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GALOIS COHOMOLOGY OF FINITELY GENERATED MODULES

Dedicated to Professor T. Tannaka on his 60th birthday

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The purpose of this paper is to generalize Tate's theorem concerning Galois cohomology of finite modules. It will be shown here that a large part of the theorem holds also for finitely generated modules. In section 2 we shall consider, particularly, unramified cohomology of finitely generated modules over local fields. In the final section, we shall study the relation between local and global cohomology. I wish to express my thanks to Dr. K. Uchida for his useful suggestions.

1. Notation. Let R be a Dedekind ring with field of fractions k. Let Ω be the union of all finite extensions K of k in which the integral closure of R is unramified over R, and let $\bar{R}$ denote the integral closure of R in Ω. Let G_R denote the Galois group of the extension Ω/k. For any discrete G_R-module A, we put

$$H^r(R,A)=H^r(G_R,A) \qquad (r\in \boldsymbol{Z})$$

$$\hat{H}^r(R,A)=\begin{cases} H^r(G_R,A) & (r\geqq 1)\\ \hat{H}^r(G_R,A) & (r\leqq 0)\end{cases}$$

(cf. [5]). By M we shall always understand a finitely generated discrete G_R-module such that the order of the torsion part of M is invertible in R. Such a module M is said to be a Galois module over R. We put $T=$ [the torsion part of M], $F=M/T$ and $M'=\mathrm{Hom}(M,\bar{R}^\times)$ where $\bar{R}^\times$ is the group of units of $\bar{R}$. For any locally compact abelian group H, we let H^* denote its Pontrjagin character group.

2. Local fields. Let k be a local field (i.e., a non-discrete locally compact field). Then we have isomorphisms

$$\hat{H}^r(k,M)\cong\hat{H}^{2-r}(k,M')^*$$

for all $r \in \boldsymbol{Z}$ (cf. [2; Chap. II, Théorème 6]). Suppose k is a non-archimedean local field with valuation ring $\mathfrak{o}$. Let M be a Galois module over $\mathfrak{o}$.

THEOREM 1. i) $H^r(\mathfrak{o}, M') = 0 \quad (r \geqq 2)$.

ii) *The inflation map* $H^2(\mathfrak{o}, M) \to H^2(k, M)$ *and the canonical homomorphism* $H^0(\mathfrak{o}, M') \to \hat{H}^0(k, M')$ *are injective. The subgroups* $H^2(\mathfrak{o}, M)$ *of* $H^2(k, M)$ *and* $H^0(\mathfrak{o}, M')$ *of* $\hat{H}^0(k, M')$ *are the exact annihilators of each other.*

iii) *The inflation map* $H^1(\mathfrak{o}, M) \to H^1(k, M)$ *and the homomorphism* $H^1(\mathfrak{o}, M') \to H^1(k, M')$ *by the inflation map and the injection* $\bar{\mathfrak{o}}^\times \to \bar{k}^\times$ *are injective. The subgroups* $H^1(\mathfrak{o}, M)$ *of* $H^1(k, M)$ *and* $H^1(\mathfrak{o}, M')$ *of* $H^1(k, M')$ *are the exact annihilators of each other.*

PROOF. i) Since $\operatorname{cd} G_{\mathfrak{o}} = 1$, we have $H^r(\mathfrak{o}, T') = 0$ for $r \geqq 2$, and $H^r(\mathfrak{o}, M') = 0$ for $r \geqq 3$. Since $\bar{\mathfrak{o}}^\times$ is cohomologically trivial, we have $H^r(\mathfrak{o}, F') = 0$ for $r \geqq 1$. By the exact sequence

$$H^2(\mathfrak{o}, F') \to H^2(\mathfrak{o}, M') \to H^2(\mathfrak{o}, T'),$$

we get $H^2(\mathfrak{o}, M') = 0$.

ii) Consider a commutative diagram:

$$\begin{array}{ccc} H^2(k, M) & \longrightarrow & H^2(k, F) \\ \uparrow \text{inf} & & \uparrow \text{inf} \\ H^2(\mathfrak{o}, M) & \longrightarrow & H^2(\mathfrak{o}, F) \end{array} .$$

Since $H^2(\mathfrak{o}, M)$ is isomorphic to $H^2(\mathfrak{o}, F)$ and the inflation map $H^2(\mathfrak{o}, F) \to H^2(k, F)$ is injective, the inflation map $H^2(\mathfrak{o}, M) \to H^2(k, M)$ is injective. For any finite extension K of k, let $\hat{K}^\times$ denote the compactification of $K^\times$, and we put $\bar{k}^{\times\wedge} = \bigcup_K \hat{K}^\times$, the union taken over all finite separable extensions K of k. The injectivity of the map $H^0(\mathfrak{o}, M') \to \hat{H}^0(k, M')$ is an immediate consequence of the fact $\hat{H}^0(k, M') = H^0(k, \operatorname{Hom}(M, \bar{k}^{\times\wedge}))$. Let k_{nr} denote the maximal unramified extension of k. Now by the G-split exact sequence

$$0 \longrightarrow \bar{\mathfrak{o}}^\times \longrightarrow \hat{k}_{nr}^\times \longrightarrow \hat{\boldsymbol{Z}} \longrightarrow 0$$

and $H^0(\mathfrak{o}, \operatorname{Hom}(M, \hat{k}_{nr}^\times)) = H^0(k, \operatorname{Hom}(M, \bar{k}^{\times\wedge}))$, we get an exact sequence

$$0 \longrightarrow H^0(\mathfrak{o}, M') \longrightarrow H^0(k, \operatorname{Hom}(M, \bar{k}^{\times\wedge})) \longrightarrow H^0(\mathfrak{o}, \operatorname{Hom}(M, \hat{\boldsymbol{Z}})) \longrightarrow 0 .$$

Hence we get an exact sequence

$$0 \longrightarrow H^0(\mathfrak{o}, M') \longrightarrow \hat{H}^0(k, M') \longrightarrow H^2(\mathfrak{o}, M)^* \longrightarrow 0,$$

because $\hat{\boldsymbol{Z}}$ is a "module dualisant" for the group G_0 ($\cong \hat{\boldsymbol{Z}}$) (cf. [2; Chap. I, Annexe]).

iii) Consider a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 = H^2(\mathfrak{o}, T)^* & \longrightarrow & H^1(\mathfrak{o}, F)^* & \longrightarrow & H^1(\mathfrak{o}, M)^* & \longrightarrow & H^1(\mathfrak{o}, T)^* & & \\ & & \uparrow \wr & & \uparrow & & \uparrow & & \\ & & H^1(k, F') & \longrightarrow & H^1(k, M') & \longrightarrow & H^1(k, T') & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & 0 = & H^1(\mathfrak{o}, F') & \longrightarrow & H^1(\mathfrak{o}, M') & \longrightarrow & H^1(\mathfrak{o}, T') & \longrightarrow & H^2(\mathfrak{o}, F') = 0 \end{array}$$

where $H^1(k, F')$ ($\cong H^1(k, F)^*$) $\cong H^1(\mathfrak{o}, F)^*$. The sequence $H^1(\mathfrak{o}, T') \to H^1(k, T') \to H^1(\mathfrak{o}, T)^*$ is exact by [3; Theorem 2.4]. Now it is easily verified that the sequence

$$0 \longrightarrow H^1(\mathfrak{o}, M') \longrightarrow H^1(k, M') \longrightarrow H^1(\mathfrak{o}, M)^* \longrightarrow 0$$

is exact and the theorem is proved.

3. Global fields. Let k be a finite extension of $\boldsymbol{Q}$, or a function field in one variable over a finite field, let S be a non-empty set of primes of k, including the archimedean ones, and k_S denote the ring of elements in k which are integers at all primes not in S. For each prime v in S, let k_v denote the completion of k at v. Throughout this section, M will be a Galois module over k_S. Let $P^r(k_S, M)$ (resp. $P^r(k_S, M')$) be the restricted direct product of $H^r(k_v, M)$ (resp. $H^r(k_v, M')$ ($v \in S$) relative to the subgroups $H^r(\mathfrak{o}_v, M)$ (resp. $H^r(\mathfrak{o}_v, M')$). Since $H^1(\mathfrak{o}_v, M)$ and $H^1(\mathfrak{o}_v, M')$ are finite, $P^1(k_S, M)$ and $P^1(k_S, M')$ are locally compact. By Theorem 1, $P^r(k_S, M')$ is the direct sum for $r \geqq 2$. Since scd $G_{k_v} = 2$ if v is non-archimedean, $P^r(k_S, M)$ and $P^r(k_S, M')$ are equal to $\prod_{v \text{ arch}} H^r(k_v, M)$ and $\prod_{v \text{ arch}} H^r(k_v, M')$ respectively for $r \geqq 3$. The localization maps $H^r(k_S, M) \to H^r(k_v, M)$ and $H^r(k_S, M') \to H^r(k_v, M')$ give canonical maps:

$$f_r : H^r(k_S, M) \longrightarrow P^r(k_S, M),$$

$$f'_r : H^r(k_S, M') \longrightarrow P^r(k_S, M').$$

By Theorem 1, local duality yields an isomorphism

$$(*) \qquad P^1(k_S, M) \cong P^1(k_S, M')^* .$$

Hence by duality we obtain maps:

$$f_1^* : P^1(k_S, M') \longrightarrow H^1(k_S, M)^* ,$$
$$f_1'^* : P^1(k_S, M) \longrightarrow H^1(k_S, M')^* .$$

Let Ω be the maximal extension of k unramified outside S, and let G be the Galois group of the extension Ω/k. Let J denote the projection to S of the idèle group of Ω, and we put $C = J/\bar{k}_S^\times$. Then C is a class formation for extensions of k unramified outside S. For simplicity, we put $J(M) = \mathrm{Hom}(M, J)$ and $C(M) = \mathrm{Hom}(M, C)$. Let l be a prime number such that $l k_S = k_S$.

LEMMA 1. $H^r(k_S, C(F))(l) = 0 \quad (r \geqq 3)$.

PROOF. By Nakayama-Tate's Theorem, we have a commutative diagram whose horizontal arrows are isomorphisms

$$\begin{array}{ccc} H^{r-2}(L/k, \mathrm{Hom}(F, \boldsymbol{Z})) & \xrightarrow{\sim} & H^r(L/k, \mathrm{Hom}(F, \boldsymbol{Z}) \otimes H^0(L_S, C)) \\ \uparrow [L:K]\ \mathrm{inf} & & \uparrow \mathrm{inf} \\ H^{r-2}(K/k, \mathrm{Hom}(F, \boldsymbol{Z})) & \xrightarrow{\sim} & H^r(K/k, \mathrm{Hom}(F, \boldsymbol{Z}) \otimes H^0(K_S, C)) \end{array}$$

for $r \geqq 3$, where $L \supset K$ are sufficiently large Galois extensions of k unramified outside S. Since $l^\infty | [\Omega : k]$, we obtain $H^r(k_S, C(F))(l) = H^r(k_S, \mathrm{Hom}(F, \boldsymbol{Z}) \otimes C)(l) = \varinjlim_K H^r(K/k, \mathrm{Hom}(F, \boldsymbol{Z}) \otimes H^0(K_S, C))(l) = 0$.

LEMMA 2. i) $\quad H^1(k_S, J(F)) = P^1(k_S, F')$.

ii) $\quad H^r(k_S, J(F))(l) = P^r(k_S, F')(l) \quad (r \geqq 2)$.

PROOF. By Shapiro's Lemma, we have

$$H^r(k_S, J(F)) = \sum_{v \in S} H^r(G_v, \mathrm{Hom}(F, \Omega_v^\times)) \quad (r \geqq 1)$$

where G_v is the decomposition subgroup for a place lying above v and Ω_v is the extension of k_v corresponding to G_v. We remark $P^r(k_S, F')$ is the direct sum for $r \geqq 1$. Of course, if v is archimedean, $G_v = G_{k_v}$.

i) Let v be a non-archimedean prime in S. Consider the inflation-restriction sequence:

$$0 \longrightarrow H^1(G_v, \mathrm{Hom}(F, \Omega_v^\times)) \longrightarrow H^1(k_v, F') \longrightarrow H^1(\Omega_v, F').$$

Since G_{Ω_v} acts trivially on F, we get $H^1(\Omega_v, F') = 0$, hence

$$H^1(G_v, \mathrm{Hom}(F, \Omega_v^\times)) = H^1(k_v, F').$$

ii) Let v be non-archimedean. Since $l^\infty | [\Omega_v : k_v]$ and G_{Ω_v} acts trivially on F, we get $H^2(\Omega_v, F')(l)=0$. Since scd $G_{k_v} = 2$, we get $H^r(\Omega_v, F') = 0$ for $r \geqq 3$. Hence we obtain

$$H^r(G_v, \mathrm{Hom}(F, \Omega_v^\times))(l) = H^r(k_v, F')(l) \quad (r \geqq 2)$$

by the inflation-restriction sequences. Q.E.D.

THEOREM 2. *Let l be a prime number such that $l k_S = k_S$. Then*

$$f'_r : H^r(k_S, M')(l) \cong \prod_{v \text{ arch}} H^r(k_v, M')(l) \quad (r \geqq 3).$$

PROOF. a) Consider an exact sequence:

$$H^{r-1}(k_S, C(F)) \longrightarrow H^r(k_S, F') \longrightarrow H^r(k_S, J(F)) \longrightarrow H^r(k_S, C(F)).$$

By Lemmas 1 and 2, we get the theorem in case $M=F$ and $r \geqq 4$.

b) Consider a commutative exact diagram:

$$\begin{array}{ccccccccc} H^{r-1}(T')(l) & \longrightarrow & H^r(F')(l) & \longrightarrow & H^r(M')(l) & \longrightarrow & H^r(T')(l) & \longrightarrow & H^{r+1}(F')(l) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P^{r-1}(T')(l) & \longrightarrow & P^r(F')(l) & \longrightarrow & P^r(M')(l) & \longrightarrow & P^r(T')(l) & \longrightarrow & P^{r+1}(F')(l) \end{array}$$

for $r \geqq 4$, where $H^i(\ ) = H^i(k_S,\ )$ and $P^i(\ ) = P^i(k_S,\ )$. By a) and [3; Theorem 3.1 (c)], each vertical map except the middle is isomorphic. Hence by Five Lemma the middle is also isomorphic.

c) Finally we must prove the theorem for $r = 3$. We can find an open subgroup U of G such that its invariant field K is totally imaginary and $M^U = M$. We have an exact sequence:

$$0 \longrightarrow M \longrightarrow Q \longrightarrow A \longrightarrow 0$$

where Q is the induced module $\mathfrak{M}_G^U(M)$ (cf. [2; Chap. I, n° 2.5]) and $A=Q/M$. Consider a commutative exact diagram:

$$\begin{array}{ccccccc} H^3(Q')(l) & \longrightarrow & H^3(M')(l) & \longrightarrow & H^4(A')(l) & \longrightarrow & H^4(Q')(l) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P^3(Q')(l) & \longrightarrow & P^3(M')(l) & \longrightarrow & P^4(A')(l) & \longrightarrow & P^4(Q')(l)\,. \end{array}$$

By Shapiro's Lemma, we have $H^r(k_S, Q') = H^r(K_S, M')$ and $P^r(k_S, Q') = P^r(K_S, M')$. Since K is totally imaginary, $P^r(K_S, M') = 0$ for $r \geqq 3$. On the other hand, we have $H^r(K_S, F')(l) = 0$ for $r \geqq 3$, because G_{K_S} $(=U)$ acts trivially on F and $H^r(K_S, \overline{K}_S^\times)(l) = 0$ for $r \geqq 3$. Hence we have $H^r(K_S, M')(l) = H^r(K_S, T')(l)$ for $r \geqq 3$. Since $\mathrm{cd}_l\, G_{K_S} = 2$, $H^r(K_S, T')(l) = 0$ for $r \geqq 3$. Thus we get $H^3(k_S, M')(l) \cong P^3(k_S, M')(l)$ by the above diagram. Q.E.D.

REMARK 1. The proof of Tate's Theorem [3; Theorem 3.1 (c)] which has been used in the above proof has been unpublished. It can be proved as follows: In the exact sequence $0 \to T' \to J(T) \to C(T)$, the universal norms of $J(T)$ are mapped isomorphically onto the universal norms of $C(T)$. Hence we get an exact sequence: $\hat{H}^{-1}(k_S, T') \to \hat{H}^{-1}(k_S, J(T)) \to \hat{H}^{-1}(k_S, C(T)) \to \hat{H}^0(k_S, T') \to \hat{H}^0(k_S, J(T))$ (cf. [5]). Since T' has no universal norms, $\hat{H}^{-1}(k_S, T') = 0$. It is easily shown that $\hat{H}^0(k_S, T') \to \hat{H}^0(k_S, J(T))$ is injective, and $\hat{H}^{-1}(k_S, J(T)) = \prod_{v\ \mathrm{arch}} \hat{H}^{-1}(k_v, T')$. Hence we get $\hat{H}^3(k_S, T) \cong \hat{H}^{-1}(k_S, C(T))^* \cong \hat{H}^{-1}(k_S, J(T))^* \cong \prod_{v\ \mathrm{arch}} \hat{H}^{-1}(k_v, T')^* \cong \prod_{v\ \mathrm{arch}} H^3(k_v, T)$. Let K, Q and A be as in the proof of Theorem 2 respectively, and $M=T$. Then Q and A are also finite. Consider a commutative exact diagram:

$$\begin{array}{ccccccc} 0 = H^{r-1}(Q) & \longrightarrow & H^{r-1}(A) & \longrightarrow & H^r(T) & \longrightarrow & H^r(Q) = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 = P^{r-1}(Q) & \longrightarrow & P^{r-1}(A) & \longrightarrow & P^r(T) & \longrightarrow & P^r(Q) = 0 \end{array}$$

for $r \geqq 4$. By induction we get the theorem.

LEMMA 3. *Suppose that there exists an open subgroup of G which has strict cohomological dimension 2 for l. Then*

$$f_r : H^r(k_S, M)(l) \cong P^r(k_S, M)(l) \quad (r \geqq 3)\,.$$

PROOF. By Theorem 2 we obtain $H^r(k_S, N)(l) = P^r(k_S, N)(l)$ $(r \geqq 3)$ for any module N of torsion. Using the exact sequence: $0 \to F \otimes \boldsymbol{Z}_l \to F \otimes \boldsymbol{Q}_l \to F \otimes \boldsymbol{Q}_l/\boldsymbol{Z}_l \to 0$ and the above isomorphism, we get $H^r(k_S, F)(l) \cong P^r(k_S, F)(l)$

for $r \geqq 4$. Now the lemma can be proved similarly as Theorem 2 and Remark 1.

THEOREM 3. i) *If k is a number field, then we have*

$$f_r: H^r(k, M) \cong \prod_{v \text{ arch}} H^r(k_v, M) \qquad (r \geqq 3).$$

ii) *If k is a function field, then we have*

$$H^r(k_S, M) = 0 \qquad (r \geqq 3).$$

PROOF. i) It is well known that $\operatorname{scd}_l G_k = 2$ if k is totally imaginary (in case $l = 2$). ii) If k is a function field, C has no universal norm. Hence $\operatorname{scd} G_{k_S} = 2$.

REMARK 2. In general case, Tate [3] has asserted that the group G_{k_S} has strict cohomological dimension 2 for l such that $l k_S = k_S$, except if $l = 2$ and k is not totally imaginary (the proof still remains unpublished).

THEOREM 4. $\operatorname{Im} f_1$ *and* $\operatorname{Im} f_1'$ *are the exact annihilators of each other in our duality* (*). *That is, the sequence*

$$H^1(k_S, M) \xrightarrow{f_1} P^1(k_S, M) \xrightarrow{f_1'^*} H^1(k_S, M')^*$$

is exact.

PROOF. a) In case $M = T$, the theorem was obtained by Tate [3; Theorem 3.1 (b)]. We give here an outline of the proof:

For finite S, one can prove the equality

$$(1) \qquad \frac{[H^0(k_S, T)]\,[H^2(k_S, T)]}{[H^1(k_S, T)]} = \prod_{v \text{ arch}} \frac{[H^0(k_S, T)]}{|[T]|_v}$$

by using Theorem 2 (cf. [4]). We have two exact sequences:

$$0 \longrightarrow H^0(k_S, T) \longrightarrow \prod_{v \in S} \hat{H}^0(k_v, T) \longrightarrow H^2(k_S, T')^* \longrightarrow H^1(k_S, T) \longrightarrow P^1(k_S, T),$$

$$0 \longleftarrow H^0(k_S, T')^* \longleftarrow P^2(k_S, T) \longleftarrow H^2(k_S, T) \longleftarrow H^1(k_S, T')^* \longleftarrow P^1(k_S, T)$$

(cf. [5]) and a null sequence:

$$(2) \qquad H^1(k_S, T) \longrightarrow P^1(k_S, T) \longrightarrow H^1(k_S, T')^*.$$

By the equality (1) we conclude the sequence (2) is exact. The passage to infinite S is not difficult.

b) We have an exact sequence $H^1(k_S, F') \to H^1(k_S, J(F)) \to H^1(k_S, C(F))$, and two isomorphisms $H^1(k_S, J(F)) \cong P^1(k_S, F')$ (Lemma 2) and $H^1(k_S, C(F)) \cong H^1(k_S, F)^*$ (cf. [5]). Hence the theorem is proved in case $M = F$.

c) Let M_J (resp. M_C) denote the cokernel of $H^1(k_S, J(M)) \to P^1(k_S, M')$ (resp. $H^1(k_S, C(M)) \to H^1(k_S, M)^*$). For any module A of torsion, we put $A(T) = \sum_{p|[T]} A(p)$. We get following three commutative exact diagrams:

$$
(3)\qquad
\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
H^1(T') & \longrightarrow & H^1(J(T)) & \longrightarrow & H^1(C(T)) & \longrightarrow & H^2(T') \\
\| & & \downarrow & & \downarrow & & \| \\
H^1(T') & \longrightarrow & P^1(T') & \longrightarrow & H^1(T)^* & \longrightarrow & H^2(T') \\
 & & \downarrow & & \downarrow & & \\
 & & T_J & \longrightarrow & T_C & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

$$
(4)\qquad
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
H^1(J(F)) & \longrightarrow & H^1(J(M)) & \longrightarrow & H^1(J(T)) & \longrightarrow & H^2(J(F))(T) \\
\downarrow \wr \text{ iso} & & \downarrow & & \downarrow & & \downarrow \wr \text{ iso} \\
P^1(F') & \longrightarrow & P^1(M') & \longrightarrow & P^1(T') & \longrightarrow & P^2(F')(T) \\
 & & \downarrow & & \downarrow & & \\
 & & M_J & \longrightarrow & T_J & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

$$
(5)\qquad
\begin{array}{ccccccccc}
H^0(C(T)) & \longrightarrow & H^1(C(F)) & \longrightarrow & H^1(C(M)) & \longrightarrow & H^1(C(T)) & \longrightarrow & H^2(C(F)) \\
\downarrow \text{ epi} & & \downarrow \wr \text{ iso} & & \downarrow & & \downarrow & & \downarrow \wr \text{ iso} \\
H^2(T)^* & \longrightarrow & H^1(F)^* & \longrightarrow & H^1(M)^* & \longrightarrow & H^1(T)^* & \longrightarrow & \hat{H}^0(F)^* \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M_C & \longrightarrow & T_C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $H^r(\ )=H^r(k_S,\ )$ and $P^r(\ )=P^r(k_S,\ )$. In the diagram (3), $H^1(C(T)) \to H^1(T')^*$ is necessarily injective, hence $H^1(C(M)) \to H^1(M)^*$ is also injective by the diagram (5). Since all rows of the above diagrams are exact, we get exact sequences:

$$0 \longrightarrow T_J \longrightarrow T_C, \quad 0 \longrightarrow M_J \longrightarrow T_J \quad \text{and} \quad 0 \longrightarrow M_C \longrightarrow T_C .$$

A commutative diagram

$$\begin{array}{ccc} P^1(T') & \xrightarrow{f_1^*} & H^1(T)^* \\ \uparrow & & \uparrow \\ P^1(M') & \xrightarrow{f_1^*} & H^1(M)^* \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} T_J & \longrightarrow & T_C \\ \uparrow & & \uparrow \\ M_J & \longrightarrow & M_C . \end{array}$$

Hence $M_J \to M_C$ is injective. Finally consider a commutative diagram:

$$\begin{array}{ccccc} 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow \\ H^1(M') & \longrightarrow & H^1(J(M)) & \longrightarrow & H^1(C(M)) \\ \| & & \downarrow & & \downarrow \\ H^1(M') & \longrightarrow & P^1(M') & \longrightarrow & H^1(M)^* \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M_J & \longrightarrow & M_C \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

where all sequences are exact except the middle row. Hence the middle row is also exact. Q.E.D.

REMARK 3. Combining Theorem 4 with [5; Theorem 2], we have an exact sequence

$$H^1(k_S, M) \longrightarrow P^1(k_S, M) \longrightarrow H^1(k_S, M')^* \longrightarrow H^2(k_S, M) \longrightarrow P^2(k_S, M) .$$

ADDED IN PROOF: Recently, the author has given the proof of the Tate's assertion in Remark 2. See Proc. Japan Acad., 44(1968), 771-775.

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