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AN EXTENSION OF TANNO'S FORM OF THE CONVOLUTION TRANSFORMATION

J. N. PANDEY*) AND A. H. ZEMANIAN

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1. Introduction. The convolution transformation of Hirschman and Widder [1]:

$$F(x) = \int_{-\infty}^{\infty} f(t)\,G(x-t)\,dt \qquad -\infty < x < \infty \tag{1}$$

has been extended by Tanno [2]—[4] to the case where $f(t)$ is a suitably restricted conventional function and the two-sided Laplace transform of $G(t)$ is a meromorphic function with only real poles and zeros. The particular form of Tanno's kernel that we consider here is the following:

$$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{st}}{J(s)}\,ds\,, \tag{2}$$

$$J(s) = \prod_{k=1}^{\infty} \frac{1-s^2/a_k^2}{1-s^2/c_k^2}\,. \tag{3}$$

The a_k and c_k are real positive numbers such that

$$0 < a_1 \leqq a_2 \leqq \cdots < \infty\,;\ c_1 \leqq c_2 \leqq \cdots;\ a_k \leqq c_k\,;$$

$$0 \leqq \overline{\lim_{k\to\infty}}\,\frac{k}{c_k} < \overline{\lim_{k\to\infty}}\,\frac{k}{a_k} < \infty\,.$$

From a certain point on, all c_k may be equal to infinity.

Our objective is to extend (1) to the case where f is a generalized function of a certain type and to establish in this case the validity of Tanno's inversion formula in a certain weak sense.

2. The Extended Transformation. Let c and d denote real numbers, let $L_{c,d}$ be the testing-function space defined in [5], and let $L'_{c,d}$ be the

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corresponding dual space of generalized functions. Now, Tanno [2; p. 41] has shown that $G(t)$ is a C^∞ function having the following asymptotic properties. With $p(t)$ denoting some polynomial, we have

$$G(t) = p(t)\,e^{-a_1 t} + R_+(t) = p(-t)\,e^{a_1 t} + R_-(t) \tag{4}$$

where for any given nonnegative integer n and for some $\epsilon > 0$,

$$D^n R_+(t) = O(e^{-(a_1+\epsilon)t}) \qquad t \to \infty\,,$$

$$D^n R_-(t) = O(e^{(a_1+\epsilon)t}) \qquad t \to -\infty\,.$$

Here, $D = D_t = d/dt$. It follows that, if $c < a_1$ and $d > -a_1$, then, for each fixed x and any nonnegative integer k, $D^k_{x-t}G(x-t)$ as a function of t is a member of $L_{c,d}$. Hence, we can define the convolution transform of any $f \in L'_{c,d}$ by

$$F(x) = \langle f(t),\, G(x-t) \rangle \qquad -\infty < x < \infty\,. \tag{5}$$

Precisely the same proof as that of [5; Theorem 4.1] establishes the

PROPOSITION. *Let $f \in L'_{c,d}$ for some $c < a_1$ and some $d > -a_1$ and let $F(x)$ be defined by* (5). *Also, let a and b be any real numbers such that $a < \min(a_1, -c)$ and $b > \max(-a_1, -d)$. Then, $F(x) \in L_{a,b}$. Moreover,*

$$D^k F(x) = \langle f(t),\, D^k_x G(x-t) \rangle\,.$$

Note that our conclusion implies that, for each nonnegative integer k, there exists an $\eta > 0$ such that

$$D^k F(x) = O(e^{(a_1-\eta)|x|}) \qquad |x| \to \infty\,. \tag{6}$$

3. **An Integrodifferential Operator.** Let α be a real number and define an integration operator on a continuous function $\Psi(x)$ by

$$\frac{1}{1-D/\alpha}\Psi(x) = \begin{cases} \alpha e^{\alpha x} \displaystyle\int_x^\infty \Psi(y)\,e^{-\alpha y}\,dy & \alpha > 0 \\ -\alpha e^{\alpha x} \displaystyle\int_{-\infty}^x \Psi(y)\,e^{-\alpha y}\,dy & \alpha < 0\,. \end{cases}$$

The right-hand side exists if $\Psi(x) = O(\exp[b|x|])$ as $|x| \to \infty$ and if $|b| < |\alpha|$. In this case we also have

$$\frac{1}{1-D/\alpha}\Psi(x) = O(e^{b|x|}) \qquad |x| \to \infty . \tag{7}$$

Moreover,

$$\frac{1}{1-D_x/\alpha}\Psi(x-t) = \frac{1}{1-D_{x-t}/\alpha}\Psi(x-t) .$$

Next, we define the integrodifferential operator $P_n(D)$ by

$$P_n(D) = \prod_{k=1}^{n} \frac{(1-D/a_k)(1+D/a_k)}{(1-D/c_k)(1+D/c_k)} \qquad n = 1, 2, 3, \cdots .$$

When $P_n(D)$ is applied to a C^∞ function $\Psi(x)$ satisfying (7) with $|b| < a_1$, the order of application of the differentiation and integration operators within $P_n(D)$ can be changed in any fashion. Indeed, an integration by parts shows that, for $|\alpha| \geqq a_1$,

$$D\left(\frac{1}{1-D/\alpha}\Psi\right) = \frac{1}{1-D/\alpha}(D\Psi) .$$

Furthermore, for $|\alpha| \geqq a_1$, $|\beta| \geqq a_1$,

$$f_1 = \frac{1}{1-D/\alpha} \cdot \frac{1}{1-D/\beta}\Psi ,$$

and

$$f_2 = \frac{1}{1-D/\beta} \cdot \frac{1}{1-D/\alpha}\Psi ,$$

a computation shows that

$$(1 - D/\alpha)(1 - D/\beta)(f_1 - f_2) = 0 .$$

Hence, $f_1 - f_2 = A\exp[\alpha x] + B\exp[\beta x]$. But, $f_1 - f_2 = O(\exp[b|x|])$ as $|x| \to \infty$, and therefore $A = B = 0$. Thus, $f_1 = f_2$. These results imply our assertion.

4. Inversion. We now show that Tanno's inversion formula can be extended to the present case if the limit therein is understood in the sense of weak distributional convergence.

THEOREM. *Let $f \in L'_{c,d}$ for some $c < a_1$ and some $d > -a_1$, and let $F(x)$ be defined by* (5). *Then, for every C^∞ function $\varphi(x)$ of compact support,*

$$\lim_{n\to\infty} <P_n(D) F, \varphi> = <f, \varphi> \tag{8}$$

PROOF. The proof consists in establishing the validity of the following manipulations, wherein $D = d/dx$:

$$<P_n(D) F, \varphi> = <F, P_n(D) \varphi> \tag{9}$$

$$= <P_n(D) \varphi(x), <f(t), G(x-t)>> \tag{10}$$

$$= <f(t), <P_n(D)\varphi(x), G(x-t)>> \tag{11}$$

$$= <f(t), <\varphi(x), P_n(D) G(x-t)>> \tag{12}$$

and then in showing that $<\varphi(x), P_n(D) G(x-t)>$ converges in $L_{c,d}$ to $\varphi(t)$ as $n \to \infty$.

The equality in (9) is established by repeatedly integrating by parts and using the order estimates (6) and (7) to show that the limit terms in each integration by parts are equal to zero. Obviously, (9) is equal to (10).

To see that (10) is equal to (11), we first note that (7) implies that for each $\gamma > 0$

$$P_n(D) \varphi(x) = O(e^{(\gamma - c_1)|x|}) \qquad |x| \to \infty . \tag{13}$$

This fact coupled with (4) imples that

$$\left(\int_{-\infty}^{-X} + \int_{X}^{\infty}\right) G(x-t) P_n(D) \varphi(x)\, dx$$

converges in $L_{c,d}$ to zero as $X \to \infty$. These results and the estimate (6) imply in turn that, given any $\epsilon > 0$, we can choose X so large that

$$\left| \left(\int_{-\infty}^{-X} + \int_{X}^{\infty}\right) <f(t), G(x-t)> P_n(D) \varphi(x)\, dx \right| < \epsilon$$

and simultaneously

$$\left| <f(t), \left(\int_{-\infty}^{-X} + \int_{X}^{\infty}\right) G(x-t) P_n(D) \varphi(x)\, dx> \right| < \epsilon .$$

Finally, we can show that

$$\int_{-X}^{X} <f(t), G(x-t)> P_n(D) \varphi(x)\, dx = <f(t), \int_{-X}^{X} G(x-t) P_n(D) \varphi(x)\, dx>$$

by using Riemann sums in exactly the same way as in [6]. This verifies that (10) is equal to (11).

We get from (11) to (12) by repeatedly integrating by parts and using the estimates (4), (7), and (13) to set all limit terms equal to zero.

Now for the last step: let $G_n = P_n(D)G$, $n = 1, 2, 3, \cdots$; G_n is a C^∞ function. Tanno [2; pp. 43, 45, and 48] has shown that

$$G_n(t) \geqq 0 \qquad -\infty < t < \infty, \tag{14}$$

$$\int_{-\infty}^{\infty} G_n(t)\, dt = 1, \tag{15}$$

$$\operatorname{sgn} D\, G_n(t) = -\operatorname{sgn} t, \tag{16}$$

$$\lim_{n\to\infty} G_n(t) = 0 \qquad 0 < |t| < \infty. \tag{17}$$

We also need two lemmas.

LEMMA 1. *For any* $\delta > 0$,

$$\lim_{n\to\infty} \int_{|t|>\delta} G_n(t)\, dt = 0.$$

PROOF.

$$0 \leqq \int_{|t|>\delta} G_n(t)\, dt \leqq \delta^{-2} \int_{|t|>\delta} t^2 G_n(t)\, dt \leqq \delta^{-2} \int_{-\infty}^{\infty} t^2 G_n(t)\, dt.$$

By [2; Theorem 4A], the right-hand side tends to zero as $n \to \infty$.

LEMMA 2. *Let* $\delta > 0$, $c < a_1$, *and* $d > -a_1$. *Then, as* $n \to \infty$, $\exp[-ct]\, G_n(t)$ *tends uniformly to zero on* $-\infty < t \leqq -\delta$, *and* $\exp[-dt] G_n(t)$ *tends uniformly to zero on* $\delta \leqq t < \infty$.

PROOF. If $c \leqq 0$, our assertion about $\exp[-ct]\, G_n(t)$ follows directly from (14), (16), and (17). So, assume $0 < c < a_1$. Also, choose the real number α such that $1 < \alpha < a_1/c$. By virtue of (14) and (16), we may write for $t < 0$

$$0 \leqq (\alpha - 1)\left(-\frac{t}{\alpha}\right)^3 e^{-ct}\, G_n(t) \leqq \int_t^{t/\alpha} u^2 e^{-c\alpha u}\, G_n(u)\, du \leqq \int_{-\infty}^{\infty} u^2 e^{-c\alpha u}\, G_n(u)\, du. \tag{18}$$

Our assertion concerning $\exp[-ct]\, G_n(t)$ will be completely established when we show that the right-hand side of (18) tends to zero as $n \to \infty$. Set $s = c\alpha$. By using [2; p. 45, Eq. (3)] and differentiating twice with respect to s, we find that the right-hand side of (18) is equal to

$$D_s^2 \prod_{k=n+1}^{\infty} \frac{1-s^2/c_k^2}{1-s^2/a_k^2}. \tag{19}$$

Then, by using logarithmic differentiation, we find (19) to be equal to

$$\frac{2}{J_n(s)} \sum_{k=n+1}^{\infty} \left(\frac{1}{a_k^2-s^2} - \frac{1}{c_k^2-s^2} \right) + \frac{4s^2}{J_n(s)} \left[\sum_{k=n+1}^{\infty} \left(\frac{1}{a_k^2-s^2} - \frac{1}{c_k^2-s^2} \right) \right]^2 + \frac{4s^2}{J_n(s)} \sum_{k=n+1}^{\infty} \left[\frac{1}{(a_k^2-s^2)^2} - \frac{1}{(c_k^2-s^2)^2} \right] \tag{20}$$

where

$$J_n(s) = \prod_{k=n+1}^{\infty} \frac{1-s^2/a_k^2}{1-s^2/c_k^2}.$$

The last manipulation is justified by the fact that each arising infinite series converges uniformly for s is restricted to any compact subset of the open interval $(-a_1, a_1)$. For $s = c\alpha$, (20) tends to zero as $n \to \infty$, which is what we had to show.

The conclusion of Lemma 2 concerning $\exp[-dt] G_n(t)$ is established in a similar way.

The above properties of $G_n = P_n(D) G$ are precisely the ones needed to show that $\langle \varphi(x), P_n(D) G(x-t) \rangle$ converges in $L_{c,d}$ to $\varphi(t)$ as $n \to \infty$. The proof is the same as that in [5; Theorem 5.3] and is therefore omitted.

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CARLETON UNIVERSITY
OTTAWA, CANADA

AND

THE STATE UNIVERSITY OF NEW YORK
STONY BROOK, U. S. A.