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SOME TAUBERIAN THEOREMS CONCERNING (S^*, μ) TRANSFORMATIONS

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1. Introduction. The regular series to sequence (S^*, μ) transform of a series $\sum_{i=0}^{\infty} a_i$ is defined as follows:

$$S_n^*(\beta)=\sum_{i=0}^{\infty} a_i \sum_{k=i}^{\infty} \binom{k+n}{n} \int_0^1 (1-t)^k t^{n+1} d\beta(t),\ n\geqq 0, \tag{1.1}$$

where $\beta(t)$ satisfies

(1.2) $\beta(t)$ is of bounded variation in $[0,1]$, $\beta(1)-\beta(0+)=1$ and $\beta(1)=\beta(1-)$.

The series to sequence (S^*, μ) transformation is the series to sequence analogues of the sequence to sequence (S^*, μ) transformation defined by Ramanujan [7] §4. We shall be interested in finding Tauberian estimates of the following form. For a series $\sum_{i=0}^{\infty} a_i$ denote $s_n=\sum_{i=0}^{n} a_i$, then what is the best possible constant A satisfying

$$\limsup_{\lambda\to\infty} |S^*_{n(\lambda)}(\beta)-s_{m(\lambda)}| \leqq A \limsup_{n\to\infty} |na_n|$$

where $n(\lambda)$, $m(\lambda)$ are given functions assuming integral values only, and all series $\sum_{i=0}^{\infty} a_i$ satisfying the Tauberian condition

$$\limsup_{n\to\infty} |na_n| < \infty. \tag{1.3}$$

What is the best constant B satisfying

$$\limsup_{\lambda\to\infty} |S^*_{n(\lambda)}(\beta)-S^*_{m(\lambda)}(\gamma)| \leqq B \limsup_{n\to\infty} |na_n|$$

where $\gamma(t)$ is another function satisfying (1.2), $n(\lambda)$, $m(\lambda)$ are as before and

$\sum_{i=0}^{\infty} a_i$ satisfies (1. 3). What is the best constant C satisfying

$$\limsup_{n\to\infty} |S^*_{n(\lambda)}(\beta) - s_{m(\lambda)}| \leqq C \limsup_{\lambda\to\infty} |b_n|$$

where $n(\lambda), m(\lambda)$ are as before, $b_n = (a_1 + 2a_2 + \cdots + na_n)/(n+1)$ $n \geqq 1$ and our series satisfies the Tauberian condition weaker than (1. 3),

$$\limsup_{n\to\infty} |b_n| < \infty. \tag{1. 4}$$

In order to simplify the notation we write instead of (1.2),

(1. 5) $\beta(t)$ is of bounded variation in [0,1], $\beta(0+) = 0$ and $\beta(1) = \beta(1-) = 1$.
We shall restrict ourselves to function $\beta(t)$ satisfying

$$\beta(0) = 0, \quad \int_0^1 x^{-1} |\beta(x)| \, dx < \infty, \int_0^1 (1-x)^{-1} |1 - \beta(x)| \, dx < \infty. \tag{1. 6}$$

By inspecting the Tauberian estimates obtained in the following sections one sees that the condition (1. 6) is necessary in order to obtain finite constants A, B, or C and thus not much of a restriction. Recently the first problem was discussed by S. Sherif [8] under an additional assumption that $\beta(t)$ is non-decreasing in [0, 1].

2. Main results.

THEOREM 1. *For a series* $\sum_{i=0}^{\infty} a_i$ *satisfying* (1. 3) *and a function* $\beta(t)$ *satisfying* (1. 5) *and* (1. 6) *we have for each* q, $0 < q < \infty$ *and any two functions* $n(\lambda) \to \infty$ $m(\lambda) \to \infty$ *assuming integral values only and satisfying* $m(\lambda)/n(\lambda) \to q$ *as* $\lambda \to \infty$,

$$\limsup_{\lambda\to\infty} |S^*_{n(\lambda)}(\beta) - s_{m(\lambda)}| \leqq A_q \limsup_{n\to\infty} |na_n| \tag{2. 1}$$

where

$$A_q = \int_0^{1/(q+1)} \frac{|\beta(x)|}{x(1-x)} \, dx + \int_{1/(q+1)}^1 \frac{|1-\beta(x)|}{x(1-x)} \, dx. \tag{2. 2}$$

The constant A_q *is the best possible in the following sense. There exists a series* $\sum_{i=0}^{\infty} a_i$ *satisfying* (1. 3) *and the members of inequality* (2. 1) *are equal.*

Theorem 1 for a non-decreasing $\beta(t)$ was proved by S.Sherif [8]. For the function $\beta(t)=0$ for $0\leqq t<1-\alpha$ and $\beta(t)=1$ for $1-\alpha\leqq t\leqq 1$ $(0<\alpha<1)$ the series to sequence (S^*, μ) transform of a series $\sum_{i=0}^{\infty} a_i$ is the sequence to sequence S_α transform of the sequence $s_n=\left\{\sum_{i=0}^{n} a_i\right\}(n\geqq 0)$ defined by Meyer-König [5]. Theorem 1 for the S_α transformation was proved by Biegert [2].

THEOREM 2. *For a series* $\sum_{i=0}^{\infty} a_i$ *satisfying* (1.3) *and functions* $\beta(t)$ *and* $\gamma(t)$ *satisfying* (1.5) *and* (1.6) *we have for each* q, $0<q<\infty$ *and any two functions* $n(\lambda)\to\infty$ $m(\lambda)\to\infty$ *assuming integral values only and satisfying* $m(\lambda)/n(\lambda)\to q$ *as* $\lambda\to\infty$,

$$\limsup_{\lambda\to\infty}|S^*_{n(\lambda)}(\beta)-S^*_{m(\lambda)}(\gamma)|\leqq B_q \limsup_{n\to\infty}|na_n| \tag{2.3}$$

where

$$B_q=\int_0^1 \frac{\left|\beta\left(\frac{t}{q-(q-1)t}\right)-\gamma(t)\right|}{t(1-t)}\,dt. \tag{2.4}$$

The constant is the best possible in the following sense. There exists a series $\sum_{i=0}^{\infty} a_i$ *satisfing* (1.3) *and the members of inequality* (2.3) *are equal.*

REMARK. We note that the constant B_q is better than estimates we could obtain by Theorem 1, by introducing a function $p(\lambda)$ assuming integral values only and such that $p(\lambda)/m(\lambda)\to a$, $0<a<\infty$ and estimating $|S^*_n(\beta)-S^*_m(\gamma)|$ by

$$|S^*_n(\beta)-S^*_m(\gamma)|\leqq|S^*_n(\beta)-s_p|+|s_p-S^*_m(\gamma)|.$$

The computations are left to the reader.

THEOREM 3. *For a series* $\sum_{i=0}^{\infty} a_i$ *satisfying* (1.4) *and a continuous function* $\beta(t)$ *satisfying* (1.5) *and* (1.6), *we have for each* q, $0<q<\infty$ *and any two functions* $n(\lambda)\to\infty$, $m(\lambda)\to\infty$ *assuming integral values only and satisfying* $m(\lambda)/n(\lambda)\to q$ *as* $\lambda\to\infty$,

$$\limsup_{\lambda\to\infty}|S^*_{n(\lambda)}(\beta)-s_{m(\lambda)}|\leqq C_q \limsup_{n\to\infty}|b_n| \tag{2.5}$$

where

$$(2.6)\quad C_q = 1 + \int_{0+}^{1/(1+q)} \frac{1-t}{t}\left|d\left[\frac{t}{1-t}\beta(t)\right]\right| + \int_{1/(1+q)}^{1-} \frac{1-t}{t}\left|d\left[\frac{t}{1-t}(1-\beta(t))\right]\right|$$

$$\left(\text{where } \int_{0+} = \lim_{\rho\downarrow 0}\int_{\rho} \text{ and } \int^{1-} = \lim_{\eta\uparrow 1}\int^{\eta}\right).$$

The constant C_q is the best possible in the following sense. There exists a series $\sum_{i=0}^{\infty} a_i$ satisfying (1.4) *and the members of inequality* (2.5) *are equal.*

The following is an immediate consequence of Theorem 3.

COROLLARY. *If, in addition to the assumption of Theorem* 3, *the functions $\frac{t}{1-t}\beta(t)$ and $\frac{t}{1-t}(1-\beta(t))$ are non-decreasing for $0<t<1$, then*

$$C_q = \int_0^{1/(1+q)} \frac{\beta(x)}{x(1-x)}\,dx + \int_{1/(1+q)}^{1} \frac{1-\beta(x)}{x(1-x)}\,dx + 2\beta\left(\frac{1}{1+q}\right).$$

Inasmuch as $0 \leqq \beta(t) \leqq 1$ for $0 \leqq t \leqq 1$ we have

$$C_q = A_q + 2\beta\left(\frac{1}{1+q}\right).$$

THEOREM 4. *For a series $\sum_{i=0}^{\infty} a_i$ satisfying* (1.4) *and a function $\beta(t)$ satisfying* (1.5), (1.6) *and $\beta(t) = \frac{1}{2}[\beta(t+)+\beta(t-)]$ for $0<t<1$, we have for each q, $0<q<\infty$,*

$$(2.7)\qquad \limsup_{n\to\infty} |S_n^*(\beta) - s_{[nq]}| \leqq C_q \limsup_{n\to\infty} |b_n|$$

where C_q is defined by (2.6) *and is the best constant possible in the following sense. There exists a series $\sum_{i=0}^{\infty} a_i$ satisfying* (1.6) *and the members of inequality* (2.7) *are equal.*

3. Proofs of Theorem 1 and 2.

PROOF OF THEOREM 1. First we prove that whenever the series $\sum_{i=0}^{\infty} a_i$ satisfies (1.3) and $\beta(t)$ satisfies (1.5) and (1.6), then $S_n^*(\beta)$ exists for every $n \geqq 0$. It follows immediately that for $0 \leqq t \leqq 1$ and $i > 0$

$$\frac{d}{dt} \sum_{k=i}^{\infty} \binom{k+n}{n} (1-t)^k t^{n+1} = \frac{d}{dt}\left[1 - \sum_{k=0}^{i-1} \binom{k+n}{n} (1-t)^k t^{n+1}\right]$$

$$= -it^{-1}(1-t)^{-1}\binom{i+n}{n}(1-t)^i t^{n+1},$$

whence

$$\sum_{k=i}^{\infty} \binom{k+n}{n} (1-t)^k t^{n+1} = i\int_t^1 u^{-1}(1-u)^{-1}\binom{i+n}{n}(1-u)^i u^{n+1} du. \tag{3.1}$$

By Beppo-Levi's theorem for every $i \geqq 0$

$$\sum_{k=i}^{\infty} \binom{k+n}{n} \int_0^1 (1-t)^k t^{n+1} \left|d\beta(t)\right| = \int_0^1 \sum_{k=i}^{\infty} \binom{k+n}{n} (1-t)^k t^{n+1} \left|d\beta(t)\right|$$

$$\leqq \int_0^1 \left|d\beta(t)\right| < \infty,$$

hence by (3.1) and integration by parts we obtain

$$\sum_{k=i}^{\infty} \binom{k+n}{n} \int_0^1 (1-t)^k t^{n+1} d\beta(t) = \int_0^1 \sum_{k=i}^{\infty} \binom{k+n}{n} (1-t)^k t^{n+1} d\beta(t) \tag{3.2}$$

$$= i\int_0^1 \int_t^1 u^{-1}(1-u)^{-1}\binom{i+n}{n}(1-u)^i u^{n+1} du\, d\beta(t)$$

$$= i\int_0^1 \frac{\beta(u)}{u(1-u)} \binom{i+n}{n}(1-u)^i u^{n+1} du.$$

Denote $\Delta_{ni} = \sum_{k=i}^{\infty} \binom{k+n}{n} \int_0^1 (1-t)^k t^{n+1} d\beta(t)$, then by (3.2)

$$\sum_{i=0}^{\infty} \left|a_i\right| \left|\Delta_{ni}\right| \leqq \left|a_0\right| + \sum_{i=1}^{\infty} \left|ia_i\right| \int_0^1 \frac{|\beta(u)|}{u(1-u)} \binom{i+n}{n}(1-u)^i u^{n+1} du.$$

By (1. 3) and Beppo-Levi's theorem

$$\leqq \left|a_0\right| + L\int_0^1 \frac{|\beta(u)|}{u(1-u)}\sum_{i=1}^{\infty}\binom{i+n}{n}(1-u)^i u^{n+1}du$$

and by (1. 6)

$$= \left|a_0\right| + L\int_0^1 \frac{|\beta(u)|}{u(1-u)}(1-u^{n+1})\ du < \infty.$$

Therefore we have proved the existence of $S_n^*(\beta)$ for every $n \geqq 0$.

Now

$$S_n^*(\beta) - s_m = \sum_{i=0}^{\infty} a_i\Delta_{ni} - \sum_{i=0}^{m} a_i = -\sum_{i=0}^{m} a_i(1-\Delta_{ni}) + \sum_{i=m+1}^{\infty} a_i\Delta_{ni}. \tag{3. 3}$$

By Agnew's theorem [1] we have to show that when $n \equiv n(\lambda)$ and $m \equiv m(\lambda)$

$$\lim_{\lambda\to\infty} \frac{1}{i}\sum_{k=0}^{i-1}\binom{k+n}{n}\int_0^1 (1-t)^k t^{n+1} d\beta(t) = 0 \text{ for } i = 1,2,\cdots \tag{3. 4}$$

and

$$\limsup_{\lambda\to\infty}\left\{\sum_{i=1}^{m}\frac{1}{i}\left|1-\Delta_{ni}\right| + \sum_{i=m+1}^{\infty}\frac{1}{i}\left|\Delta_{ni}\right|\right\} = A_q. \tag{3. 5}$$

Now (3. 4) follows by (1. 5) in the same manner in which the regularity of (S^*, μ) is proved (see [7] §4). In a way similar to the proof of (3. 2) we obtain

$$\frac{1}{i}\sum_{k=0}^{i-1}\binom{k+n}{n}\int_0^1 (1-t)^k t^{n+1} d\beta(t) = \int_0^1 \frac{1-\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1}du, \tag{3. 6}$$

and thus by (3. 2) and (3. 6) it follows that

$$\sum_{i=1}^{m}\frac{1}{i}\left|1-\Delta_{ni}\right| + \sum_{i=m+1}^{\infty}\frac{1}{i}\left|\Delta_{ni}\right| = \sum_{i=1}^{m}\left|\int_0^1 \frac{1-\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1}du\right|$$
$$+ \sum_{i=m+1}^{\infty}\left|\int_0^1 \frac{\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1}du\right|.$$

For $u \leqq \dfrac{n}{n+m}$

$$\frac{d}{du}\sum_{i=1}^{m}\binom{i+n}{n}(1-u)^i u^n=\sum_{i=1}^{m}\binom{i+n}{n}(1-u)^{i-1}u^{n-1}[(n-(n+i)u]\geqq 0,$$

and for $1>u\geqq\dfrac{n+1}{n+m+1}$

$$\frac{d}{du}\sum_{i=m+1}^{\infty}\binom{i+n}{n}(1-u)^{i-1}u^{n+1}=\sum_{i=m+1}^{\infty}\binom{i+n}{n}(1-u)^{i-2}u^n[n+1-(n+i)u]\leqq 0.$$

Since $m(\lambda)/n(\lambda)\to q$ as $\lambda\to\infty$, for $\lambda\geqq\lambda_0$ we have $\dfrac{n}{n+m}\geqq\dfrac{1}{2(1+q)}$ and $\dfrac{n+1}{n+m+1}$ $\leqq\dfrac{1}{2}\left[\dfrac{1}{1+q}+1\right]$, consequently for $\lambda\geqq\lambda_0$ the functions $u^{-1}\sum_{i=1}^{m}\binom{i+n}{n}(1-u)^i u^{n+1}$ are non-decreasing in $0<u\leqq\dfrac{1}{2(1+q)}$ and the functions $(1-u)^{-1}\sum_{i=m+1}^{\infty}\binom{i+n}{n}$ $\left(1-u\right)^i u^{n+1}$ are non-increasing in $\dfrac{1}{2}\left[\dfrac{1}{1+q}+1\right]\leqq u<1.$

Applying the approximation properties of the Bernstein power series of Meyer-König and Zeller [6] we obtain

$$\lim_{\lambda\to\infty}\frac{1-\beta(u)}{u(1-u)}\sum_{i=1}^{m}\binom{i+n}{n}(1-u)^i u^{n+1}=\begin{cases}\dfrac{1-\beta(u)}{u(1-u)} & \text{if } \dfrac{1}{1+q}<u<1\\[2ex] 0 \text{ if} & 0<u<\dfrac{1}{1+q}\end{cases}$$

and the convergence is dominated by the integrable function $K\dfrac{1-\beta(u)}{1-u}$ for some constant K. Similarly

$$\lim_{\lambda\to\infty}\frac{\beta(u)}{u(1-u)}\sum_{i=m+1}^{\infty}\binom{i+u}{n}(1-u)^i u^{n+1}=\begin{cases}\dfrac{\beta(n)}{u(1-u)} \text{ if } 0<u<\dfrac{1}{1+q}\\[2ex] 0 \text{ if } \dfrac{1}{1+q}<u<1\end{cases}$$

and the convergence is dominated by the integrable function $H\dfrac{\beta(u)}{u}$ for some constant H.

A proof similar to that of Theorem 2.1 of [3] enables us to conclude that

$$\lim_{\lambda\to\infty}\sum_{i=1}^{m}\left|\int_0^1\frac{1-\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1}du\right|=\int_{1/(1+q)}^{1}\frac{|1-\beta(u)|}{u(1-u)}\,du$$

and

$$\lim_{\lambda\to\infty}\sum_{i=m+1}^{\infty}\left|\int_0^1 \frac{\beta(u)}{u(1-u)}\binom{i+n}{u}(1-n)^i u^{n+1}du\right| = \int_0^{1/(1+q)} \frac{|\beta(u)|}{u(1-u)}du.$$

This proves (3. 5) and completes the proof of our theorem.

REMARK. If $\beta(t)$ is of bounded variation in [0, 1] and satisfies

$$\int_0^1 x^{-1}\int_0^x |d\beta(u)|\,dx < \infty, \tag{3.7}$$

and the series $\sum_{i=0}^{\infty} a_i$ satisfies (1. 3), then $\{S_n^*(\beta)\}$ is the sequence to sequence (S^*,μ) transform defined by Ramanujan [7] §4. This is exactly the case if $\beta(t)$ is non-decreasing in [0, 1] and satisfies (1. 5) and (1. 6).

PROOF. By (1. 3) and (3. 2) for the function $\gamma(u) = \int_0^u |d\beta(t)|$ we obtain

$$\sum_{i=0}^{\infty}|a_i|\sum_{k=i}^{\infty}\binom{k+n}{n}\int_0^1 (1-t)^k t^{n+1}|d\beta(t)| \tag{3.8}$$

$$\leqq |a_0| + L\sum_{i=1}^{\infty}\int_0^1 u^{-1}(1-u)^{-1}\int_0^u |d\beta(t)|\binom{i+n}{n}(1-u)^i u^{n+1}du,$$

and applying Beppo-Levi's theorem and (3. 7) we get

$$= |a_0| + L\int_0^1 u^{-1}(1-u)^{-1}(1-u^{n+1})\int_0^u |d\beta(t)|\,du < \infty.$$

Hence

$$\sum_{i=0}^{\infty} a_i\sum_{k=i}^{\infty}\int_0^1\binom{k+n}{n}(1-t)^k t^{n+1}d\beta(t) = \sum_{k=0}^{\infty}\binom{k+n}{n}\int_0^1 (1-t)^k t^{n+1}d\beta(t)\sum_{i=0}^{k} a_i$$

where the change of order of summation is justified by (3. 8)

$$= \sum_{k=0}^{\infty}\binom{k+n}{n}\int_0^1 (1-t)^k t^{n+1}d\beta(t)s_k.$$

This concludes our proof.

For the proof of Theorem 2 we need the following lemma.

LEMMA. *Let $0<\eta<1$ be fixed and suppose that $\beta(t)$ satisfies* (1.5) *and* (1.6). *For $0<\delta<\eta$ define*

$$\alpha_{\delta,\eta}^{(\beta)}(t)=\begin{cases}\dfrac{\beta(t)}{t(1-t)} & \delta\leqq t\leqq\eta\\ 0 & \textit{elsewhere.}\end{cases}\tag{3.9}$$

Then for every $\varepsilon>0$ there exists $0<\delta_0(\varepsilon)<\eta$ such that for any δ, $0<\delta\leqq\delta_0$ there exists $n_0(\varepsilon,\delta)$ such that

$$I=\sum_{i=1}^{\infty}\left|\int_0^{\eta}\frac{\beta(t)}{t(1-t)}\binom{i+n}{n}(1-t)^i t^{n+1}dt-\alpha_{\delta,\eta}^{(\beta)}\left(\frac{n}{i+n}\right)\int_0^1\binom{i+n}{n}(1-t)^i t^{n+1}dt\right|<\varepsilon$$

provided $n\geqq n_0$.

PROOF. Given $\varepsilon>0$ choose $\delta_0(\varepsilon)$ such that for any δ, $0<\delta\leqq\delta_0$ we have

$$\int_0^{\delta}\frac{|\beta(t)|}{t(1-t)}\,dt<\frac{\varepsilon}{4}.\tag{3.10}$$

This is possible since $\beta(t)$ satisfies (1.6). Let δ, $0<\delta\leqq\delta_0$ be fixed, then by (3.10)

$$\begin{aligned}0\leqq I\leqq\frac{\varepsilon}{4}&+\left\{\int_0^{\delta}+\int_{\eta}^{1}\right\}\sum_{i=1}^{\infty}\binom{i+n}{n}(1-t)^i t^{n+1}\left|\alpha_{\delta,\eta}^{(\beta)}\left(\frac{n}{i+n}\right)\right|dt\\&+\int_{\delta}^{\eta}\sum_{i=1}^{\infty}\binom{i+n}{n}(1-t)^i t^{n+1}\left|\frac{\beta(t)}{t(1-t)}-\alpha_{\delta,\eta}^{(\beta)}\left(\frac{n}{i+n}\right)\right|dt\\&=\frac{\varepsilon}{4}+I_1+I_2,\ \text{say.}\end{aligned}\tag{3.11}$$

The Bernstein power series of Meyer-König and Zeller [6] admit the following approximation property. If $f(s)$ is bounded in $[0,1]$, then at each point of continuity, $0<s<1$, of $f(s)$ we have

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\binom{i+n}{n}(1-s)^i s^{n+1}\,f\left(\frac{n}{i+n}\right)=f(s).\tag{3.12}$$

Our $\alpha_{\delta,\eta}^{(\beta)}(t)$ is bounded in $[0,1]$ since $\sup_{0\leqq t\leqq 1}\left|\alpha_{\delta,\eta}^{(\beta)}(t)\right|=\sup_{\delta\leqq t\leqq\eta}\left|\frac{\beta(t)}{t(1-t)}\right|=M<\infty$. So by

(3.12) we obtain

$$(3.13)\qquad \lim \sum_{i=1}^{\infty}\binom{i+n}{n}(1-t)^i t^{n+1}\alpha_{\delta,\eta}^{(\beta)}\left(\frac{n}{i+n}\right)=0 \text{ for } 0<t<\delta \text{ and } \eta<t<1.$$

Moreover the convergence in (3.13) is dominated by M whence

$$(3.14)\qquad \lim_{n\to\infty} I_1 = 0.$$

For a fixed t, $\delta\leqq t\leqq\eta$ the function $f(s)=\left|\frac{\beta(t)}{t(1-t)}-\alpha_{\delta,\eta}^{(\beta)}(s)\right|$ is bounded for $0\leqq s \leqq 1$, hence at each point of continuity $\delta\leqq s\leqq\eta$ of $f(s)$ we have by (3.12)

$$(3.15)\qquad \lim_{n\to\infty}\sum_{i=1}^{\infty}\binom{i+n}{n}(1-s)^i s^{n+1}\left|\frac{\beta(t)}{t(1-t)}-\alpha_{\delta,\eta}^{(\beta)}\left(\frac{n}{i+n}\right)\right|=\left|\frac{\beta(t)}{t(1-t)}-\frac{\beta(s)}{s(1-s)}\right|.$$

For $\delta<s<\eta$, $f(s)$ is continuous if and only if $\beta(s)$ is continuous, that is almost everywhere in $[\delta,\eta]$ whence by (3.15)

$$(3.16)\qquad \lim_{n\to\infty}\sum_{i=1}^{\infty}\binom{i+n}{n}(1-t)^i t^{n+1}\left|\frac{\beta(t)}{t(1-t)}-\alpha_{\delta,\eta}^{(\beta)}\left(\frac{n}{i+n}\right)\right|=0$$

almost everywhere in $[\delta, \eta]$. Once again the convergence in (3.16) is dominated by $2M$ and so we obtain

$$(3.17)\qquad \lim_{n\to\infty} I_2 = 0.$$

Combining (3.11), (3.14) and (3.17), our lemma is proved.

PROOF OF THEOREM 2. By (3.2) we obtain

$$S_n^*(\beta)-S_m^*(\gamma)=\sum_{i=1}^{\infty} i a_i\int_0^1\left[\frac{\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1}\right.$$
$$\left.-\frac{\gamma(u)}{u(1-u)}\binom{i+m}{m}(1-u)^i u^{m+1}\right]du.$$

By Agnew's theorem we have to show that

$$(3.18)\qquad \lim_{\lambda\to\infty}\int_0^1\left[\frac{\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1}-\frac{\gamma(u)}{u(1-u)}\binom{i+m}{m}(1-u)^i u^{m+1}\right]du=0$$

for $i=1,2,\cdots,$

and

$$\text{(3.19)} \qquad \limsup_{\lambda\to\infty}\sum_{i=1}^{\infty}\left|\int_0^1\left[\frac{\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1} - \frac{\gamma(u)}{u(1-u)}\binom{i+m}{m}(1-u)^i u^{m+1}\right]du\right| = B_q.$$

Now (3.18) is proved like (3.4) and we have to prove only (3.19). Let $0<\eta<1$ be chosen such that if $\theta=\eta/[q-(q-1)\eta]$, then

$$\text{(3.20)} \qquad \int_\theta^1 \frac{|1-\beta(x)|}{x(1-x)}\,dx<\varepsilon, \qquad \int_\eta^1 \frac{|1-\gamma(x)|}{x(1-x)}\,dx<\varepsilon.$$

This is possible by (1.6) and the fact that $\theta\to 1$ as $\eta\to 1$. By Agnew's theorem

$$\limsup_{\lambda\to\infty}\sum_{i=1}^{\infty}\left|\int_\theta^1\frac{\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1}du-\int_\eta^1\frac{\gamma(u)}{u(1-u)}\binom{i+m}{m}(1-u)^i u^{m+1}du\right|$$
$$=\limsup_{\lambda\to\infty}|S^*_{n(\lambda)}(\bar\beta)-S^*_{m(\lambda)}(\bar\gamma)|$$

where

$$\bar\beta(t)=\begin{cases}\beta(t) & \text{if } \theta\leqq t\leqq 1\\ 0 & \text{if } 0\leqq t<\theta\end{cases}$$

and

$$\bar\gamma(t)=\begin{cases}\gamma(t) & \text{if } \eta\leqq t\leqq 1\\ 0 & \text{if } 0\leqq t<\eta.\end{cases}$$

Let $p(\lambda)=\left[\frac{1-\eta}{\eta}m(\lambda)\right]$ (where $[x]$ denotes the largest integer not greater than x), then $p(\lambda)/m(\lambda)\to\frac{1-\eta}{\eta}$ as $\lambda\to\infty$ and $p(\lambda)/n(\lambda)\to\frac{1-\eta}{\eta}q$ as $\lambda\to\infty$. Now

$$|S^*_n(\bar\beta)-S^*_m(\bar\gamma)\leqq|S^*_n(\bar\beta)-s_p|+|s_p-S^*_m(\bar\gamma)|$$

and it follows by Theorem 1 and (3.20) that

$$\text{(3.21)} \qquad \limsup_{\lambda\to\infty}|S^*_n(\bar\beta)-S^*_m(\bar\gamma)|\leqq\limsup_{\lambda\to\infty}|S^*_n(\bar\beta)-s_p|+\limsup_{\lambda\to\infty}|s_p-S^*_m(\bar\gamma)|$$
$$=\int_\theta^1\frac{|1-\beta(x)|}{x(1-x)}dx+\int_\eta^1\frac{|1-\gamma(x)|}{x(1-x)}dx<2\varepsilon.$$

Let $\delta>0$ be chosen such that if $\rho=\delta/[q-(q-1)\delta]$, then

(3. 22)

$$\begin{cases} \int_0^\rho \frac{|\beta(t)|}{t(1-t)}dt<\varepsilon, \int_0^\delta \frac{|\gamma(t)|}{t(1-t)}dt<\varepsilon, \\ \limsup_{n\to\infty}\sum_{i=1}^\infty\left|\int_0^\theta \binom{i+n}{n}(1-t)^i t^{n+1}\frac{\beta(t)}{t(1-t)}dt-\alpha_{\rho,\theta}^{(\beta)}\left(\frac{n}{i+n}\right)\int_0^1\binom{i+n}{n}(1-t)^i t^{n+1}dt\right|<\varepsilon, \\ \limsup_{m\to\infty}\sum_{i=1}^\infty\left|\int_0^\eta \binom{i+m}{m}(1-t)^i t^{m+1}\frac{\gamma(t)}{t(1-t)}dt-\alpha_{\delta,\eta}^{(\gamma)}\left(\frac{m}{i+m}\right)\int_0^1\binom{i+m}{m}(1-t)^i t^{m+1}dt\right|<\varepsilon. \end{cases}$$

The existence of δ is guaranteed by our lemma, (1. 6) and the fact that $\rho\to0$ as $\delta\to0$. Denote

$$\beta_{\rho,\theta}(t)=\begin{cases}\beta(t) \text{ if } \rho\leqq t\leqq\theta \\ 0 \text{ if elsewhere,}\end{cases}$$

and let $0<t\leqq1$ be a fixed point of continuity of $\beta_{\rho,\theta}(t)$. For $\zeta>0$ let $\tau(\zeta)>0$ be such that $|\beta_{\rho,\theta}(t)-\beta_{\rho,\theta}(t_1)|<\zeta$ provided $|t-t_1|<2\tau$ and $\tau<\frac{\rho}{2}$, $\tau<\frac{1-\theta}{2}$. For $\lambda\geqq\lambda_0$ we have $\left|\frac{m}{n}-q\right|<\frac{2\tau}{q}$ and $\frac{m}{n}>\frac{q}{2}$ which imply $\left|\frac{n}{i+n}-\frac{m}{qi+m}\right|<\tau$. Now

$$I=\sum_{i=1}^\infty \frac{\left|\beta_{\rho,\theta}\left(\frac{n}{i+n}\right)-\beta_{\rho,\theta}\left(\frac{m}{qi+m}\right)\right|}{\frac{n}{i+n}\ \frac{i}{i+n}}\binom{i+n}{n}(1-t)^i t^{n+1}$$

$$=\left\{\sum_{\left|\frac{n}{i+n}-t\right|\geqq\tau}+\sum_{\left|\frac{n}{i+n}-t\right|<\tau}\right\}\frac{\left|\beta_{\rho,\theta}\left(\frac{n}{i+n}\right)-\beta_{\rho,\theta}\left(\frac{m}{qi+m}\right)\right|}{\frac{n}{i+n}\ \frac{i}{i+n}}\binom{i+n}{n}(1-t)^i t^{n+1}$$

$$=I_1+I_2,\quad \text{say.}$$

It follows by (3. 12) that

$$(3.23)\qquad 0\leqq I_1\leqq\frac{8M}{\rho(1-\theta)\tau^2}\sum_{i=0}^{\infty}\left(\frac{n}{i+n}-t\right)^2\binom{i+n}{n}(1-t)^i t^{n+1}\to 0 \text{ as } n\to\infty,$$

where $M=\sup_{0\leqq t\leqq 1}|\beta(t)|$, and

$$(3.24)\qquad 0\leqq I_2\leqq\zeta\,\frac{4}{\rho(1-\theta)}\sum_{i=0}^{\infty}\binom{i+n}{n}(1-t)^i t^{n+1}=\frac{4\zeta}{\rho(1-\theta)}.$$

It follows by (3.23) and (3.24) that $0\leqq\limsup_{\lambda\to\infty}\; I\leqq\frac{4\zeta}{\rho(1-\theta)}$ for every $\zeta>0$, consequently $\lim_{\lambda\to\infty} I=0$ for every point of continuity $0<t\leqq 1$ of $\beta_{\rho,\theta}(t)$, this is almost everywhere in [0, 1], and since the convergence is dominated by $\frac{8M}{\rho(1-\theta)}$ we obtain

$$(3.25)\qquad \lim_{\lambda\to\infty}\sum_{i=1}^{\infty}\frac{\left|\beta_{\rho,\theta}\left(\frac{n}{i+n}\right)-\beta_{\rho,\theta}\left(\frac{m}{qi+m}\right)\right|}{\frac{n}{i+n}\;\frac{i}{i+n}}\int^1\binom{i+n}{n}(1-t)^i t^{n+1}dt$$

$$=\lim_{\lambda\to\infty}\int_0^1\left[\sum_{i=1}^{\infty}\frac{\left|\beta_{\rho,\theta}\left(\frac{n}{i+n}\right)-\beta_{\rho,\theta}\left(\frac{m}{qi+m}\right)\right|}{\frac{n}{i+n}\;\frac{i}{i+n}}\binom{i+n}{n}(1-t)^i t^{n+1}\right]dt=0.$$

Since $\int_0^1\binom{i+n}{n}(1-t)^i t^{n+1}dt=\frac{n+1}{(i+n+1)(i+n+2)}$ we obtain similarly

$$(3.26)\qquad \lim_{\lambda\to\infty}\sum_{i=1}^{\infty}\beta_{\rho,\theta}\left(\frac{m}{qi+m}\right)\left|\frac{\int_0^1\binom{i+n}{n}(1-t)^i t^{n+1}dt}{\frac{n}{i+n}\;\frac{i}{i+n}}-\frac{\int_0^1\binom{i+m}{m}(1-t)^i t^{m+1}dt}{\frac{m}{i+m}\;\frac{i}{i+m}}\right|=0.$$

Now $\beta_{\rho,\theta}\left(\frac{m}{qi+m}\right)=\beta_{\rho,\theta}\left(\frac{\frac{m}{i+m}}{q-(q-1)\frac{m}{i+m}}\right)$, so if we denote

$\varphi(t)=\beta\left(\frac{t}{q-(q-1)t}\right)$, then it follows that $\varphi_{\delta,\eta}(t)=\beta_{\rho,\theta}\left(\frac{t}{q-(q-1)t}\right)$

and therefore $\alpha_{\delta,\eta}^{(\varphi)}(t) = \dfrac{\beta_{\rho,\theta}\left(\dfrac{t}{q-(q-1)t}\right)}{t(1-t)}$. Combining (3. 25) and (3. 26) we obtain

$$\begin{aligned}(3.27)\qquad & \lim_{\lambda\to\infty}\sum_{i=1}^{\infty}\left|\alpha_{\rho,\theta}^{(\beta)}\left(\frac{n}{i+n}\right)\int_0^1\binom{i+n}{n}(1-t)^i t^{n+1}dt\right. \\ & \qquad\left.-\alpha_{\delta,\eta}^{(\gamma)}\left(\frac{m}{i+m}\right)\int_0^1\binom{i+m}{m}(1-t)^i t^{m+1}dt\right| \\ & =\lim_{\lambda\to\infty}\sum_{i=1}^{\infty}\left|\alpha_{\delta,\eta}^{(\varphi)}\left(\frac{m}{i+m}\right)-\alpha_{\delta,\eta}^{(\gamma)}\left(\frac{m}{i+m}\right)\right|\int_0^1\binom{i+m}{m}(1-t)^i t^{m+1}dt \\ & =\lim_{\lambda\to\infty}\int_0^1\sum_{i=1}^{\infty}\left|\alpha_{\delta,\eta}^{(\varphi)}\left(\frac{m}{i+m}\right)-\alpha_{\delta,\eta}^{(\gamma)}\left(\frac{m}{i+m}\right)\right|\binom{i+m}{m}(1-t)^i t^{m+1}dt \\ & =\int_\delta^\eta\frac{\left|\beta\left(\dfrac{t}{q-(q-1)t}\right)-\gamma(t)\right|}{t(1-t)}\,dt,\end{aligned}$$

since by (3. 12)

$$\begin{aligned}& \lim_{\lambda\to\infty}\sum_{i=1}^{\infty}\left|\alpha_{\delta,\eta}^{(\varphi)}\left(\frac{m}{i+m}\right)-\alpha_{\delta,\eta}^{(\gamma)}\left(\frac{m}{i+m}\right)\right|\binom{i+m}{m}(1-t)^i t^{m+1} \\ & =\begin{cases}\alpha_{\delta,\eta}^{(\varphi)}(t)-\alpha_{\delta,\eta}^{(\gamma)}(t) & \text{if } \delta\leqq t\leqq\eta \\ 0 & \text{elsewhere}\end{cases}\end{aligned}$$

almost everywhere in $[0, 1]$ and the convergence is dominated. By (3. 20), (3. 21), (3. 22) and (3. 27) it follows that

$$\begin{aligned}(3.28)\qquad & \limsup_{\lambda\to\infty}\left|\sum_{i=1}^{\infty}\left|\int_0^1\left[\frac{\beta(u)}{u(1-u)}\binom{i+n}{n}(1-u)^i u^{n+1}\right.\right.\right. \\ & \qquad\left.\left.\left.-\frac{\gamma(u)}{u(1-u)}\binom{i+m}{m}(1-u)^i u^{m+1}\right]du\right|-B_q\right|\leqq 8\varepsilon.\end{aligned}$$

Since (3. 28) holds for every $\varepsilon>0$, our theorem is proved.

4. Proofs of Theorems 3 and 4.

PROOF OF THEOREM 3. By (3.3) and the straightforward identities $a_n=\left(1+\frac{1}{n}\right)b_n-b_{n-1}(n>0)$, $(b_0=0)$ it follows that

$$(4.1)\quad S_n^*(\beta)-s_m=-\sum_{i=1}^{m}\left[\left(1+\frac{1}{i}\right)b_i-b_{i-1}\right](1-\Delta_{ni})+\sum_{i=m+1}^{\infty}\left[\left(1+\frac{1}{i}\right)b_i-b_{i-1}\right]\Delta_{ni}.$$

Now

$$\sum_{i=m+1}^{N}\left[\left(1+\frac{1}{i}\right)b_i-b_{i-1}\right]\Delta_{ni}=\sum_{i=m+1}^{N}b_i\left[\left(1+\frac{1}{i}\right)\Delta_{ni}-\Delta_{n,i+1}\right]-b_m\Delta_{n,m+1}+b_N\Delta_{n,N+1}$$

and by (1.4), $\lim_{N\to\infty} b_N\Delta_{n,N+1}=0$, hence

$$\sum_{i=m+1}^{\infty}\left[\left(1+\frac{1}{i}\right)b_i-b_{i-1}\right]\Delta_{ni}=\sum_{i=m+1}^{\infty}b_i\left[\left(1+\frac{1}{i}\right)\Delta_{ni}-\Delta_{n,i+1}\right]-b_m\Delta_{n,m+1}.$$

Thus by (4.1)

$$\begin{aligned}S_n^*(\beta)-s_m&=-\sum_{i=1}^{m-1}b_i\left[\left(1+\frac{1}{i}\right)(1-\Delta_{ni})-(1-\Delta_{n,i+1})\right]\\&\quad-b_m\left[\left(1+\frac{1}{m}\right)(1-\Delta_{nm})+\Delta_{n,m+1}\right]+\sum_{i=m+1}^{\infty}b_i\left[\left(1+\frac{1}{i}\right)\Delta_{ni}-\Delta_{n,i+1}\right]\\&=\sum_{i=1}^{\infty}\gamma_{nmi}b_i,\ \text{say}.\end{aligned}$$

By Agnew's theorem [1] we have to show that

$$(4.2)\qquad \lim_{\lambda\to\infty}\gamma_{nmi}=0\quad\text{for}\quad i=1,2,\cdots$$

and

$$(4.3)\qquad \lim_{\lambda\to\infty}\sum_{i=1}^{\infty}\left|\gamma_{nmi}\right|=C_q.$$

Now (4.2) follows exactly as (3.4) and we have to prove (4.3). By (3.6) it follows that for $1\leqq i\leqq m$

$$(4.4)\quad \left(1+\frac{1}{i}\right)(1-\Delta_{ni})-(1-\Delta_{n,i+1})=\frac{1}{i}(1-\Delta_{ni})-\binom{i+n}{n}\int_0^1(1-t)^i t^{n+1}d\beta(t)$$

$$=\int_0^1 \binom{i+n}{n}(1-t)^i t^{n+1} d\left\{1-\beta(t)-\int_t^1 \frac{1-\beta(u)}{u(1-u)}du\right\}$$

$$=-\int_{0+}^{1-} \binom{i+n}{n}(1-t)^i t^{n+1} d\left\{\int_t^1 \frac{1-u}{u} d\left[\frac{u}{1-u}(1-\beta(u))\right]\right\}.$$

Similarly by (3.2) we obtain for $i \geqq m+1$

$$\left(1+\frac{1}{i}\right)\Delta_{ni}-\Delta_{n,i+1}=\frac{1}{i}\Delta_{ni}+\binom{i+n}{n}\int_0^1 (1-t)^i t^{n+1} d\beta(t) \tag{4.5}$$

$$=\int_{0+}^{1-} \binom{i+n}{n}(1-t)^i t^{n+1} d\left\{\int_0 \frac{1-u}{u} d\left[\frac{u}{1-u}\beta(u)\right]\right\}.$$

Applying the technique we have used in the proof of Theorem 1, it follows by (4.4) that for $\lambda \geqq \lambda_0$

$$\left|\left(1+\frac{1}{m}\right)(1-\Delta_{nm})-(1-\Delta_{n,m+1})\right|$$

$$\leqq \int_0^1 \binom{m+n}{n}(1-t)^m t^{n+1}|d\beta(t)|+\int_0^1 \binom{m+n}{n}(1-t)^m t^{n+1} \frac{|1-\beta(t)|}{t(1-t)}dt$$

$$\leqq \int_0^1 \sum_{k=[n(q-\varepsilon)]}^{[n(q+\varepsilon)]} \binom{k+n}{n}(1-t)^k t^{n+1}|d\beta(t)|$$

$$+\int_0^1 \sum_{k=[n(q-\varepsilon)]}^{[n(q+\varepsilon)]} \binom{k+n}{n}(1-t)^k t^{n+1} \frac{|1-\beta(t)|}{t(1-t)}\, dt$$

and as $\beta(t)$ is continuous we obtain

$$\longrightarrow \int_{1/(1+q+\varepsilon)}^{1/(1+q-\varepsilon)} |d\beta(t)| + \int_{1/(1+q+\varepsilon)}^{1/(1+q-\varepsilon)} \frac{|1-\beta(t)|}{t(1-t)}\, dt \quad \text{as} \quad \lambda \longrightarrow \infty.$$

Having this for every $\varepsilon>0$ it follows by the continuity of $\beta(t)$ that

$$\lim_{\lambda\to\infty}\left[\left(1+\frac{1}{m}\right)(1-\Delta_{nm})-(1-\Delta_{n,m+1})\right]=0. \tag{4.6}$$

Again the technique we have used in the proof of Theorem 1 and a proof similar to that of Theorem 2.2 of [3] enable us to conclude that

$$\lim_{\lambda\to\infty}\sum_{i=1}^{m-1}|\gamma_{nmi}| = \int_{1/(1+q)}^{1-}\left|\frac{1-u}{u}\right| d\left[\frac{u}{1-u}(1-\beta(u))\right]\Bigg| \tag{4.7}$$

and

$$\lim_{\lambda\to\infty}\sum_{i=m+1}^{\infty}|\gamma_{nmi}| = \int_{0+}^{1/(1+q)}\left|\frac{1-u}{u}\right| d\left[\frac{u}{1-u}\beta(u)\right]\Bigg|. \tag{4.8}$$

Our theorem follows now by (4.6), (4.7) and (4.8).

PROOF OF THEOREM 4. The proof is similar to that of Theorem 3, applying Remark 2.1 of [3] instead of Theorem 2.2 of [3]. It remains to prove that for each function $f(t)$ bounded in $[0,1]$, we have at each point $t=x$, $0<x<1$ where $f(x\pm)$ exist,

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}\binom{k+n}{n}(1-x)^k x^{n+1} f\left(\frac{n}{k+n}\right) = \frac{1}{2}[f(x+)+f(x-)].$$

The proof is similar to the proof of the same property for the Bernstein polynomials (see [4] Theorem 1.9.1).

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