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ON THE TYPE OF AN ASSOCIATIVE H-SPACE OF RANK TWO

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An associative H-space is a space X equipped with a continuous map $\mu: X\times X\rightarrow X$ providing X with the structure of a monoid. If X is an associative H-space and $H_*(X; \boldsymbol{Z})$ is finitely generated as an abelian group, then by a classical theorem of Hopf [4], [5], $H^*(X, \boldsymbol{Q})$ is an exterior algebra on a finite number of odd dimensional generators. The number of such generators is called the rank of X; this is consistent with the standard usage when X is a Lie group. The dimensions in which the generators occur is called the type of X.

For example $SU(3)$ has rank 2 and type $(3, 5)$.

THEOREM. *Let X be a connected associative H-space with $H_*(X, \boldsymbol{Z})$ finitely generated as an abelian group. If the rank of X is 2 then the type of X is either* $(1, 1)$, $(1, 3)$, $(3, 3)$, $(3, 5)$, $(3, 7)$ *or* $(3, 11)$.

Indeed, each of the above types does occur, examples being given by the compact Lie groups, $S^1\times S^1$, $S^1\times S^3$, $S^3\times S^3$, $SU(3)$, $Sp(2)$ and G_2 respectively.

The proof of the above theorem will be accomplished by applying a result of A. Clark [1] and some number theoretic considerations.

I am greatly indebted to Shōji Ochiai for bringing this problem to my attention.

1. Unstable Polyalgebras over $\mathcal{A}^*(p)$.

NOTATION. Let p be a prime. We denote by $\mathcal{A}^*(p)$ the mod-p Steenrod algebra [8]. The reduced p^{th}-powers are denoted by P_p^j, and the Bockstein by β. When $p=2$ we set $\beta=Sq^1$ and $P_2^j=Sq^{2j}$.

DEFINITION. An unstable algebra over the Steenrod algebra is an algebra B that is a left $\mathcal{A}^*(p)$-module satisfying

(1) $$P_p^n x = 0 \quad \text{if} \quad 2n > \deg x$$

(2) $$P_p^n x = x^p \quad \text{if} \quad 2n = \deg x\,,$$

(3) $$P_p^n(xy) = \Sigma_{i+j=n} P_p^i x\, P_p^j y \quad (P_p^0 = 1)\,,$$

$$\beta(xy) = (\beta x)y + (-1)^{\deg x} x(\beta y) . \tag{4}$$

If the underlying algebra of B is a polynomial algebra we say that B is an unstable polyalgebra over $\mathcal{A}^*(p)$.

The proof of the following theorem may be found in [1].

THEOREM 1.1 (A. Clark). *Let p be a prime and let B be an unstable polyalgebra over $\mathcal{A}^*(p)$. If B has a generator of degree $2m$, $m \not\equiv 0 \bmod p$, then B has a generator of degree $2n$, for some integer n with $n \equiv 1-p \bmod m$.*

COROLLARY 1.2. *Let p be an odd prime and let B be an unstable polyalgebra over $\mathcal{A}^*(p)$ on two generators x, y with $\deg x = 4$, $\deg y = 2n$, and $p > n > 1$. Then either*

$$n \mid p-1 \quad \textit{or}$$

$$n \mid p+1 .$$

PROOF. Since $n \not\equiv 0 \bmod p$ it follows from Theorem 1.1 that either

$$2 \equiv 1 - p \mod n \quad \text{or}$$

$$n \equiv 1 - p \mod n .$$

In the first case $p+1 = p-1+2 \equiv 0 \bmod n$, i.e., $n \mid p+1$. In the second case $p-1 \equiv -n \equiv 0 \bmod n$, and hence $n \mid p-1$. ■

ACKNOWLEDGEMENT. I am indebted to Shōji Ochiai for bringing Corollary 1.2 to my attention [7, Theorem 1a].

Theorem 1.1 has the following useful application to associative H-spaces. Again we refer the reader to [1] for the proof.

THEOREM 1.2 (A. Clark). *Let X be a simply connected associative H-space with $H_*(X; \mathbf{Z})$ finitely generated as an abelian group. Then $H_*(X; \mathbf{Z})$ has a generator of degree 3.*

2. Some Number Theory.

PROPOSITION 2.1. *Let n be a positive integer. If for all sufficiently large primes p either*

$$n \mid p-1 \quad \textit{or}$$

$$n \mid p+1 ,$$

then $n=1, 2, 3, 4$ *or* 6.

Before turning to the proof of Proposition 2.1 we recall the following classical result of Dirichlet [2].

THEOREM 2.2 (Dirichlet). *Every arithmetic series whose initial term and difference are relatively prime contains an infinite number of primes.*

PROOF OF PROPOSITION 2.1. We say an integer n has property (*) if for all sufficiently large primes p, either

$$n \mid p-1, \quad \text{or}$$

$$n \mid p+1 .$$

Suppose q is a prime. If $q \neq 2, 3$, then q and $q-2$ are relatively prime. Consider the arithmetic series $\{a_j \mid a_j = q-2+j\cdot q\}$. By Dirichlet's theorem this series contains infinitely many primes. If p is such a prime then

$$p \equiv q-2 \mod q .$$

Hence since $q>3$,

$$p+1 \equiv q-1 \not\equiv 0 \mod q$$

$$p-1 \equiv q-3 \not\equiv 0 \mod q .$$

Therefore q cannot have property (*).

If n has property (*) then so does any divisor of n. Hence if n has property (*) then $n=2^r 3^s$.

Suppose $n=2^r 3^s > 6$. Then n and $n-5$ are relatively prime. Consider the arithmetic series $\{b_j \mid b_j = n-5+j\cdot n\}$. By Dirichlet's theorem this series contains infinitely many primes. If p is such a prime, then

$$p \equiv n-5 \mod n .$$

Hence since $n>6$

$$p-1 \equiv n-6 \not\equiv 0 \mod n$$

$$p+1 \equiv n-4 \not\equiv 0 \mod n .$$

Therefore the only possible integers with property (*) are 1, 2, 3, 4 or 6. That these actually occur may be seen by a simple case check as follows:

1, 2: trivial.

3: If $3|p-1$ there is nothing to prove. So suppose $3 \nmid p-1$. Then either

$$p-1 \equiv 1 \mod 3 \quad \text{or}$$
$$p-1 \equiv 2 \mod 3 .$$

If $p-1 \equiv 1 \mod 3$, then $p+1 = p-1+2 \equiv 1+2 \equiv 0 \mod 3$, i.e., $3|p+1$. On the other hand the case $p-1 \equiv 2 \mod 3$ cannot occur. For if $p-1 \equiv 2 \mod 3$ then $p = p-1+1 \equiv 2+1 \equiv 0 \mod 3$, contrary to the assumption that p is a large prime.

4: If $4|p-1$ there is nothing to prove. So suppose $4 \nmid p-1$. Then either

$$p-1 \equiv 1, 2 \quad \text{or} \ 3 \mod 4 .$$

However since $p-1 \equiv 0 \mod 2$ it follows that

$$p-1 \equiv 2 \mod 4 .$$

Hence $p+1 = p-1+2 \equiv 2+2 \equiv 0 \mod 4$, i.e., $4|p+1$.

6: If $6|p-1$ there is nothing to prove. So suppose $6 \nmid p-1$. Then

$$p-1 \equiv 1, 2, 3, 4 \quad \text{or} \ 5 \mod 6 .$$

Since $p-1 \equiv 0 \mod 2$ it follows that

$$p-1 \equiv 2, 4 \mod 6 .$$

If $p-1 \equiv 4 \mod 6$ then $p+1 = p-1+2 \equiv 4+2 \equiv 0 \mod 6$, i.e., $6|p+1$. Finally we must show that the case $p-1 \equiv 2 \mod 6$ cannot occur. For suppose it does. Then $p-1 = 6t+2$. Therefore $p-1 \equiv 2 \mod 3$ and hence $p = p-1+1 \equiv 2+1 \equiv 0 \mod 3$ contrary to the assumption that p is a large prime. ■

3. Associative *H*-Spaces of Rank 2. We turn now to the proof of the theorem announced in the introduction.

Let X be a connected simply connected H-space of rank 2 with $H_*(X; \mathbf{Z})$ finitely generated as an abelian group. It follows from the theorems of Hopf [4] and Clark (Theorem 1.3) that

$$H^*(X; \mathbf{Q}) = E[x, y]$$

where $\deg x = 3$, $\deg y = 2n-1$,

Let BX be the classifying space of X [3]. $H_*(X; \boldsymbol{Z})$ is finitely generated and therefore has p-torsion for only a finite number of primes. It therefore follows (essentially) from the Borel transgression theorem [6] that for all sufficiently large primes p

$$H^*(BX;\ \boldsymbol{Z}_p) \cong P[u, v]$$

where $\deg u=4$, $\deg v=2n$.

From Corollary 1.2 it follows that for all sufficiently large primes p either

$$n \mid p-1 \quad \text{or}$$

$$n \mid p+1 .$$

Hence from Proposition 2.1 $n = 1, 3, 4, 6$. The case $n=1$ is excluded since we assumed X simply connected. Thus the type of X is either (3, 3), (3, 5), (3, 7) or (3, 11).

If X is not simply connected and does not have type (1, 1) then applying Clark's theorem to the universal covering of X, we readily deduce that X has type (1, 3).

Thus the only possible types of an associative H-space of rank 2 with finitely generated integral homology are (1, 1), (1, 3), (3, 3), (3, 5), (3, 7) or (3, 11), as was to be shown. ■

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