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APPLICATION OF MORSE THEORY TO SOME HOMOGENEOUS SPACES*)

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Introduction. R. Bott [2] applies Morse theory to compute the betti numbers of the complex flag manifold. The Morse inequalities become equalities simply because the indices of critical point of the distance function are even. (This shows that the complex flag manifold has no torsion.) This method is applied to several homogeneous spaces and except in three cases the Morse inequalities become equalities for the same reason. For these three we use some results on P. A. Smith theory, in particular results of E. E. Floyd [1] to show that Morse inequalities are equalities. This method is due to T. T. Frankel [5]. The results on betti numbers obtained here are contained in Bott and Samelson [3, Theorem VI]. But they obtain these results as a consequence of a theorem on "loop spaces". We apply Morse theory directly to the spaces themselves using the ideas of Bott and Samelson. It should be pointed out that we obtain a cell-decomposition for these homogeneous spaces and as a corollary we obtain the betti numbers. The basic results of Morse theory can be found in [2 or 6].

Description of the spaces. Let K be a compact Lie group acting on differentiable manifold M, i.e., there is a differentiable map $\pi: K\times M \to M$ such that (1) $\pi(e,x)=x$, where e is the identity in K and (2) $\pi(k, \pi(k', x)) = \pi(kk', x)$ for $k, k' \in K$ and $x \in M$. Let K_x be the *stability group* of $x \in M$, i.e., $K_x = \{k \in K \mid \pi(k, x) = x\}$. Then the K-orbit of x is a regular submanifold of M homeomorphic to K/K_x.

Let G be a compact connected Lie group. We will take G to be a classical group. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g}$ will be considered to be the tangent space at $e\in G$. A maximal abelian subgroup of G is called a maximal torus (denoted by T) and the tangent space to T at e will be denoted by $\mathfrak{t}$. It is well-known that $\mathfrak{t}$ is abelian, namely $[\mathfrak{t}, \mathfrak{t}]=0$ where [,], denoted the usual Lie product, i.e., $[X,Y]=XY-YX$. A point $P\in \mathfrak{t}$ will be called a *general point* if $\mathfrak{g}_P = \{X\in \mathfrak{g} \mid [X, P]=0\} = \mathfrak{t}$.

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The group G acts on its Lie algebra $\mathfrak{g}$ by adjoint action. The following spaces arise as orbits for this action.

$\dfrac{\boldsymbol{U(n)}}{\boldsymbol{U(n_1)\times\cdots\times U(n_k)}}=\boldsymbol{W(n_1,\cdots,n_k)},\ \boldsymbol{n_1+\cdots+n_k=n}$. **(Complex flag manifold)**. The Lie algebra of $U(n)$ (denoted $\mathfrak{u}(n)$) consists of $n\times n$ skew-hermitian matrices. Let $I_{n_1,\dots,n_k}$ be a $n\times n$ diagonal matrix with the same first n_1 entries (say all equal to 1), the same second n_2 entries (all 2), $\cdots$, the same last n_k entries (all k). Consider $X=iI_{n_1,\dots,n_k}\in\mathfrak{u}(n)$. The orbit of X under the action of Ad $U(n)$ on $\mathfrak{u}(n)$ is $W(n_1,\cdots,n_k)$, because the stability group of X is $U(n_1)\times\cdots\times U(n_k)$. Clearly, there is no unique choice of X. For example if we take $2X$ we get an orbit homeomorphic to $W(n_1,\cdots,n_k)$.

The algebra $\mathfrak{u}(n)$ has a natural Riemannian structure defined by $(X, Y)=-\mathrm{Rl}\ \mathrm{tr}\ X\circ Y$, $X, Y\in\mathfrak{u}(n)$. Consequently $\mathfrak{u}(n)$ can be thought of as an Euclidean space in which the orbit is imbedded. For this reason all our orbits will be considered as imbedded in a suitable Euclidean space.

If P is a general point (corresponding to X) with $n_1=n_2=\cdots=n_k=1$ then the corresponding orbit is an orbit of maximal dimension which is $U(n)/T$. In this case $\mathfrak{t}$ consists of all purely imaginary diagonal matrices and $T=e^{i\theta_1}\times\cdots\times e^{i\theta_n}$.

$\dfrac{\boldsymbol{SO(2n)}}{\boldsymbol{U(n_1)\times\cdots\times U(n_k)}}$. The algebra $\mathfrak{so}(n)$ consists of all $n\times n$ skew-symmetric matrices. Consider $X\in\mathfrak{so}(2n)$ where $X=\begin{pmatrix}0 & I_{n_1,\dots,n_k}\\ -I_{n_1,\dots,n_k} & 0\end{pmatrix}$. For the action of $\mathrm{Ad}SO(2n)$ on $\mathfrak{so}(2n)$ the orbit of X is $\dfrac{SO(2n)}{U(n_1)\times\cdots\times U(n_k)}$ for it is easy to verify that the stability group of X is $U(n_1)\times\cdots\times U(n_k)$. Here we consider $U(n)$ as imbedded in $SO(2n)$ under the correspondence $A+Bi\longleftrightarrow\begin{pmatrix}A & B\\ -B & A\end{pmatrix}$. Again for $n_1=n_2=\cdots=n_k=1$ we get an orbit of maximal dimension. This orbit is $\dfrac{SO(2n)}{T}$, where

$$T=\begin{pmatrix}\cos\theta_1 & & 0 & \sin\theta_1 & & 0\\ & \ddots & & & \ddots & \\ 0 & & \cos\theta_n & 0 & & \sin\theta_n\\ -\sin\theta_1 & & 0 & \cos\theta_1 & & 0\\ & \ddots & & & \ddots & \\ 0 & & -\sin\theta_n & 0 & & \cos\theta_n\end{pmatrix}.$$

The algebra $\mathfrak{t}$ consists of matrices having non-zero entries in $(j, n+j)$th place and $(n+j, j)$th place $j=1,\cdots,n$.

$\dfrac{\boldsymbol{SO(2n+1)}}{\boldsymbol{U(n_1)\times\cdots\times U(n_k)\times 1}}$. Consider the action of $\mathrm{Ad}SO(2n+1)$ on $\mathfrak{so}(2n+1)$. Let

$$X=\begin{array}{c} \begin{array}{ccc} n\times n & n\times n & n\times 1 \end{array} \\ \begin{pmatrix} 0 & I_{n_1,\cdots,n_k} & 0 \\ -I_{n_1,\cdots,n_k} & 0 & \\ & 0 & 0 \end{pmatrix} \end{array}.$$

The orbit of X is $\dfrac{SO(2n+1)}{U(n_1)\times\cdots\times U(n_k)\times 1}$.

$\dfrac{\boldsymbol{Sp(n)}}{\boldsymbol{U(n_1)\times\cdots\times U(n_k)}}$. The group $Sp(n)$ will be thought of as imbedded in $U(2n)$ under the correspondence $A+Bj \longleftrightarrow \begin{pmatrix} A & B \\ -\bar{B} & \bar{A}\end{pmatrix}$. We consider the quaternions as a 2-dimensional vector space over the complex numbers with basis 1, j. The algebra $Sp(n)$ consists of all matrices of the form

$$\begin{pmatrix} Z_1 & Z_2 \\ -\bar{Z}_2 & \bar{Z}_1 \end{pmatrix}, \quad \begin{array}{ll} Z_1: n\times n & \text{skew-hermitian} \\ Z_2: n\times n & \text{complex symmetric.} \end{array}$$

Let $X=\begin{pmatrix} iI_{n_1,\cdots,n_k} & 0 \\ 0 & -iI_{n_1,\cdots,n_k}\end{pmatrix}\in\mathfrak{sp}(n)$. Then the orbit of X under the action of $\mathrm{Ad}Sp(n)$ on $\mathfrak{sp}(n)$ is $\dfrac{Sp(n)}{U(n_1)\times\cdots\times U(n_k)}$. The algebra $\mathfrak{t}$ is obtained by taking $Z_2=0$, and Z_1 to be diagonal.

In order to describe the other orbits we make the following well-known definitions.

Definition. The pair (G, K) is called a *symmetric pair* if 1) G is compact connected Lie group and 2) K is the full fixed set of an involution σ (automorphism of order 2) on G.

The Lie algebra $\mathfrak{g}$ splits into a natural direct sum $\mathfrak{k}=\{X\in\mathfrak{g}\,|\,s(X)=X\}$ and $\mathfrak{p}=\{X\in\mathfrak{g}\,|\,s(X)=-X\}$ where s is the endomorphism induced on $\mathfrak{g}$ by σ. A maximal subalgebra $\mathfrak{t}$ of $\mathfrak{p}$ is called a *Cartan subalgebra*. It is abelian and the dimension of $\mathfrak{t}$ is the rank of (G,K). If $\mathfrak{t}$ is such an algebra, $A\in\mathfrak{t}$ is called a *general point* if $\mathfrak{p}_A=\{Z\in\mathfrak{p}\,|\,[Z,A]=0\}=\mathfrak{t}$ [4, p. 1019]. As before it can be verified that orbits of general point are orbits of maximal dimension. We choose and fix $\mathfrak{t}$ for each case. The two cases of handling classical groups and symmetric spaces can be dispensed with if we consider Lie groups as symmetric spaces.

It is known that K acts on $\mathfrak{p}$ by conjugation. The following homogeneous spaces arise as orbits for the action of AdK on $\mathfrak{p}$. As before the orbits will be considered to be imbedded in a suitable Euclidean space (namely $\mathfrak{p}$).

$\dfrac{\boldsymbol{O(n)}}{\boldsymbol{O(n_1)\times\cdots\times O(n_k)}}=\boldsymbol{G(n_1,\cdots,n_k)},\ \boldsymbol{n_1+\cdots+n_k=n}$ **(Real flag manifold).** Consider the involution σ on $U(n)$ defined by $\sigma(X)=\overline{X}$, $X\in U(n)$. (Bar means complex conjugation.) The full fixed set is $O(n)$, and $\mathfrak{p}$ consists of all $n\times n$ purely imaginary symmetric matrices. Let $X=iI_{n_1,\cdots,n_k}\in\mathfrak{p}$. The orbit of X under the action of Ad$O(n)$ is a real flag manifold. A maximal subalgebra of $\mathfrak{p}$ consists of all pure imaginary diagonal matrices.

$\dfrac{\boldsymbol{U(n)}}{\boldsymbol{O(n_1)\times\cdots\times O(n_k)}}$. The group $Sp(n)$ is imbedded in $U(2n)$ by $A+Bj \longleftrightarrow \begin{pmatrix} A & B\\ -\overline{B} & \overline{A}\end{pmatrix}$. Let σ be an involution on $Sp(n)$ defined by $\sigma(X)=\overline{X}$. The fixed set consists of all matrices of the form $\begin{pmatrix} A & B\\ -B & A\end{pmatrix}$ which is precisely $U(n)$ $\subset SO(2n)$. The space $\mathfrak{p}=\left\{\begin{pmatrix} Z_1 & Z_2\\ Z_2 & -Z_1\end{pmatrix} Z_1,\ Z_2 \text{ purely imaginary symmetric matrices}\right\}$ and $\mathfrak{t}$ is obtained by taking $Z_2=0$ and Z_1 to be diagonal. Let $X=\begin{pmatrix} iI_{n_1,\cdots,n_k} & 0\\ 0 & -iI_{n_1,\cdots,n_k}\end{pmatrix}$ $\in\mathfrak{p}$. The orbit of X under the action of Ad$U(n)$ is $\dfrac{U(n)}{O(n_1)\times\cdots\times O(n_k)}$.

$\dfrac{\boldsymbol{Sp(n)}}{\boldsymbol{Sp(n_1)\times\cdots\times Sp(n_k)}}$ **(The quaternionic flag manifold).** Consider the involution σ on $U(2n)$ defined by $\sigma(X)=J_n\overline{X}J_n^{-1}$, $J_n=\begin{pmatrix} 0 & I_n\\ -I_n & 0\end{pmatrix}$, $I_n=n\times n$ identity matrix. The fixed set is $\begin{pmatrix} A & B\\ -\overline{B} & \overline{A}\end{pmatrix}$, i. e., $Sp(n)$. The space $\mathfrak{p}=\left\{\begin{pmatrix} Z_1 & Z_2\\ \overline{Z}_2 & -\overline{Z}_1\end{pmatrix} \begin{matrix} Z_1 \text{ skew-hermitian}\\ Z_2 \text{ complex, skew-symmetric}\end{matrix}\right\}$, and the diagonal matrices in $\mathfrak{p}$ form $\mathfrak{t}$. Consider $X=\begin{pmatrix} iI_{n_1,\cdots,n_k} & 0\\ 0 & iI_{n_1,\cdots,n_k}\end{pmatrix}\in\mathfrak{p}$. The orbit of X under Ad$Sp(n)$ is the quaternionic flag manifold.

$\dfrac{\boldsymbol{U(2n)}}{\boldsymbol{Sp(n_1)\times\cdots\times Sp(n_k)}}$. Consider the involution σ on $SO(2n)$ defined by $\sigma(X)=J_nXJ_n^{-1}$. The fixed set is $U(n)$. The space $\mathfrak{p}=\left\{\begin{pmatrix} A & B\\ B & -A\end{pmatrix} A, B \text{ skew-}\right.$

symmetric matrices $\Big\}$. Let $X = \begin{pmatrix} 0 & 0 & 0 & I_{n_1,\cdots,n_k} \\ 0 & 0 & -I_{n_1,\cdots,n_k} & 0 \\ 0 & I_{n_1,\cdots,n_k} & 0 & 0 \\ -I_{n_1,\cdots,n_k} & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{p}$

of the pair $(SO(4n), U(2n))$. The orbit of X under the action of $\mathrm{Ad}U(2n)$ on $\mathfrak{p}$ is $\dfrac{U(2n)}{Sp(n_1)\times\cdots\times Sp(n_k)}$.

Here $Sp(n)$ is considered imbedded in $SO(4n)$ under the correspondence

$$A+Bi+Cj+Dk \longleftrightarrow \begin{pmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{pmatrix}.$$

REMARK. For special values of k, we get the classical structures of irreducible Riemannian symmetric spaces and spaces G/T, where G is a classical group and T is a maximal torus. Also we get G/Γ where

$$\Gamma = \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix} = \begin{pmatrix} Z_2 & & 0 \\ & \ddots & \\ 0 & & Z_2 \end{pmatrix}, \text{ for } G=O(n) \text{ and } U(n).$$

Morse theory. Let M^n be a differentiable manifold of dimension n differentiably imbedded in a real Euclidean space R^{n+k}. Let $(x_1, \cdots, x_{n+k})$ denote the co-ordinate system in R^{n+k}. Let the Morse function on M^n be the square of the distance of the points on M^n from a fixed point $P \in R^{n+k} - M^n$. Let us denote this function by $L_P(x)$, $x \in M^n$.

With the usual notations $L_P(x) = (\vec{x}, \vec{x})$, taking P as origin for R^{n+k}. Taking differentials, $dL_P(x) = 2(\vec{dx}, \vec{x})$. Thus $dL_P(x) = 0$ if and only if $\vec{dx}$ is perpendicular to $\vec{x}$. Hence Q is critical point for L_P if and only if $\overrightarrow{PQ}$ is normal to M^n at Q.

Consider the Hessian quadratic form

$$d^2L_P(x) = 2(\vec{x}, \vec{d^2x}) + 2(\vec{dx}, \vec{dx}).$$

At a critical point Q, $\vec{x} = |\vec{x}|\vec{N}$, $\vec{N}$ unit normal vector. Hence $d^2L_P(x)/2 = (\vec{dx}, \vec{dx}) + (|\vec{x}|\vec{N}, \vec{d^2x})$. The first term is called the first fundamental quadratic form for M^n and the second term is called the second fundamental quadratic form for the normal direction $\vec{N}$.

We choose local co-ordinates near Q as follows: $Q=(0,\cdots,0)$. Let $x_1,\cdots,x_n$ be the local co-ordinates for M^n near 0.

Then in R^{n+k}, M^n is given by $g_l(x_1,\cdots,x_n)=x_{n+l}$, $l=1,\cdots,k$, where g_l are differentiable. Since $\overrightarrow{PQ}$ is perpendicular to M^n we can take $P=(0,\cdots,0,p_1,\cdots,p_k)$. Let $tp=(0,\cdots,0,tp_1,\cdots,tp_k)$. Let $HL_{tp}(0)=$ Hessian of L_{tp} at 0. By a direct computation $HL_{tp}(0)=I-\sum^k tp_l\frac{\partial^2 g}{\partial x_i\partial x_j}(0)$, where I is $n\times n$ identity matrix.

By well-known results on quadratic forms, it is possible to find a basis for TM_Q (tangent space to M^n at Q) such that $\left(-\sum_{l=1}^k tp_l\frac{\partial^2 g}{\partial x_i\partial x_j}(0)\right)$ is reduced to diagonal form. With respect to this basis

$$HL_{tp}(0)=\begin{pmatrix} ta_{11}+1 & & & 0 \\ & ta_{22}+1 & & \\ & & \ddots & \\ 0 & & & ta_{nn}+1\end{pmatrix}.$$

It is easy to see that Q is non-degenerate critical point if and only if $t\neq -1/a_{ii}$ for all i. Thus for only finitely many values of t, Q is degenerate. The values $a_{11},\cdots,a_{nn}$ are called the *principal curvatures* of M^n at Q corresponding to the normal $\vec{N}$. The reciprocals of $a_{11},\cdots,a_{nn}$ are called the *principal radii of curvatures*. Of course these need not be all distinct. Suppose $t_1,\cdots,t_m$ are the distinct values, $t_1=1/a_{11},\cdots,t_m=1/a_{mm}$ then $t_1\vec{N},\cdots,t_m\vec{N}$ are called the *centers of principal curvature*. The set of all centers of principal curvatures for all normals is called the *focal set*. Intuitively, a focal point is where nearly normals meet. A point which is not a focal point is called *regular* point. The focal set has "measure" zero.

As t increases from 0 to 1 we get the segment $\overrightarrow{QP}$. If $t=0$ then $HL_{tp}(0)$ is positive definite and the index of $HL_{tp}(0)$ is zero. Also the index of $HL_{tp}(0)$ is an increasing function of t. The entries of $HL_{tp}(0)$ change sign at $-t_1p,\cdots,-t_mp$, i.e., at centers of principal curvature. The number of changes in sign (from plus to minus) at t_ip is $(HL_{t_ip}(0))=$ dimension of nullity of $HL_{t_ip}(0)$, $i=1,\cdots,m$. Hence

THEOREM [2,p.23]. *Let M^n be a differentiable manifold, differentiably imbedded in R^{n+k}. Let $P\in R^{n+k}-M^n$ and $L_P(x)=$ square of the distance from $x\in M$ to P. Let Q be a critical point (degenerate or not) of $L_P(x)$. Then the index of the Hessian $HL_P(Q)=\sum_{0<t<1}(HL_{(1-t)P+tQ})$.*

Since the focal set has measure zero, for almost all points $P \in R^{n+k}$ the distance function L_P has non-degenerate critical points.

Critical points and their indices. Here we closely follow the method due to Bott [2, 5]. The proofs of the Lemmas are reproduced only for the sake of completeness. The statements and proofs are also in [4, Chapter IV].

LEMMA 1a. *For the action of* AdG *on* $\mathfrak{g}$, *the tangent space to* M_X (*the orbit of* X) *is* $T_X = \{[X, Y] | Y \in \mathfrak{g}\} = \mathrm{ad}X \cdot \mathfrak{g}$. *The transversal space* $N_X = \{Z \in \mathfrak{g} | [X, Z] = 0\} = \mathfrak{g}_X$ (*centralizer of* X *in* $\mathfrak{g}$).

PROOF. The first statement is a consequence of $d/dt(\mathrm{Ad}e^{tY} \cdot X)|_{t=0} = [Y, X]$ for $Y \in \mathfrak{g}$. The second statement follows from $([X, Y], Z) = -(Y, [X, Z])$, where (,) is the Riemannian metric on $\mathfrak{g}$.

LEMMA 1b. *For the action of* AdK *on* $\mathfrak{p}$, $T_X = \mathrm{ad}\, X(\mathfrak{k})$, $N_X = \{Z \in \mathfrak{p} | [X, Z] = 0\} = \mathfrak{p}_X$.

PROOF. Same as above. For the second part we use the fact $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

The following three lemmas are true for both actions.

LEMMA 2. *If* X *is a general point* $N_X = \mathfrak{t}$.
Follows from the definition of general point.

LEMMA 3. *If a line is perpendicular to an orbit, then it is perpendicular to all orbits which it intersects.*

This lemma is a consequence of a more general statement [4, p. 967, Prop. 2.2].

PROOF. Let $B + At$ be perpendicular to M_B at B. By Lemma 1 $([X, B], A) = 0$ all $X \in R^m (= \mathfrak{g}$ or $\mathfrak{p}$ as the case may be). Since $([X, A], A) = (X, [A, A]) = 0$ all $X \in R^m$, $([X, B + tA], A) = ([X, B], A) + t([X, A], A) = 0$ all $X \in R^m$.

To apply Morse theory, we have to choose a point P for the function L_P. This point is taken to be a general point.

LEMMA 4. *Let* M_X *be any orbit. Then the critical points of the function* L_P *in* M_X *are* $M \cap \mathfrak{t}$. *In particular these points do not depend on* P.

PROOF. Let $A \in M_X$ be critical for L_P. Then by Morse theory PA is perpendicular to M_X at A. By previous lemma PA is perpendicular to M_P at P. By Lemma 2, PA is in $\mathfrak{t}$, since P is a general point. Hence $A \in \mathfrak{t}$. Conversely, if $A \in M_X \cap \mathfrak{t}$, then $[A, \mathfrak{t}]=0$ and PA is in $\mathfrak{t}$. Also, by Lemma 2, PA is perpendicular to M_X at A.

Finally since M_X is compact the function L_P has critical points. Therefore all orbits intersect $\mathfrak{t}$. In order for the critical points to be non-degenerate, P must not be a focal point. The above lemma tells us that such a choice of P does not affect the critical set of M_X.

In order to compute the indices of the critical points we give the following definitions found in [4]. (The full details may be found in this paper.)

Let a compact Lie group K act on a differentiable manifold M from left. Let $X \in \mathfrak{k}$ (the Lie algebra of K). Let $h: R \to K$ be the corresponding 1-parameter subgroup with $\dot{h}(0)=X$. For $x \in M$, let $h_x : R \to M$ be the curve defined by $h_x(\alpha)=h(\alpha)\cdot x$ for all $\alpha \in R$. The assignment $x \to \dot{h}_x(0)$ defines a vector-field X on M. This vector-field is called the *infinitesimal K-motion* corresponding to X.

A geodesic g of M is called *K-transversal* if for each $t \in R$ the tangent vector $\dot{g}(t)$ is orthogonal to the K-orbit of the point $g(t)$. By [4, Prop. 2.2, p.967] transversality holds, provided it holds at one point.

A *geodesic variation* V_α of a geodesic g in M is a C^∞-map $V: R \times I \to M$, where I is an open interval containing 0, such that (1) for each $\alpha \in I$, the map $V_\alpha : R \to M$ defined by $V_\alpha(t)=V(t,\alpha)$ is a geodesic and (2) $V_0=g$. The vector-field along g defined by $\eta(t)=\dfrac{\partial V}{\partial \alpha}(t, 0)$ is called the *Jacobi field* (J-field) determined by the geodesic variation V. The J-fields along a geodesic g will be denoted by J_g. For any $t_0 \in R$, let $\Lambda_g(t_0)=\{\eta \in J_g \mid \eta(t_0)=0\}$.

Suppose g is a K-transversal geodesic. A J-field is called *transversal* if it is derived from a geodesic variation V_α of g in which all V_α are transversal geodesics. By [4, Prop. 6. 6, p. 974] if g is transversal then the restriction of any infinitesimal K-motion to g is a transversal J-field. Let g be a transversal geodesic and let $t_0 \in R$. Then $J_g^\pi(t_0)=\{\eta \in J_g \mid \eta(t_0)$ is tangent to the K-orbit of $g(t_0)\}$.

The action of K on M is *variationally complete* if every transversal J-field η which is tangent to the K-orbits for two different points on g (i. e., $\eta \in J_g^\pi(t_0) \cap J_g^\pi(t_1)$, $t_0 \neq t_1$) is induced by K (i. e., is the restriction to g of an infinitesimal K-motion).

Variational completeness was introduced by Bott [3] and Bott showed that the action of AdG on $\mathfrak{g}$ is variationally complete. Bott and Samelson generalized this result and showed that the action of AdK on $\mathfrak{p}$ is variationally complete, [4, p. 986]. The proof is quite easy because in $\mathfrak{g}$ or $\mathfrak{p}$ the geodesics are straight lines.

Bott also proved the following result [3, Prop. 6.1]. Let the action of K on M be variationally complete. Let N be an orbit of any point of M under K. Let P be a regular point of $M-N$ on an orbit of maximal dimension. (The regular points in $M-N$ are "plentiful".) Let $Q \in N$ be a critical point of the function L_P on N. Then as a non-degenerate critical point Q has index $=\Sigma(\dim M_P - \dim M_R) = \Sigma(\dim K_R - \dim K_P)$, where K_R=stability group of R, for all focal points $R \in PQ$. (Here M_X means the orbit of X under K). Hence to find the index of Q we have only to find where PQ intersects orbit of lower dimension.

In our cases PQ lies in a torus $\mathfrak{t}$ and hence the points R can be found easily. The index of the segment PQ is the same as "defect function" of PQ as defined by Bott and Samelson.

Applications of fixed point theory. The (weak) Morse inequalities are $b_i(M) \leqq$ number of critical points of index i. To show that these Morse inequalities become equalities for the cases of real flag manifold, $\dfrac{U(n)}{O(n_1)\times\cdots\times O(n_k)}$ and $\dfrac{U(2n)}{Sp(n_1)\times\cdots\times Sp(n_k)}$ we use Smith theory of fixed points. We use the following two theorems which are special cases of results of E. E. Floyd. More general results and details are found in [1].

THEOREM A. *If Z_2 acts on a compact manifold, if F is the fixed set then* $\sum_i b_i(F\,;\, Z_2) \leqq \sum b_i(M\,;\, Z_2)$.

By repeated application of this theorem we get that if $\Gamma = Z_2\times\cdots\times Z_2$ (n copies) acts on M and if F is the full fixed set then $\sum_i b_i(F\,;\, Z_2) \leqq \sum b_i(M\,;\, Z_2)$. For the adjoint action of Γ on the real flag manifold and $\dfrac{U(n)}{O(n_1)\times\cdots\times O(n_k)}$, the fixed sets are precisely the intersection of these orbits with $\mathfrak{t}$ (i.e., the critical points) because Γ commutes with $\mathfrak{t}$. Hence Morse inequalities combined with Floyd inequalities give equalities for these two cases.

For the case $\dfrac{U(2n)}{Sp(n_1)\times\cdots\times Sp(n_k)}$ we use

THEOREM B. *If a toral group acts on a compact manifold and if F is the fixed set then*

$$\sum b_i(F\,;K) \leqq \sum b_i(M\,;\, K)$$

where $K=R$ or Z_p, p prime.

Consider the torus

$$T=\left(\begin{array}{cccc}
\begin{matrix}\cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_n\end{matrix} & \begin{matrix}\sin\theta_1 & & \\ & \ddots & \\ & & \sin\theta_n\end{matrix} & 0 & 0 \\
\begin{matrix}-\sin\theta_1 & & \\ & \ddots & \\ & & -\sin\theta_n\end{matrix} & \begin{matrix}\cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_n\end{matrix} & 0 & 0 \\
0 & 0 & \begin{matrix}\cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_n\end{matrix} & \begin{matrix}\sin\theta_1 & & \\ & \ddots & \\ & & \sin\theta_n\end{matrix} \\
0 & 0 & \begin{matrix}-\sin\theta_1 & & \\ & \ddots & \\ & & -\sin\theta_n\end{matrix} & \begin{matrix}\cos\theta_1 & & \\ & \ddots & \\ & & \cos\theta_n\end{matrix}
\end{array}\right)$$

acting on $N=\dfrac{U(2n)}{Sp(n_1)\times\cdots\times Sp(n_k)}$ by adjoint action. The fixed set F is precisely the set of critical points ($N\cap \mathfrak{t}$) because T and $\mathfrak{t}$ commute.

Hence in this case we get Morse inequalities to be equalities. This space has no torsion for according to the above theorem we can take the coefficients to be $K=Z_p$ or R.

In the other cases the (strong) Morse inequalities become equalities because the indices of the critical points are even [6, p. 31].

Lastly we list the number of critical points obtained in each case. Let $r=\dfrac{n!}{n_1!\,n_2!\cdots n_k!}$.

The space	Number of critical points
$\dfrac{O(n)}{O(n_1)\times\cdots\times O(n_k)}$	r
$\dfrac{U(n)}{U(n_1)\times\cdots\times U(n_k)}$	r
$\dfrac{Sp(n)}{Sp(n_1)\times\cdots\times Sp(n_k)}$	r

$\dfrac{SO(2n)}{U(n_1)\times\cdots\times U(n_k)}$	$2^{n-1}r$
$\dfrac{SO(2n+1)}{U(n_1)\times\cdots\times U(n_k)\times 1}$	$2^n r$
$\dfrac{U(n)}{O(n_1)\times\cdots\times O(n_k)}$	$2^n r$
$\dfrac{Sp(n)}{U(n_1)\times\cdots\times U(n_k)}$	$2^n r$
$\dfrac{U(2n)}{Sp(n_1)\times\cdots\times Sp(n_k)}$	$2^n r$

ADDED IN THE PROOF.

The spaces studied in this paper have also been considered by Kobayashi [Tôhoku Math. J. 19(1967), 63-70] and Takeuchi and Kobayashi [J. Differential Geometry, 2(1968), 203-215]. In the latter paper it is shown that the imbeddings are "minimal". The Morse function used by Takeuchi and Kobayashi is essentially the same as our length function.

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