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OPERATING FUNCTIONS ON $B_0(\widehat{G})$ IN PLANE REGIONS

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Throughout this paper, let G be any infinite compact abelian group, and $\widehat{G}$ its dual. We shall respectively denote by $M(G)$, $M_0(G)$, and $M_A(G)$, the measure algebra of all bounded regular measures on G, the closed ideal of those measures μ whose Fourier-Stieltjes transforms $\hat{\mu}$ vanish at the infinity of $\widehat{G}$, and that of the measures absolutely continuous with respect to the Haar measure of G. We shall also denote by $B(\widehat{G})$, $B_0(\widehat{G})$, and $A(\widehat{G})$, the function algebras on $\widehat{G}$ consisting of the Fourier-Stieltjes transforms of the measures in $M(G)$, $M_0(G)$, and $M_A(G)$ respectively. Let us introduce a norm on $B(\widehat{G})$ by $\|\hat{\mu}\| = \|\mu\|$.

Suppose now that C is a subset of $B(\widehat{G})$, and that $F(z)$ is a complex-valued function defined on some set E in the complex plane. We say that $F(z)$ operates on C if

$$F(f) = F\circ f \in B(\widehat{G})$$

for every function $f \in C$ whose range lies in E.

N.Th. Varopoulos [2] has shown the following.

THEOREM 1. *For every G, there exists $f \in B_0(\widehat{G})$ with the property; if $F(z)$ is a function defined on the interval $(-1,1)$, and if $F(z)$ operates on the subalgebra of $B(\widehat{G})$ generated by $A(\widehat{G})$ and f, then $F(z)$ coincides with an entire function in some neighborhood of* 0.

In this paper we shall point out that an analogous result also holds for operating functions defined in a plane region.

THEOREM 2. *For every G, there exist f_1, f_2, g_1 and g_2 in $B_0(\widehat{G})$ with the property; if $F(z)$ is a function on the unit disc $\{z : |z| \leqq 1\}$ in the complex plane, and if $F(z)$ operates on the closed subalgebra generated by $A(\widehat{G})$ and f_1, f_2, g_1, g_2 then $F(z)$ coincides with a real-entire function in some neighborhood of* 0.

We need a lemma.

LEMMA. *For every positive intger k, there exist non-negative non-zero measures $\mu_1, \cdots, \mu_k$ in $M_0(G)$ such that*:

(i) *If $(m_1, \cdots, m_k)$ and $(n_1, \cdots, n_k)$ are two distinct ordered k-tuples of non-negative integers, then the measures*

$$\mu_1{}^{m_1} * \cdots * \mu_k{}^{m_k} \text{ and } \mu_1{}^{n_1} * \cdots * \mu_k{}^{n_k}$$

are mutually singular;

(ii) *For all $j=1, \cdots, k$, $\hat{\mu}_j \geqq 0$.*

PROOF. Since $\widehat{G}$ is a infinite (discrete) group, it contains a countably infinite subgroup $\widehat{I}$. If H is the annihilator of $\widehat{I}$, it follows that the quotient group $I=G/H$ is an infinite compact group. Since $\widehat{I}$ is countable, and since the dual of I is $\widehat{I}$, I is metrizable. It follows from Theorem R of [3] that there is a non-negative measure λ in $M_0(I)$ whose closed support $S(\lambda)$ is independent. Thus for every positive integer k, we can find non-negative non-zero measures $\lambda_1, \cdots \lambda_k$ in $M_0(I)$ such that

$$\bigcup_{j=1}^{k} S(\lambda_j) \subset S(\lambda)$$

and the sets $S(\lambda_j)$ are pairwise disjoint. Define

$$\nu_j = \lambda_j + \lambda_j{}^*$$

for each $j=1, \cdots, k$. It follows that these measures $\nu_1, \cdots, \nu_k$ satisfy condition (i) in the lemma [1 : p-105], since all the measures ν_j are continuous [1 : p-118]. For each $j=1, \cdots, k$, let μ_j be the measure in $M_0(G)$ uniquely defined by the requirement that

$$\hat{\mu}_j(\gamma) = \begin{cases} \{\widehat{\nu_j}(\gamma)\}^2 & (\gamma \in \widehat{I}) \\ 0 & (\gamma \notin \widehat{I}). \end{cases}$$

It is easy to see that these measures μ_j have both of the required properties. This completes the proof.

PROOF OF THEOREM 2. Let $\mu_1, \mu_2, \mu_3, \mu_4 \in M_0(G)$ be as in the lemma for $k = 4$, and put

$$f_1 = \hat{\mu}_1, f_2 = \hat{\mu}_2, g_1 = \hat{\mu}_3, \text{ and } g_2 = \hat{\mu}_4 .$$

To show that these functions in $B_0(\widehat{G})$ have the required property in the Theorem 2, let $F(z)$ be any function defined on the unit disc $\{z : |z| \leqq 1\}$ which operates on the closed subalgebra of $B_0(\widehat{G})$ generated by $A(\widehat{G})$ and f_1, f_2, g_1 and g_2. Since $F(z)$ operates on $A(\widehat{G})$, it can be expressed on some neighborhood of 0 in the form

$$(1) \qquad F(s, t) = F(s + it) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} s^j t^k ,$$

the series in the right-hand side being absolutely convergent in some neighborhood of 0. To show that this series absolutely converges for all values of s and t, we may clearly assume, by considering $F_1(z) = F(cz)$ for a small contact $c > 0$ in place of $F(z)$, that the series absolutely converges in the square

$$(2) \qquad E = \{(s, t) ;\ -\pi \leqq s \leqq \pi, -\pi \leqq t \leqq \pi\}$$

and that the equality in (1) holds there. We may also assume that

$$\|f_1\| = \|f_2\| = \|g_1\| = \|g_2\| = 1 .$$

Let now $C > 0$ be any constant and put

$$(3) \qquad h_{st}(\gamma) = 2f_1(\gamma) \cos\{Cf_2(\gamma) + s\} + i2g_1(\gamma) \cos\{Cg_2(\gamma) + t\}.$$

Then the set

$$(4) \qquad \{h_{st};\ (s, t) \in E\} \subset B_0(\widehat{G})$$

is a continuous image of the compact set E, and so that it is a compact subset of the closed subalgebra generated by $\{f_1, f_2, g_1, g_2\}$. Thus we can find a positive constant K_C such that

$$(5) \qquad \|F(h_{st})\| \leqq K_C \qquad ((s, t) \in E).$$

Hence if we set for $(p, q) \in Z^2$ (the set of all ordered pairs of non-negative integers)

$$(6) \qquad l_{pq}(\gamma) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-ips} e^{-iqt} F(h_{st}(\gamma)) ds dt ,$$

it follows that

$$(7) \qquad l_{pq} \in B(\hat{G}) \quad \text{and} \quad \|l_{pq}\| \leqq K_C \ \ ((p, q) \in Z^2),$$

since $F(s, t)$ is continuous in E by our assumption. On the other hand, we see from (1) that

$$(8) \qquad l_{pq} = \exp(ipCf_2 + iqCg_2)\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp(-i(ps+qt))F(2f_1\cos s, 2g_1\cos t)$$

$$= \exp(ipCf_2 + iqCg_2)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{jk}b_j(p)b_k(q)f_1{}^j g_1{}^k$$

for each $\gamma \in \hat{G}$, where

$$(9) \qquad b_j(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(2\cos s)^j e^{-ips}ds.$$

But since the series in the right-hand side of (8) converges in the norm of $B(\hat{G})$, it follows from the assumptions on f_1, f_2, g_1 and g_2 that

$$(10) \qquad \|l_{pq}\| = \exp((p+q)C)\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|a_{jk}|\,|b_j(p)|\,|b_k(q)| \quad ((p, q) \in Z^2).$$

Thus, in particular, we have

$$(11) \qquad \exp((p+q)C)|a_{pq}|\,|b_p(p)|\,|b_q(q)| \leqq \|l_{pq}\|.$$

Since $b_p(p)=1$ for all non-negative integers p, and since $\|l_{pq}\| \leqq K_c$ for all $(p, q) \in Z^2$, we conclude that

$$(12) \qquad |a_{pq}| \leqq K_c \exp(-(p+q)C) \quad ((p, q) \in Z^2).$$

This assures that the series in the right-hand side of (1) is absolutely convergent in the square $\max(|s|, |t|) < e^C$. Since $C > 0$ can be taken arbitrarily large, it follows that the series in (1) absolutely converges for all values of s and t, which yields the desired conclusion.

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