Tôhoku Math. Journ.
21(1969), 451-455

ON A THEOREM OF HAAR

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(Received December 20, 1968)

Let $\{\varphi_n\}$ be a set of functions which are orthonormal on [0, 1]. W. Rudin [3] (cf. A. Haar [1]) proved the following result.

PROPOSITION. (I) *If* $\varphi_n \in BV\,[0, 1]$ $(n = 1, 2, \cdots)$, *then* $V(\varphi_n') > An^{1/2}$, *and*

(II) *if* $\varphi_n \in \Lambda_1[0, 1]$ $(n = 1, 2, \cdots)$, *then* $N(\varphi_n'') > A'n$,

where A *and* A' *are positive constants,* $V(f)$ *is the total variation of* $f \in BV[0,1]$, $N(f) = \sup_{x \neq y} |f(x) - f(y)|/|x-y|$ *for* $f \in \Lambda_1[0, 1]$ *and* $\{\varphi_n'\}$ $(\{\varphi_n''\})$ *is a rearrangement of* $\{\varphi_n\}$ *according to non-decreasing* $V(\varphi_n)$ $(N(\varphi_n))$.

Recently, J. J. Price [2] showed (I) under the conditions $\sum_{m \neq n} |(\varphi_m, \varphi_n)| < \infty$ and $\|\varphi_n\|_2 = 1$ $(n = 1, 2, \cdots)$ instead of $\{\varphi_n\} \in ON[0, 1]$.

In this note we extend W. Rudin's result under a much weaker hypotheses than those of J. J. Price.

Let $\{\varphi_n\}$ be a set of functions in $L^2[0, 1]$ such that

$$\|\varphi_n\|_2 = 1 \qquad (n = 1, 2, \cdots) \tag{1}$$

and

$$\sum_{m \neq n} |(\varphi_m, \varphi_n)|^2 < \infty. \tag{2}$$

For $f \in \Lambda_\alpha^p[0, 1]$ $(0 < \alpha \leqq 1,\ 1 \leqq p \leqq \infty)$ (cf. [4]) we put

$$N_\alpha^p(f) = \sup \|f(\cdot + h) - f(\cdot)\|_p / |h|^\alpha. \tag{3}$$

Then, we have the following.

THEOREM. *Under the conditions* (1) *and* (2), *if* $\varphi_n \in \Lambda_\alpha^p[0,1]$ $(n=1,2,\cdots)$, *then*

(i) *for* $1 \leqq p \leqq 2$ *and* $1 \geqq \alpha > 1/p - 1/2$,

$$N_\alpha^p(\tilde{\varphi}_n) > An^{\alpha+1/2-1/p}$$

and

(ii) *for* $2 \leqq p \leqq \infty$ *and* $1 \geqq \alpha > 0$,

$$N_\alpha^p(\tilde{\varphi}_n) > A'n^\alpha,$$

where A *and* A' *are positive constants depending on the sum* (2), p *and* α, *and* $\{\tilde{\varphi}_n\}$ *is a rearrangement of* $\{\varphi_n\}$ *according to non-decreasing* $N_\alpha^p(\varphi_n)$.

(1) and (2) are satisfied for $\{\varphi_n\} \in ON[0,1]$. So we have (I) from (i) and $N_1^1(f) \leqq A''V(f)$ for $f \in BV[0,1]$. We get (II) from (ii) for $p=\infty$ and $\alpha=1$. Further, we have the result of J. J. Price by (2) and (i).

Now, we need the following lemma.

LEMMA. *If* $f \in \Lambda_\alpha^p(-\pi,\pi)$ *and* f *is* 2π*-periodic, then*

(i′) *for* $1 \leqq p \leqq 2$ *and* $1 \geqq \alpha > 1/p - 1/2$

$$\|f - S_n(f)\|_2 = O(n^{-(\alpha+1/2-1/p)}) \tag{4}$$

and

(ii′) *for* $2 \leqq p \leqq \infty$ *and* $1 \geqq \alpha > 0$

$$\|f - S_n(f)\|_2 = O(n^{-\alpha}),$$

$$\textit{where } f \sim \sum_{-\infty}^{\infty} \hat{f}_n e^{inx} \quad \textit{and} \quad S_n(f) = \sum_{|k| \leqq n} \hat{f}_k e^{ikx}.$$

PROOF OF LEMMA. (i′) We have

$$f(x+h) - f(x-h) \sim 2i \sum_{-\infty}^{\infty} \hat{f}_n \sin(nh) e^{inx}.$$

From Hausdorff-Young's inequality we get

$$\left(\sum_{-\infty}^{\infty} |\hat{f}_n \sin(nh)|^{p'}\right)^{1/p'} \leqq A_p \|f(\cdot+h)-f(\cdot-h)\|_p$$

$$= O(h^\alpha) \quad \text{for} \quad 1 \leqq p \leqq 2,$$

where $1/p+1/p'=1$. Putting $h=\pi/2^{\nu+1}$, then

$$\text{const.}\left(\sum_{2^{\nu-1}<|n|\leqq 2^\nu} |\hat{f}_n|^{p'}\right)^{1/p'} \leqq \left(\sum_{2^{\nu-1}<|n|\leqq 2^\nu} |\hat{f}_n \sin(nh)|^{p'}\right)^{1/p'}$$

$$= O(2^{-\nu\alpha}).$$

For $2^{\mu-1} < |n| \leqq 2^\mu$,

$$\|f-S_n(f)\|_2{}^2 \leqq \sum_{|n|>2^\mu} |\hat{f}_n|^2 = \sum_{\nu=\mu}^{\infty} \sum_{2^{\nu-1}<|n|\leqq 2^\nu} |\hat{f}_n|^2$$

$$\leqq \sum_{\nu=\mu}^{\infty}\left(\sum_{2^{\nu-1}<|n|\leqq 2^\nu} |\hat{f}_n|^{p'}\right)^{2/p'} \cdot 2^{\nu(2-p)/p}$$

$$\leqq B_{p,\alpha} \sum_{\nu=\mu}^{\infty} 2^{-\nu(2\alpha+1-2/p)}.$$

From $2\alpha+1>2/p$, we have

$$\|f-S_n(f)\|_2 = O(2^{-\mu(\alpha+1/2-1/p)}) = O(n^{-(\alpha+1/2-1/p)}).$$

(ii′) $\Lambda_\alpha^p \subset \Lambda_\alpha^2$ for $p \geqq 2$, so that by (i′)

$$\|f-S_n(f)\|_2 = O(n^{-\alpha}) \qquad \text{for} \quad f \in \Lambda_\alpha^p \quad (p \geqq 2).$$

This is the best possible, because (cf. A. Zygmund [4])

$$f = \sum_{n=1}^{\infty} e^{icn(\log n)} e^{inx}/n^{1/2+\alpha} \in \Lambda_\alpha \ (0<\alpha<1),$$

but

$$\|f-S_n(f)\|_2{}^2 > A \sum_{k>n} 1/k^{1+2\alpha} \sim n^{-2\alpha}.$$

PROOF OF THEOREM. From (1) and (2), we have easily the following

$$(5)\qquad \sum_{k=1}^{n} c_k{}^2 \leqq M\|f\|_2{}^2 \quad \text{for} \quad f\in L^2[0,\ 1] \quad \text{and} \quad c_k=(f,\varphi_k),$$

where M is a positive constant depending on the sum (2). Let $\{\psi_k\}$ be a cosine set on [0, 1] and we put

$$\lambda_n^{p,\alpha} = \sup_{0<N_\alpha^p(f)<\infty} \|f-S_n(f)\|_2/N_\alpha^p(f) \quad \text{for} \quad f\in \Lambda_\alpha^p,$$

where $f\sim\sum \widehat{f}_n\psi_n$, $S_n(f)=\sum_{k=0}^{n}\widehat{f}_k\psi_k$ and $\widehat{f}_n=(f,\ \psi_n)$. From the lemma,

we have

$$(6)\qquad \lambda_n^{p,\alpha} = \begin{cases} O(n^{-(\alpha+1/2-1/p)}) & \text{for} \quad 1\leqq p\leqq 2 \quad \text{and} \quad 1\geqq\alpha\geqq 1/p-1/2 \\ O(n^{-\alpha}) & \text{for} \quad p\geqq 2 \quad \text{and} \quad 1\geqq\alpha>0. \end{cases}$$

It follows from the definition of $\lambda_n^{p,\alpha}$ that

$$\sum_{i\geqq n+1} (\varphi_k\ ,\psi_i)^2 \leqq (\lambda_n^{p,\alpha} N_\alpha^p(\varphi_k))^2.$$

Since $\{\psi_n\}$ is a complete set,

$$1=\sum_{i=1}^{\infty} (\varphi_k,\ \psi_i)^2 \leqq (\lambda_n^{p,\alpha} N_\alpha^p(\varphi_k))^2 + \sum_{i=1}^{n} (\varphi_k,\ \psi_i)^2.$$

Adding these inequalities for $k=1,2,\cdots,m$ and applying (5), we get

$$m\leqq(\lambda_n^{p,\alpha})^2\sum_{k=1}^{m} (N_\alpha^p(\varphi_k))^2+M\cdot n \quad \text{for every } m \text{ and } n.$$

This inequality holds for $N_\alpha^p(\widetilde{\varphi}_k)$, so putting $n=[m/2M]$,

$$m/2\leqq(\lambda_n^{p,\alpha} N_\alpha^p(\widetilde{\varphi}_m))^2\cdot m.$$

From (6) we have the results.

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