And we may prove the following propositions [14]:

PROPOSITION (5.2) $\rho'^*_{B,C,A}<\beta, <\gamma, \alpha>> = p^{*-1}_{1,2,3}(p^*_2(\beta), (p^*_3(\gamma), p^*_1(\alpha)),$

$\rho'^*_{C,A,B}<\gamma, <\alpha, \beta>> = p^{*-1}_{1,2,3}(p^*_3(\gamma), (p^*_1(\alpha), p^*_2(\beta)).$

PROPOSITION (5.3) *If* $\alpha \in \pi(\Sigma A, \Omega X)$, $\beta \in \pi(\Sigma B, \Omega X)$ *and* $\gamma \in \pi(\Sigma C, \Omega X)$, *then*

$$<\alpha, <\beta, \gamma>> + \rho'^*_{B,C,A}<\beta, <\gamma, \alpha>> + \rho'^*_{C,A,B}<\gamma, <\alpha, \beta>> = 0.$$

REMARK. Note that $(X\# Y)\# Z$ and $X\#(Y\# Z)$ are identified and denoted simply $X\# Y\# Z$.

We consider the well-known natural isomorphism $k_*: \pi(P, \Omega Q) \to \pi(\Sigma P, Q)$, defined by $K_*\{f\} = \{K(f)\}$, $K(f)(x, t) = f(x)(t)$ for $x \in P$ and $f: P \to \Omega Q$. Then it easily follows that $K_*<\alpha, \beta> = [K_*(\alpha), K_*(\beta)] \in \pi(\Sigma(\Sigma A\#\Sigma B), X)$, and hence from this fact and (5.3) we obtain the Jacobi identity for the absolute GWP:

THEOREM (5.4). *If* $\alpha \in \pi(\Sigma A, X)$, $\beta \in \pi(\Sigma B, X)$ *and* $\gamma \in \pi(\Sigma C, X)$ *and* A, B *and* C *are suspensions then*

$$[\alpha, [\beta, \gamma]] + (\Sigma\rho'_{B,C,A})^*[\beta, [\gamma, \alpha]] + (\Sigma\rho'_{C,A,B})^*[\gamma, [\alpha, \beta]] = 0.$$

Now we shall consider the Jacobi identies for the various GWP's defined in section 3. We put $\rho = \rho'_{B,C,A}$ and $\tau = \rho'_{C,A,B}$, and we consider the commutative diagrams

$$\begin{tikzcd}
\Sigma(A\#B\#C) \arrow[r, "\Sigma\rho"] \arrow[d, "\iota_{A,B,C}"] & \Sigma(B\#C\#A) \arrow[d, "\iota_{B,C,A}"] \\
C\Sigma(A\#B\#C) \arrow[r, "C\Sigma\rho"] & C\Sigma(B\#C\#A),
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma(A\#B\#C) \arrow[r, "\Sigma\tau"] \arrow[d, "\iota_{A,B,C}"] & \Sigma(C\#A\#B) \arrow[d, "\iota_{C,A,B}"] \\
C\Sigma(A\#B\#C) \arrow[r, "C\Sigma\tau"] & C\Sigma(C\#A\#B).
\end{tikzcd}$$

THEOREM (5.5). *If* $\alpha \in \pi_1(\Sigma A, k)$, $\beta \in \pi(\Sigma B, X)$ *and* $\gamma \in \pi(\Sigma C, X)$, *and if* A, B *and* C *are suspensions, then*

$$[\alpha, [\beta, \gamma]]_1 + (\Sigma\rho, C\Sigma\rho)^*[\beta, [\gamma, \alpha]_1]_1 + (\Sigma\tau, C\Sigma\tau)^*[\gamma, [\alpha, \beta]_1]_1 = 0.$$

PROOF. Let

$i_1: \Sigma A \subset \Sigma A \vee \Sigma B \vee \Sigma C$, $\quad i_2: \Sigma B \subset \Sigma A \vee \Sigma B \vee \Sigma C$,

$i_3: \Sigma C \subset \Sigma A \vee \Sigma B \vee \Sigma C$, $\quad i_{1,C}: C\Sigma A \subset C\Sigma A \vee \Sigma B \vee \Sigma C$,

$i\ : \Sigma A \vee \Sigma B \vee \Sigma C \subset C\Sigma A \vee \Sigma B \vee \Sigma C$, and let $\iota_1=\{i_1\}$, $\iota_2=\{i_2\}$, $\iota_3=\{i_3\}$, $\alpha=\{(f,f')\}$, $\beta=\{g\}$ and $\gamma=\{h\}$.

We now consider the commutative diagram

$$\begin{tikzcd} \Sigma A\vee\Sigma B\vee\Sigma C \arrow[r, "p"] \arrow[d, "i"] & X \arrow[d, "k"] \\ C\Sigma A\vee\Sigma B\vee\Sigma C \arrow[r, "q"] & Y, \end{tikzcd}$$

where $p=\nabla\circ(1\vee\nabla)\circ(f\vee g\vee h)$ and $q=\nabla\circ(1\vee\nabla)\circ(f'\vee kg\vee kh)$. If we choose $\iota=\{(i_1, i_{1,C})\}\in\pi_1(\Sigma A, i)$ then $[\iota,[\iota_2,\iota_3]]_1\in\pi_1(\Sigma(A\#B\#C), i)$, $[\iota_2,[\iota_3,\iota]_1]_1\in\pi_1(\Sigma(B\#C\#A), i)$ and $[\iota_3,[\iota,\iota_2]_1]_1\in\pi_1(\Sigma(C\#A\#B), i)$.

Now we shall prove the following formula

(5.5.1) $$[\iota,[\iota_2,\iota_3]]_1+(\Sigma\rho, C\Sigma\rho)^*[\iota_2,[\iota_3,\iota]_1]_1+(\Sigma\tau, C\Sigma\tau)^*[\iota_3,[\iota,\iota_2]_1]_1=0\,.$$

In the homotopy exact sequence

$$\longrightarrow\pi_1(\Sigma\Lambda, i)\xrightarrow{\partial_i}\pi(\Sigma\Lambda,\Sigma A\vee\Sigma B\vee\Sigma C)\xrightarrow{i_*}\pi(\Sigma\Lambda, C\Sigma A\vee\Sigma B\vee\Sigma C)\,,$$

where $\Sigma\Lambda=\Sigma(A\#B\#C)$, since i_* is onto, ∂_i is a monomorphism. Hence we obtain

$$\begin{aligned}&\partial_i([\iota,[\iota_2,\iota_3]]_1+(\Sigma\rho, C\Sigma\rho)^*[\iota_2,[\iota_3,\iota]_1]_1+(\Sigma\tau, C\Sigma\tau)^*[\iota_3,[\iota,\iota_2]_1]_1)\\ &\quad=[\partial_i\iota,[\iota_2,\iota_3]]+(\Sigma\rho)^*\partial_i[\iota_2,[\iota_3,\iota]_1]_1+(\Sigma\tau)^*\partial_i[\iota_3,[\iota,\iota_2]_1]_1\\ &\quad=[\iota_1,[\iota_2,\iota_3]]+(\Sigma\rho)^*[\iota_2,\partial_i[\iota_3,\iota]_1]+(\Sigma\tau)^*[\iota_3,\partial_i[\iota,\iota_2]_1]\\ &\quad=[\iota_1,[\iota_2,\iota_3]]+(\Sigma\rho)^*[\iota_2,[\iota_3,\iota_1]]+(\Sigma\tau)^*[\iota_3,[\iota_1,\iota_2]]\\ &\quad=0\quad(\text{by }(5.4)).\end{aligned}$$

Since ∂_i is a monomorphism we obtain (5.5.1).

Next we have

$$\begin{aligned}&(p,q)_*([\iota,[\iota_2,\iota_3]]_1+(\Sigma\rho, C\Sigma\rho)^*[\iota_2,[\iota_3,\iota]_1]_1+(\Sigma\tau, C\Sigma\tau)^*[\iota_3,[\iota,\iota_2]_1]_1)\\ &\quad=[(p,q)_*\iota, p_*[\iota_2,\iota_3]]_1+(\Sigma\rho, C\Sigma\rho)^*(p,q)^*[\iota_2,[\iota_3,\iota]_1]_1\\ &\qquad\qquad+(\Sigma\tau, C\Sigma\tau)^*(p,q)_*[\iota_3,[\iota,\iota_2]_1]_1\\ &\quad=[(p,q)_*\iota,[p_*\iota_2, p_*\iota_3]]_1+(\Sigma\rho, C\Sigma\rho)^*[p_*\iota_2,[p_*\iota_3,(p,q)_*\iota]_1]_1\\ &\qquad\qquad+(\Sigma\tau, C\Sigma\tau)^*[p_*\iota_3,[(p,q)_*\iota, p^*\iota_2]_1]_1\end{aligned}$$

$$= [\alpha, [\beta, \gamma]]_1 + (\Sigma\rho, C\Sigma\rho)^*[\beta, [\gamma, \alpha]_1]_1 + (\Sigma\tau, C\Sigma\tau)^*[\gamma, [\alpha, \beta]_1]_1$$
$$= 0 \quad (\text{by } (5.5.1)).$$

Thus we deduce the desired result.

If ρ and τ denote as before we have the maps $\begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}: \Pi_{A,B,C} \to \Pi_{B,C,A}$, $\begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}: \Pi_{A,B,C} \to \Pi_{C,A,B}$, where $\Pi_{A,B,C}$, $\Pi_{C,A,B}$ are maps

$$\begin{tikzcd} \Sigma(A\#B\#C) \arrow[r, "\iota_{A,B,C}"] \arrow[d, "\iota_{A,B,C}"'] \arrow[dr, phantom, "\Downarrow\Pi_{A,B,C}"] & C\Sigma(A\#B\#C) \arrow[d, "\iota_{C\Sigma}"] \\ C\Sigma(A\#B\#C) \arrow[r, "C\iota_{A,B,C}"] & C^2\Sigma(A\#B\#C), \end{tikzcd} \quad \begin{tikzcd} \Sigma(C\#A\#B) \arrow[r, "\iota_{C,A,B}"] \arrow[d, "\iota_{C,A,B}"'] \arrow[dr, phantom, "\Downarrow\Pi_{C,A,B}"] & C\Sigma(C\#A\#B) \arrow[d, "\iota_{C\Sigma}"] \\ C\Sigma(C\#A\#B) \arrow[r, "C\iota_{C,A,B}"] & C^2\Sigma(C\#A\#B). \end{tikzcd}$$

COROLLARY (5.6). *Let A, B and C be suspensions.*

(a) *If $\alpha \in \pi_1(\Sigma A, k)$, $\beta \in \pi_1(\Sigma B, k)$ and $\gamma \in \pi(\Sigma C, X)$, then*

$$[\alpha, [\beta, \gamma]_1]_2 + (\Sigma\rho, C\Sigma\rho)^*[\beta, [\gamma, \alpha]_1]_2 + (\Sigma\tau, C\Sigma\tau)^*[\gamma, [\alpha, \beta]_2]_1 = 0.$$

(b) *If $\alpha \in \pi_1(\Sigma A, k)$, $\beta \in \pi_1(\Sigma B, k)$ and $\gamma \in \pi_1(\Sigma C, k)$, then*

$$[\alpha, [\beta, \gamma]_2]_2 + (\Sigma\rho, C\Sigma\rho)^*[\beta, [\gamma, \alpha]_2]_2 + (\Sigma\tau, C\Sigma\tau)^*[\gamma, [\alpha, \beta]_2]_2 = 0.$$

(c) *If $\alpha \in \pi_1(\Sigma A, u)$, $\beta \in \pi_1(\Sigma B, v)$ and $\gamma \in \pi(\Sigma C, W)$, then*

$$[\alpha, [\beta, \gamma]_1]_3 + \begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^*[\beta, [\gamma, \alpha]_1]_3 + \begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*[\gamma, [\alpha, \beta]_3]_4 = 0.$$

(d) *If $\alpha \in \pi_1(\Sigma A, u)$, $\beta \in \pi_1(\Sigma B, u)$ and $\gamma \in \pi_1(\Sigma C, v)$, then*

$$[\alpha, [\beta, \gamma]_3]_5 + \begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^*[\beta, [\gamma, \alpha]_3]_5 + \begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*[\gamma, [\alpha, \beta]_2]_3 = 0.$$

(e) *If $\alpha \in \pi_2(\Sigma A, \Phi)$, $\beta \in \pi(\Sigma B, W)$ and $\gamma \in \pi(\Sigma C, W)$, then*

$$[\alpha, [\beta, \gamma]]_4 + \begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^*[\beta, [\gamma, \alpha]_4]_4 + \begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*[\gamma, [\alpha, \beta]_4]_4 = 0.$$

(f) *If $\alpha \in \pi_2(\Sigma A, \Phi)$, $\beta \in \pi_1(\Sigma B, u)$ and $\gamma \in \pi(\Sigma C, W)$, then*

$$[\alpha, [\beta, \gamma]_1]_5 + \begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^*[\beta, [\gamma, \alpha]_4]_5 + \begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*[\gamma, [\alpha, \beta]_5]_4 = 0.$$

(g) *If* $\alpha \in \pi_2(\Sigma A, \Phi)$, $\beta \in \pi_1(\Sigma B, u)$ *and* $\gamma \in \pi_1(\Sigma C, u)$, *then*

$$[\alpha,[\beta,\gamma]_2]_5 + \begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^* [\beta,[\gamma,\alpha]_5]_5 + \begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^* [\gamma,[\alpha,\beta]_5]_5 = 0.$$

(h) *If* $\alpha \in \pi_2(\Sigma A, \Phi)$, $\beta \in \pi_2(\Sigma B, \Phi)$ *and* $\gamma \in \pi(\Sigma C, W)$, *then*

$$[\alpha,[\beta,\gamma]_4]_6 + \begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^* [\beta,[\gamma,\alpha]_4]_6 + \begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^* [\gamma,[\alpha,\beta]_6]_4 = 0.$$

(i) *If* $\alpha \in \pi_2(\Sigma A, \Phi)$, $\beta \in \pi_2(\Sigma B, \Phi)$ *and* $\gamma \in \pi_1(\Sigma C, u)$, *then*

$$[\alpha,[\beta,\gamma]_5]_6] + \begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^* [\beta,[\gamma,\alpha]_5]_6 + \begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^* [\gamma,[\alpha,\beta]_6]_5 = 0.$$

(j) *If* $\alpha \in \pi_2(\Sigma A, \Phi)$, $\beta \in \pi_2(\Sigma B, \Phi)$ *and* $\gamma \in \pi_2(\Sigma C, \Phi)$, *then*

$$[\alpha,[\beta,\gamma]_6]_6 + \begin{pmatrix}\Sigma\rho & C\Sigma\rho \\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^* [\beta,[\gamma,\alpha]_6]_6 + \begin{pmatrix}\Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^* [\gamma,[\alpha,\beta]_6]_6 = 0.$$

PROOF. We shall prove only (e), then the other formulas may be proved similarly when we consider the appropriate universal examples.

Let

$i_1 : \Sigma A \subset \Sigma A \vee \Sigma B \vee \Sigma C$, $i_2 : \Sigma B \subset \Sigma A \vee \Sigma B \vee \Sigma C$, $i_3 : \Sigma C \subset \Sigma A \vee \Sigma B \vee \Sigma C$,

$i_{1,C}: C\Sigma A \subset C\Sigma A \vee \Sigma B \vee \Sigma C$ and $i_{1,C^2}: C^2\Sigma A \subset C^2\Sigma A \vee \Sigma B \vee \Sigma C$,

and let $\alpha = \left\{\begin{pmatrix} f & f' \\ f'' & f''' \end{pmatrix}\right\}$, $\beta = \{g\}$ and $\gamma = \{h\}$.

Let Λ' be a map of pair $(\iota_{\Sigma A} \vee 1_{\Sigma B} \vee 1_{\Sigma C}, \iota_{C\Sigma A} \vee 1_{\Sigma B} \vee 1_{\Sigma C}) : \iota_{\Sigma A} \vee 1_{\Sigma B} \vee 1_{\Sigma C} \to C\iota_{\Sigma A} \vee 1_{\Sigma B} \vee 1_{\Sigma C}$:

$$\begin{tikzcd}
\Sigma A \vee \Sigma B \vee \Sigma C \arrow[r, "\iota_{\Sigma A} \vee 1_{\Sigma B} \vee 1_{\Sigma C}"] \arrow[d, "\iota_{\Sigma A} \vee 1_{\Sigma B} \vee 1_{\Sigma C}"'] \arrow[dr, phantom, "\Downarrow \Lambda'"] & C\Sigma A \vee \Sigma B \vee \Sigma C \arrow[d, "\iota_{C\Sigma A} \vee 1_{\Sigma B} \vee 1_{\Sigma C}"] \\
C\Sigma A \vee \Sigma B \vee \Sigma C \arrow[r, "C\iota_{\Sigma A} \vee 1_{\Sigma B} \vee 1_{\Sigma C}"] & C^2\Sigma A \vee \Sigma B \vee \Sigma C
\end{tikzcd}$$

Consider the map: $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$: $\Lambda' \to \Phi$, where $p = \nabla \circ (1 \vee \nabla) \circ (f \vee g \vee h)$, $q = \nabla \circ (1 \vee \nabla) \circ (f' \vee ug \vee uh)$, $r = \nabla \circ (1 \vee \nabla) \circ (f'' \vee vg \vee vh)$ and $s = \nabla \circ (1 \vee \nabla) \circ (f''' \vee v'ug \vee u'vg)$. Let $\iota_2 = \{i_2\}$, $\iota_3 = \{i_3\}$ and $\iota = \left\{\begin{pmatrix} i_1 & i_{1,C} \\ i_{1,C} & i_{1,C^2} \end{pmatrix}\right\} \in \pi_2(\Sigma A, \Lambda')$, then $[\iota,[\iota_2,\iota_3]]_4\in\pi_2(\Sigma(A\#B\#C),\Lambda')$, $[\iota_2,[\iota_3,\iota]_4]_4\in\pi_2(\Sigma(B\#C\#A),\Lambda')$, $[\iota_3,[\iota,\iota_2]_4]_4\in\pi_2(\Sigma(C\#A\#B),\Lambda')$.

We now shall prove the following formula

$$(5.6.1)\qquad [\iota,[\iota_2,\iota_3]]_4+\begin{pmatrix}\Sigma\rho & C\Sigma\rho\\ C\Sigma\rho & C^2\Sigma\rho\end{pmatrix}^*[\iota_2,[\iota_3,\iota]_4]_4+\begin{pmatrix}\Sigma\tau & C\Sigma\tau\\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*[\iota_3,[\iota,\iota_2]_4]_4=0\,.$$

In the homotopy sequence

$$\longrightarrow \pi_2(\Sigma\Lambda,\Lambda')\xrightarrow{\partial_{\Lambda'}}\pi_1(\Sigma\Lambda,\iota_{\Sigma A}\vee 1_{\Sigma B}\vee 1_{\Sigma C})$$
$$\xrightarrow{(\iota_{\Sigma A}\vee 1_{\Sigma B}\vee 1_{\Sigma C},\,\iota_{C\Sigma A}\vee 1_{\Sigma B}\vee 1_{\Sigma C})_*}\pi_1(\Sigma\Lambda,C\iota_{\Sigma A}\vee 1_{\Sigma B}\vee 1_{\Sigma C})\,,$$

where $\Sigma\Lambda=\Sigma(A\#B\#C)$, $\partial_{\Lambda'}$ is an isomorphism since $\pi_n(\Sigma\Lambda,C\iota_{\Sigma A}\vee 1_{\Sigma B}\vee 1_{\Sigma C})=0$, $n\geqq 1$. Then we obtain

$$\partial_{\Lambda'}([\iota,[\iota_2,\iota_3]]_4+\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[\iota_2,[\iota_3,\iota]_4]_4+\begin{pmatrix}\Sigma\tau & C\Sigma\tau\\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*[\iota_3,[\iota,\iota_2]_4]_4)$$
$$=[\partial_{\Lambda'}\iota,[\iota_2,\iota_3]]_1+(\Sigma\sigma,C\Sigma\sigma)^*\partial_{\Lambda'}[\iota_2,[\iota_3,\iota]_4]_4+(\Sigma\tau,C\Sigma\tau)^*\partial_{\Lambda'}[\iota_3,[\iota,\iota_2]_4]_4$$
$$=[\partial_{\Lambda'}\iota,[\iota_2,\iota_3]]_1+(\Sigma\sigma,C\Sigma\sigma)^*[\iota_2,[\iota_3,\partial_{\Lambda'}\iota]_1]_1+(\Sigma\tau,C\Sigma\tau)^*[\iota_3,[\partial_{\Lambda'}\iota,\iota_2]_1]_1$$
$$=0\quad(\text{by }(5.5)).$$

Since $\partial_{\Lambda'}$ is the isomorphism we obtain (5.6.1).
By using (5.6.1) we have

$$\begin{pmatrix}p & q\\ r & s\end{pmatrix}_*\left([\iota,[\iota_2,\mu_3]]_4+\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[\iota_2,[\iota_3,\iota]_4]_4+\begin{pmatrix}\Sigma\tau & C\Sigma\tau\\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*[\iota_3,[\iota,\iota_2]_4]_4\right)$$
$$=[\begin{pmatrix}p & q\\ r & s\end{pmatrix}_*\iota,p_*[\iota_2,\iota_3]]_4+\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*\begin{pmatrix}p & q\\ r & s\end{pmatrix}_*[\iota_2,[\iota_3,\iota_4]_4]_4$$
$$+\begin{pmatrix}\Sigma\tau & C\Sigma\tau\\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*\begin{pmatrix}p & q\\ r & s\end{pmatrix}_*[\iota_3,[\iota,\iota_2]_4]_4$$
$$=[\alpha,[p_*\iota_2,p_*\iota_3]]_4+\begin{pmatrix}\Sigma\sigma & C\Sigma\sigma\\ C\Sigma\sigma & C^2\Sigma\sigma\end{pmatrix}^*[p_*\iota_2,[p_*\iota_3,\begin{pmatrix}p & q\\ r & s\end{pmatrix}_*\iota]_4]_4$$
$$+\begin{pmatrix}\Sigma\tau & C\Sigma\tau\\ C\Sigma\tau & C^2\Sigma\tau\end{pmatrix}^*[p_*\iota_3,[\begin{pmatrix}p & q\\ r & s\end{pmatrix}_*\iota,p_*\iota_2]_4]_4$$

$$= [\alpha, [\beta, \gamma]]_4 + \begin{pmatrix} \Sigma\sigma & C\Sigma\sigma \\ C\Sigma\sigma & C^2\Sigma\sigma \end{pmatrix}^* [\beta, [\gamma, \alpha]_4]_4 + \begin{pmatrix} \Sigma\tau & C\Sigma\tau \\ C\Sigma\tau & C^2\Sigma\tau \end{pmatrix}^* [\gamma, [\alpha, \beta]_4]_4$$

$$= 0 .$$

Thus we deduce the desired formula.

6. The generalized Hopf invariant of a composition element. In this section, we generalize [8; Theorem 1] to the form of Theorem (6.3).

Throughout this section, we assume that all spaces are finite CW-complexes. We recall that

$$\pi(\Sigma A, \Sigma X_1 \vee \Sigma X_2) \cong \pi(\Sigma A, \Sigma X_1) + \pi(\Sigma A, \Sigma X_2) + \pi_1(\Sigma A, j_X) ,$$

where A is a suspension space and j_X: $\Sigma X_1 \vee \Sigma X_2 \subset \Sigma X_1 \times \Sigma X_2$, $X = X_1 = X_2$. Then $\pi(\Sigma A, \Sigma X_\lambda)$, $\lambda = 1, 2$, is embedded in $\pi(\Sigma A, \Sigma X_1 \vee \Sigma X_2)$ by the injection, and $\pi_1(\Sigma A, j_X)$, is embedded in $\pi(\Sigma A, \Sigma X_1 \vee \Sigma X_2)$ by the boundary monomorphism ∂_{j_X}: $\pi_1(\Sigma A, j_X) \to \pi(\Sigma A, \Sigma X_1 \vee \Sigma X_2)$.

Let ϕ_*: $\pi(\Sigma A, \Sigma X) \to \pi(\Sigma A, \Sigma X_1 \vee \Sigma X_2)$ be the homomorphism induced by the structure map ϕ: $\Sigma X \to \Sigma X \vee \Sigma X$. If ι_λ is the class of the identity map $\Sigma X \to \Sigma X$, $\iota_\lambda \in \pi(\Sigma X, \Sigma X)$, regarded as embedded in $\pi(\Sigma X, \Sigma X_1 \vee \Sigma X_2)$, $\lambda = 1, 2$, and if $\alpha \in \pi(\Sigma A, \Sigma X)$ then we have easily

(6.1) $$\phi_*(\alpha) = (\iota_1 + \iota_2) \circ \alpha ,$$

where $\circ$ is the composite operator.

Let $\{f_\lambda\} = \beta_\lambda \in \pi(\Sigma A, X)$, $\lambda = 1, 2$, and let (β_1, β_2) be the class of maps $\Sigma A_1 \vee \Sigma A_2 \to X$ ($A_\lambda = A$, $\lambda = 1, 2$) which agree on ΣA_λ with a representative of β_λ, i.e., $(\beta_1, \beta_2) = \{\nabla \circ (f_1 \vee f_2)\}$.

LEMMA (6.2). *If* $\gamma \in \pi(\Sigma A, \Sigma B)$ *and* $\beta_\lambda \in \pi(\Sigma B, \Sigma D)$, *then*

$$(\beta_1 + \beta_2) \circ \gamma = \beta_1 \circ \gamma + \beta_2 \circ \gamma + (\beta_1, \beta_2) \circ \partial_{j_B} \omega \phi_*(\gamma) ,$$

where ω: $\pi(\Sigma A, \Sigma B \vee \Sigma B) \to \pi_1(\Sigma A, j_B)$ *is the homomorphism such that* $\omega \partial_{j_B} = 1$ [1], *and* A *is a suspension*.

Proof is analogous to the proof of Lemma 2 in [8].

Now we consider the generalized Hopf invariant defined by $H^* = \varepsilon_j \omega \phi_*$:

$$\pi(\Sigma A, \Sigma B) \longrightarrow \pi(\Sigma A, \Sigma B \vee \Sigma B) \longrightarrow \pi_1(\Sigma A, j_B) \longrightarrow \pi_1(\Sigma A, \Sigma B \# \Sigma B) ,$$

where A is a suspension, and ε_j is the excision homomorphism (c.f. [1]).

In this section, we have the following theorem.:

THEOREM (6.3). *Let* $\beta \in \pi(\Sigma B, \Sigma D)$ *and* $\gamma \in \pi(\Sigma A, \Sigma B)$. *If* A *and* B *are suspensions then*

$$H^*(\beta \circ \gamma) = H^*(\beta) \circ \Sigma\gamma + (\beta \# \beta) \circ H^*(\gamma).$$

PROOF. By using Lemma (6.2) as in [8] we have

$$\begin{aligned}\partial_{j_D}\omega\phi_*(\beta \circ \gamma) &= \partial_{j_D}\omega\phi_*(\beta) \circ \gamma + (\iota_1 \circ \beta, \iota_2 \circ \beta) \circ \partial_{j_B}\omega\phi_*(\gamma) \\ &\quad + (\iota_1 \circ \beta + \iota_2 \circ \beta, \partial_{j_D}\omega\phi_*(\beta)) \circ \partial_{j_B}\omega\phi_*(\gamma).\end{aligned}$$

Since $H^* = \varepsilon_j\omega\phi_*$ we obtain

$$\begin{aligned}H^*(\beta \circ \gamma) &= \varepsilon_{j_D}\partial_{j_D}^{-1}(\partial_{j_D}\omega\phi_*(\beta) \circ \gamma) + \varepsilon_{j_D}\partial_{j_D}^{-1}((\iota_1 \circ \beta, \iota_2 \circ \beta) \circ \partial_{j_B}\omega\phi_*(\gamma)) \\ &\quad + \varepsilon_{j_D}\partial_{j_D}^{-1}((\iota_1 \circ \beta + \iota_2 \circ \beta, \partial_{j_D}\omega\phi_*(\beta)) \circ \partial_{j_B}\omega\phi_*(\gamma)).\end{aligned}$$

(a) We shall prove $\varepsilon_{jD}\partial_{jD}^{-1}(\partial_{jD}\,\omega\phi_*(\beta) \circ \gamma) = H^*(\beta) \circ \Sigma\gamma$.

Let $\gamma = \{f\} \in \pi(\Sigma A, \Sigma B)$, and we consider the commutative diagram

$$\begin{tikzcd} \Sigma A \arrow[r, "f"] \arrow[d, "\iota_{\Sigma A}"] & \Sigma B \arrow[d, "\iota_{\Sigma B}"] \\ C\Sigma A \arrow[r, "Cf"] \arrow[d, "p_{\Sigma A}"] & C\Sigma B \arrow[d, "p_{\Sigma B}"] \\ \Sigma^2 A \arrow[r, "\Sigma f"] & \Sigma^2 B \end{tikzcd},$$

where $p_{\Sigma A}$ and $p_{\Sigma B}$ are projections. Then we have the following diagram

$$\begin{tikzcd} & \pi_1(\Sigma B, \Sigma D \# \Sigma D) \arrow[r, "(\Sigma f)^*"] & \pi_1(\Sigma A, \Sigma D \# \Sigma D) \\ & \pi_1(\Sigma B, j_D) \arrow[r, "(f, Cf)^*"] \arrow[u, "\varepsilon_{j_D}"] \arrow[d, shift left, "\partial_{j_D}"] & \pi_1(\Sigma A, j_D) \arrow[u, "\varepsilon_{j_D}"] \arrow[d, "\partial_{j_D}"] \\ \pi(\Sigma B, \Sigma D) \arrow[r, "\phi_*"] \arrow[uur, "H^*"] & \pi(\Sigma B, \Sigma D \vee \Sigma D) \arrow[r, "f^*"] \arrow[u, shift left, "\omega"] & \pi(\Sigma A, \Sigma D \vee \Sigma D) \end{tikzcd},$$

where each ∂_{j_D} is a monomorphism and ω is an epimorphism. For $\beta \in \pi(\Sigma B, \Sigma D)$ we have $(\Sigma f)^* H^*(\beta) = (\Sigma f)^*(\varepsilon_{j_D}\omega\phi_*(\beta)) = \varepsilon_{j_D}(f, Cf)^*(\omega\phi_*(\beta)) = \varepsilon_{j_D}\partial_{j_D}^{-1} f^* \partial_{j_D}(\omega\phi_*(\beta))$. Hence we obtain $H^*(\beta)\circ\Sigma\gamma = \varepsilon_{j_D}\partial_{j_D}^{-1}(\partial_{j_D}\omega\phi_*(\beta)\circ\gamma)$.

(b) We shall prove $\varepsilon_{j_D}\partial_{j_D}^{-1}((\iota_1\circ\beta, \iota_2\circ\beta)\circ\partial_{j_B}\omega\phi_*(\gamma)) = (\beta\#\beta)\circ H^*(\gamma)$.

Let $\beta = \{g\} \in \pi(\Sigma B, \Sigma D)$, then $(\iota_1\circ\beta, \iota_2\circ\beta) = \beta\vee\beta = \{g\vee g\} \in \pi(\Sigma B\vee\Sigma B, \Sigma D\vee\Sigma D)$, and we consider the commutative diagram

$$\begin{array}{ccc}
\Sigma B\vee\Sigma B & \xrightarrow{g\vee g} & \Sigma D\vee\Sigma D \\
\downarrow{\scriptstyle j_B} & & \downarrow{\scriptstyle j_D} \\
\Sigma B\times\Sigma B & \xrightarrow{g\times g} & \Sigma D\times\Sigma D \\
\downarrow{\scriptstyle p_B} & & \downarrow{\scriptstyle p_D} \\
\Sigma B\#\Sigma B & \xrightarrow{g\# g} & \Sigma D\#\Sigma D\,.
\end{array}$$

We have the following diagram

$$\begin{array}{ccccc}
 & & \pi_1(\Sigma A, \Sigma B\#\Sigma B) & \xrightarrow{(g\# g)_*} & \pi_1(\Sigma A, \Sigma D\vee\Sigma D) \\
 & \nearrow{\scriptstyle H^*} & \uparrow{\scriptstyle \varepsilon_{j_B}} & & \uparrow{\scriptstyle \varepsilon_{j_D}} \\
 & & \pi_1(\Sigma A, j_B) & \xrightarrow{(g\vee g,\, g\times g)_*} & \pi_1(\Sigma A, j_D) \\
 & & {\scriptstyle\omega}\uparrow\downarrow{\scriptstyle \partial_{j_B}} & & \downarrow{\scriptstyle \partial_{j_D}} \\
\pi(\Sigma A, \Sigma B) & \xrightarrow{\phi_*} & \pi(\Sigma A, \Sigma B\vee\Sigma B) & \xrightarrow{(g\vee g)_*} & \pi(\Sigma A, \Sigma D\vee\Sigma D)\,.
\end{array}$$

For $\gamma\in\pi(\Sigma A, \Sigma B)$ we obtain $\varepsilon_{j_D}\partial_{j_D}^{-1}((g\vee g)_*(\partial_{j_B}\omega\phi_*(\gamma)) = \varepsilon_{j_D}(g\vee g, g\times g)_*\omega\phi_*(\gamma) = (g\# g)_*\varepsilon_{j_D}\omega\phi_*(\gamma) = (g\# g)_* H^*(\gamma)$. Thus we deduce $\varepsilon_{j_D}\partial_{j_D}^{-1}((\iota_1\circ\beta, \iota_2\circ\beta)\circ\partial_{j_B}\omega\phi_*(\gamma)) = (\beta\#\beta)\circ H^*(\gamma)$.

Finally we shall prove that

(c) $\varepsilon_{j_D}\partial_{j_D}^{-1}((\iota_1\circ\beta + \iota_2\circ\beta, \partial_{j_D}\omega\phi_*(\beta))\circ\partial_{j_B}\omega\phi_*(\gamma)) = 0\,.$

It is sufficient to prove that $\varepsilon_{j_D}\partial_{j_D}^{-1}(\beta_1+\beta_2, \partial_{j_D}\omega\phi_*(\beta))_*(\partial_{j_B}\omega\phi_*(\gamma)) = 0$, where $\beta_1 = \iota_1\circ\beta$, $\beta_2 = \iota_2\circ\beta$. We set $\xi = \beta_1+\beta_2$, $\eta = \partial_{j_D}\omega\phi_*(\beta)$ and $\zeta = \partial_{j_B}\omega\phi_*(\gamma)$ and again we denote by ξ, η and ζ these representatives. Since $\eta = \partial_{j_D}\omega\phi_*(\beta) \in \pi(\Sigma B, \Sigma D\vee\Sigma D)$ for $\beta\in\pi(\Sigma B, \Sigma D)$ we have $j_{D*}(\eta) = 0$ and hence $j_D\eta \simeq *$, and we obtain the following diagram

$$
\begin{tikzcd}
\Sigma B\vee\Sigma B \arrow[r, "\xi\vee\eta"] \arrow[rrr, bend right, "(\xi,\eta)"'] \arrow[d, equal] & (\Sigma D\vee\Sigma D)\vee(\Sigma D\vee\Sigma D) \arrow[rr, "\nabla"] & & \Sigma D\vee\Sigma D \arrow[d, "j_D"] \\
\Sigma B\vee\Sigma B \arrow[r, "\xi\vee *"] \arrow[d, "j_B"] & (\Sigma D\vee\Sigma D)\vee * \arrow[r, "\nabla"] & \Sigma D\vee\Sigma D \arrow[r, "j_D"] & \Sigma D\times\Sigma D \arrow[d, equal] \\
\Sigma B\times\Sigma B \arrow[rrr, "k"] & & & \Sigma D\times\Sigma D
\end{tikzcd}
$$

the upper diagram is homotopy commutative and the low diagram is commutative, where k is defined by $k(x,y)=j_D(\xi,*)(x\vee y)$ for $x,y\in\Sigma B$. Then $j_D(\xi,\eta)\simeq kj_B$, and since j_B is a cofibration there exists k': $\Sigma B\times\Sigma B\to\Sigma D\times\Sigma D$ such that $j_D(\xi,\eta)=k'j_B$, and we have the following commutative diagram

$$
\begin{tikzcd}
\Sigma B\vee\Sigma B \arrow[r, "(\xi,\eta)"] \arrow[d, "j_B"] & \Sigma D\vee\Sigma D \arrow[d, "j_D"] \\
\Sigma B\times\Sigma B \arrow[r, "k'"] \arrow[d, "p_B"] & \Sigma D\times\Sigma D \arrow[d, "p_D"] \\
\Sigma B\#\Sigma B \arrow[r, "\tilde{k}"] & \Sigma D\#\Sigma D
\end{tikzcd}
,
$$

where $\tilde{k}$ is determined by (ξ,η) and k'. Then we obtain the diagram

(6.3.1)
$$
\begin{tikzcd}
 & \pi_1(\Sigma A,\Sigma B\#\Sigma B) \arrow[r, "\tilde{k}_*"] & \pi_1(\Sigma A,\Sigma D\#\Sigma D) \\
 & \pi_1(\Sigma A,j_B) \arrow[u, "\varepsilon_{j_B}"] \arrow[r, "((\xi,\eta),k')_*"] \arrow[d, shift left, "\partial_{j_B}"] & \pi_1(\Sigma A,j_D) \arrow[u, "\varepsilon_{j_D}"'] \arrow[d, "\partial_{j_D}"] \\
\pi(\Sigma A,\Sigma B) \arrow[ruu, "H^*"] \arrow[r, "\phi_*"] & \pi(\Sigma A,\Sigma B\vee\Sigma B) \arrow[r, "(\xi,\eta)_*"] \arrow[u, shift left, "\omega"] & \pi(\Sigma A,\Sigma D\vee\Sigma D)
\end{tikzcd}
.
$$

Consider the commutative diagram

$$\begin{array}{ccc}
\pi_1(\Sigma(B\#B), \Sigma B\#\Sigma B) & \xrightarrow{k_*} & \pi_1(\Sigma(B\#B), \Sigma D\#\Sigma D) \\
\uparrow \varepsilon_{j_B} & & \uparrow \varepsilon_{j_D} \\
\pi_1(\Sigma(B\#B), j_B) & \xrightarrow{((\xi,\eta),k')_*} & \pi_1(\Sigma(B\#B), j_D) \\
\downarrow \partial_{j_B} & & \downarrow \partial_{j_D} \\
\pi(\Sigma(B\#B), \Sigma B\vee\Sigma B) & \xrightarrow{(\xi,\eta)_*} & \pi(\Sigma(B\#B), \Sigma D\vee\Sigma D) \\
\downarrow j_{B*} & & \downarrow j_{D*} \\
\pi(\Sigma(B\#B), \Sigma B\times\Sigma B) & \xrightarrow{k'_*} & \pi(\Sigma(B\#B), \Sigma D\times\Sigma D)\ .
\end{array} \tag{6.3.2}$$

If we choose $\{(\bar{\rho h}, \rho h)\} \in \pi_1(\Sigma(B\#B), j_B)$ then $\varepsilon_{j_B}\{(\bar{\rho h}, \rho h)\} = \{t\} \in \pi_1(\Sigma(B\#B), \Sigma B\#\Sigma B)$ where $t: \Sigma^2(B\#B) \to \Sigma B\#\Sigma B$ is a homotopy equivalence (c.f. [1]), and $\partial_{j_B}\{(\bar{\rho h}, \rho h)\} = \{\bar{\rho h}\} = \theta$. Since $j_{B*}(\theta) = 0$ ([12; Theorem 2.2]) we have $0 = k'_* j_{B*}(\theta) = j_{D*}(\xi, \eta)_*(\theta) = j_{D*}[\xi, \eta]$, and hence there exists $\alpha \in \pi_1(\Sigma(B\#B), j_D)$ such that $\partial_{j_D}\alpha = [\xi, \eta]$.

Next we consider the GWP $[\xi, \omega\phi_*(\beta)]_1 \in \pi_1(\Sigma(B\#B), j_D)$ of $\xi \in \pi(\Sigma B, \Sigma D\vee\Sigma D)$ and $\omega\phi_*(\beta) \in \pi_1(\Sigma B, j_D)$, then $\partial_{j_D}[\xi, \omega\phi_*(\beta)]_1 = [\xi, \partial_{j_D}\omega\phi_*(\beta)] = [\xi, \eta] = \partial_{j_D}\alpha$. Since ∂_{j_D} is a monomorphism we have

$$\alpha = [\xi, \omega\varphi_*(\beta)]_1\,. \tag{6.3.3}$$

By the commutativity of (6.3.2)

$$\begin{aligned}
\varepsilon_{j_D}((\xi, \eta), k'_*\{(\bar{\rho h}, \rho h)\} &= \varepsilon_{j_D}\partial_{j_D}^{-1}(\xi, \eta)_*\partial_{j_B}\{(\bar{\rho h}, \rho h)\} \\
&= \varepsilon_{j_D}\partial_{j_D}^{-1}(\xi, \eta)_*(\theta) \\
&= \varepsilon_{j_D}\partial_{j_D}^{-1}[\xi, \eta] \\
&= \varepsilon_{j_D}\alpha \\
&= \varepsilon_{j_D}[\xi, \omega\phi_*\beta]_1 \quad \text{(by (6.3.3))}\,.
\end{aligned}$$

While

$$\begin{aligned}
\varepsilon_{j_D}((\xi, \eta), k')_*\{(\bar{\rho h}, \rho h)\} &= \tilde{k}_*\varepsilon_{j_B}\{\bar{\rho h}, \rho h)\} \\
&= \tilde{k}_*\{t\}\,,
\end{aligned}$$

thus we have $\varepsilon_{j_D}[\xi, \omega\phi_*\beta]_1 = \tilde{k}_*\{t\}$. The excision homomorphism ε_{j_D}: $\pi_1(\Sigma(B\#B), j_D) \to \pi_1(\Sigma(B\#B), \Sigma D\#\Sigma D)$ is represented as follows:

$$\begin{tikzcd}
\Sigma(B\# B) \arrow[r] \arrow[d] & \Sigma D\vee\Sigma D \arrow[r] \arrow[d, "j_D"] & * \arrow[d] \\
C\Sigma(B\# B) \arrow[r, "l"] & \Sigma D\times\Sigma D \arrow[r, "p_D"] & \Sigma D\#\Sigma D .
\end{tikzcd}$$

Hence $\varepsilon_{j_D}[\xi, \omega\phi_*(\beta)]_1=(*, p_D l)_*[\xi, \omega\phi_*(\beta)]_1=[*, (*, p_D l)_*\omega\phi_*(\beta)]_1=0$. Therefore $\tilde{k}_*\{t\}=0$. For any element $\{f\}\in\pi(\Sigma A, \Sigma B\#\Sigma B)$, since $f\simeq t(t^{-1}f)$ we have $\tilde{k}_*\{f\}=\{(\tilde{k}f)\}=\{(\tilde{k}t)(t^{-1}f)\}=0$. Hence, by the commutativity of (6.3.1),

$$\varepsilon_{j_D}\partial_{j_D}^{-1}(\xi, \eta)_*\partial_{j_B}\omega\phi_*(\gamma)=\tilde{k}_*\varepsilon_{j_B}\omega\phi_*(\gamma)=\tilde{k}_*H^*(\gamma)=0 .$$

Therefore by (a), (b) and (c) we deduce

$$H^*(\beta\circ\gamma)=H^*(\beta)\circ\Sigma\gamma+(\beta\#\beta)\circ H^*(\gamma) .$$

In [1] we defined the generalized Hopf invariant as follows: Assume that ΣX is $(n-1)$-connected and A is a suspension space. Then

(i) *If* $\dim.\Sigma A\leqq 3n-3$, *then*

$$\begin{aligned} H=\bar{h}_*^{-1}\partial(\bar{\rho}, \rho)_*^{-1}\omega\phi_*: \pi(\Sigma A, \Sigma X)&\longrightarrow\pi(\Sigma A, \Sigma X\vee\Sigma X)\longrightarrow\pi_1(\Sigma A, j)\\ &\longrightarrow\pi_1(\Sigma A, j_Q)\longrightarrow\pi(\Sigma A, Q)\longrightarrow\pi(\Sigma A, \Sigma(X\# X)) .\end{aligned}$$

(ii) *If* $\dim.\Sigma A\leqq 2(2n-2)$, *then*

$$\begin{aligned} H=\Sigma^{-1}t_*\varepsilon_j\omega\phi_*: \pi(\Sigma A, \Sigma X)&\longrightarrow\pi(\Sigma A, \Sigma X\vee\Sigma X)\longrightarrow\pi_1(\Sigma A, j)\\ &\longrightarrow\pi_1(\Sigma A, \Sigma X\#\Sigma X)\longrightarrow\pi_1(\Sigma A, \Sigma^2(X\# X))\longrightarrow\pi(\Sigma A, \Sigma(X\# X)) .\end{aligned}$$

If $\dim\Sigma A\leqq 3n-3$, H in (ii) equal to H in (i) (c.f. [1]), and hence if we identify $\pi_1(\Sigma A, \Sigma X\#\Sigma X)$ with $\pi_1(\Sigma A, \Sigma^2(X\# X))$ under the isomorphism t_*^{-1}, the Hopf invariant H is denoted $\Sigma^{-1}\varepsilon_j\omega\phi_*$ and $H^*=\Sigma H$.

The following properties are shown easily from (6.3), where A and B are suspensions, and $\beta\in\pi(\Sigma B, \Sigma X)$, $\gamma\in\pi(\Sigma A, \Sigma B)$:

(6.4) *If* ΣB *is* $(n-1)$*-connected,* ΣX *is* $(s-1)$*-connected, and if* $\dim\Sigma B\leqq 3n-3, 3s-3$, $\dim\Sigma A\leqq 3s-3$ *and* $H(\gamma)=0$, *then*

$$H(\beta\circ\gamma)=H(\beta)\circ\gamma .$$

(6.5) *If* B *is* $(n-2)$*-connected,* X *is* $(s-2)$*-connected, and if* $\dim \Sigma A \leqq 3s-3 \leqq 3n-3$ *then for* $\beta' \in \pi(B, X)$ *we have*

$$H(\Sigma\beta' \circ \gamma) = \Sigma(\beta' \# \beta') \circ H(\gamma).$$

(6.6) *If* $\gamma \in \Sigma\pi(A, B)$, *then*

$$H^*(\beta \circ \gamma) = H^*(\beta) \circ \Sigma\gamma.$$

(6.7) *If* ΣB *is* $(n-1)$*-connected,* ΣX *is* $(s-1)$*-connected, and if* $\Sigma A \leqq 4n-4$, $\dim \Sigma B \leqq 4s-4$, *then*

$$H^*(\beta \circ \gamma) = \Sigma H(\beta) \circ \Sigma\gamma + (\beta \# \beta) \circ \Sigma H(\gamma).$$

(6.8) *If* $\beta \in \Sigma\pi(B, X)$, *then*

$$H^*(\beta \circ \gamma) = (\beta \# \beta) \circ H^*(\gamma).$$

REFERENCES

[1] H. ANDO AND K. TSUCHIDA, On excision and its applications, Sci. Rep. Hirosaki Univ., 13(1966), 92–103.
[2] M. ARKOWITZ, The generalized Whitehead product, Pacific Journ. of Math., 12(1962), 7–23.
[3] M. ARKOWITZ, Homotopy product for *H*-spaces, Michigan Math. Journ., 10(1963), 1–9.
[4] A. L. BLAKERS AND W. S. MASSEY, Products in homotopy theory, Ann. of Math., 58 (1953), 295–324.
[5] B. ECKMANN AND P. J. HILTON, Homtopy groups of maps and exact sequences, Comm. Math. Helv., 34(1960), 271–304.
[6] K. A. HARDIE, On a construction of E. C. Zeeman, Journ. London Math. Soc., 35(1960), 452–464.
[7] P. J. HILTON, Suspension theorem and generalized Hopf invariant, Proc. London Math. Soc., Ser. 3, 2(1951), 462–493.
[8] P. J. HILTON, Hopf invariant of a composition element, Journ. London Math. Soc., 29 (1954), 165–171.
[9] P. J. HILTON, On the homotopy groups of the union of spheres, Journ. London Math. Soc., 30(1955), 154–172.
[10] P. J. HILTON, Homotopy theory and duality, Mimeographed Notes, Cornell University, 1959.
[11] S. T. HU, A group multiplication for relative homotopy groups, Journ. London Math. Soc., 22(1947), 61–67.
[12] G. T. PORTER, Higher order Whitehead products, Topology, 3(1965), 123–135.
[13] G. W. WHITEHEAD, Generalization of the Hopf invariant, Ann. of Math. (2), 51(1950), 192–237.
[14] G. W. WHITWHAD, On mappings into group-like spaces, Comm. Math. Helv., 28(1954), 320–328.
[15] J. H. C. WHITEHEAD, On adding relation to homotopy group, Ann. of Math., 42(1941), 409–429.

MATHMATICAL INSTITUTE
AKITA UNIVERSITY
AKITA. JAPAN