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A NOTE ON SINGULAR IDEALS

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Introduction. In this note, A is a ring with identity, and modules are left, unitary.

Our object here is to answer (in the negative) the following question raised by F.L. Sandomierski in [5, p. 117] :

Does the condition $Z(M/Z(M)) = 0$ for every module M imply that A has a semi-simple maximal left quotient ring?

It is known that A has a semi-simple maximal left quotient ring if, and only if, $Z(A) = 0$ and the dimension of ${}_AA$ is finite (in the sense of Goldie) [5, Th. 1.6].

In fact, we characterize a ring with $Z(A) = 0$ by the above given condition.

This characterization also proves that the homomorphic image of an injective module over a ring A with $Z(A) = 0$ has its singular submodule as a direct summand.

Finally, we give a characterization of a self-injective regular ring and a characterization of a self-injective non-regular ring.

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Let us recall

DEFINITION. The closure of a submodule N in M is $Cl_M(N) = \{x \in M/ (N:x)$ is essential in $A\}$. Then $Cl_M(0)$ is the singular submodule of M denoted by $Z(M)$. Write $Cl(N)$ when no ambiguity arises.

The fact that $Cl_M(N)$ contains every essential extension of N in M follows from the following known

LEMMA 0. *If P is an essential submodule of M, for any $x \in M$, $(P:x)$ is essential in A.*

PROOF. Let $0 \neq b \in A$. If $bx = 0$, $b \in (P:x)$. If $bx \neq 0$, there is a $t \in A$

such that $0 \neq tbx \in P$. Hence $0 \neq tb \in (P:x)$ which proves the lemma.

LEMMA 1. *Let N be a submodule of M. Then $ClCl(N)$ is the unique maximal essential extension of $Cl(N)$ in M.*

PROOF. Since $ClCl(N)$ contains every essential extension of $Cl(N)$ in M, it is sufficient to prove that $Cl(N)$ is essential in $ClCl(N)$.

Suppose it is not : then there is a nonzero $x \in ClCl(N)$ such that $Ax \cap Cl(N) = 0$. Since $(Cl(N):x)$ is essential in A, for any $0 \neq a_0 \in A$, there is a $t \in A$ such that $0 \neq ta_0 \in (Cl(N):x)$. Then $ta_0x \in Ax \cap Cl(N) = 0$ which proves that $x \in Z(M)$. Therefore $x \in Ax \cap Cl(N)$ which is a contradiction.

LEMMA 2. *Let N be a submodule of M. For any submodule P of N, N is essential in $Cl_M(N)$ if, and only if, $Cl_N(P)$ is essential in $Cl_M(P)$.*

PROOF. If N is essential in $Cl_M(N)$, then obviously $Cl_N(P)$ is essential in $Cl_M(P)$.

Conversely, suppose N is not essential in $Cl_M(N)$. There is a nonzero $x \in Cl_M(N)$ such that $N \cap Ax = 0$. Then $(0:x) = (N:x)$ which implies $(0:x)$ is essential in A. Thus $x \in Z(M) \subseteq Cl_M(P)$. Now $Ax \cap Cl_N(P) \subseteq Ax \cap N = 0$ which proves that $Cl_N(P)$ is not essential in $Cl_M(P)$.

LEMMA 3. *Let M be an A-module which contains an element whose left annihilator is zero. If $Z(M)$ is a direct summand of M, then $Z(A) = 0$.*

PROOF. Let $x \in M$ such that $(0:x) = 0$. Since $M = Z(M) \oplus N$, where N is a submodule of M, let $x = z + y$, where $z \in Z(M)$, $y \in N$. y is nonzero since $x \notin Z(M)$. Now $(0:z) \cap (0:y) \subseteq (0:z+y) = (0:x)$ and therefore $(0:z) \cap (0:y) = 0$ by hypothesis. This implies that $(0:y) = 0$ since $z \in Z(M)$. Let $0 \neq b \in A$. Since $0 \neq by \in N$, then $by \notin Z(M)$, which means there is a nonzero $c \in A$ such that $(0:by) \cap Ac = 0$. This implies $(0:b) \cap Ac = 0$ which proves that $b \notin Z(A)$, and therefore $Z(A) = 0$.

THEOREM 4. *Let M be a module, N a submodule of M. Consider the following statements :*

(1) $Z(A) = 0$;

(2) $ClCl(N) = Cl(N)$ (*or equivalently*, $Z(M/Cl(N)) = 0$) ;

(3) *M injective implies $ClCl(N) = Cl(N)$* ;

(4) *M injective implies $Cl(N)$ injective* ;

(5) *$Cl(N)$ is essential in M implies $Cl(N) = M$.*

Then, if (1) *is true,* (2) *through* (5) *hold for arbitrary M, N. Conversely, if one of* (2) *through* (5) *holds for arbitrary M when $N = 0$, then* (1) *is true.*

PROOF. (1) implies (2). Suppose $Z(A)=0$. It is sufficient to show $ClCl(N)\subseteq Cl(N)$. Let $x\in M$, $x\notin Cl(N)$. We prove $x\notin ClCl(N)$. Since $(N:x)$ is not essential in A, there is a nonzero $b\in A$ such that $Ab\cap(N:x)=0$. We show $(Cl(N):x)\cap Ab=0$ and this will prove that $x\notin ClCl(N)$.

Let $c\in(Cl(N):x)\cap Ab$. Then $c=ab$, where $abx\in Cl(N)$. So for any $d\in A$, there is a $t\in A$ such that $0\neq td\in(N:abx)$.

Hence $tdab\in(N:x)\cap Ab=0$, which shows that $c=ab\in Z(A)=0$.

(2) implies (3) obviously.

(3) implies (4). If M is injective, let E be an injective hull of $Cl(N)$ in M.

Then $E=ClCl(N)$ by Lemma 1, and hence $Cl(N)=ClCl(N)=E$ is injective.

(4) implies (5). Let $Cl_M(N)$ be essential in M. Let $\widehat{M}$ be an injective hull fo M. Then $Cl_M(N)\subseteq Cl_{\widehat{M}}(N)$ and since $Cl_M(N)$ is essential in $\widehat{M}$, so is $Cl_{\widehat{M}}(N)$.

By (4), $Cl_{\widehat{M}}(N)$ is injective and therefore $Cl_{\widehat{M}}(N)=\widehat{M}$, which proves that $Cl_M(N)=M$.

Here, we remark that (4) for $N=0$ implies (5) for $N=0$.

Suppose now that (5) holds for $N=0$. Let E be an injective hull of $Z(A)$ in $\widehat{A}$, the injective hull of A.

Then $Z(A)\subseteq Z(E)$ and therefore $Z(E)$ is essential in E which implies $Z(E)=E$ by (5).

Since $Z(A)$ is essential in $Z(\widehat{A})$ by Lemma 2, and $Z(A)\subseteq Z(E)\subseteq Z(\widehat{A})$, $E=Z(E)$ is essential in $Z(\widehat{A})$ which proves that $E=Z(\widehat{A})$. By Lemma 3, $Z(A)=0$, which shows that (5) implies (1).

REMARK 1. The equivalence of (1) and (2) in Theorem 4 (taking $N=0$) answers in the negative a question raised by Sandomierski (see Introduction) since a ring with identity, having zero singular ideal, is not necessarily of finite dimension. (For an example, see [1, p. 219]).

Also this characterization provides the following generalization of [5, Corollary to Theorem 2.10]:

PROPOSITION 5. *If $Z(A)=0$ and $M\to Q\to 0$ is an exact sequence of A-modules with M injective, then $Z(Q)$ is a direct summand of Q.*

For a proof, see [5, Theorem 2.10].

Now what can we say about a ring with a non-zero singular ideal?

LEMMA 6. *Let M be a module with $Z(M)=0$. For any $x\in M$, and any ideal I of A, Ix is essential in $Cl_A(I)x$.*

PROOF. Let $0 \neq y \in Cl_A(I)x$. Then $y = bx$; $b \in Cl(I)$. Since $Z(M) = 0$, there is a nonzero $c \in A$ such that $Ac \cap (0 : y) = 0$.

Since $(I : b)$ is essential in A, there is a $t \in A$ such that $0 \neq tc \in (I : b)$. Therefore $0 \neq tcy = tcbx \in Ix$, which proves that Ix is essential in $Cl(I)x$.

PROPOSITION 7. *$Z(A) \neq 0$ if, and only if, for any module M with $Z(M) = 0$, $(0 : x) \neq 0$ for every $x \in M$.*

PROOF If $Z(A) \neq 0$, for a module M with $Z(M) = 0$, if $x \in M$, by Lemma 6, 0 is essential in $Z(A)x$, which proves that $(0 : x) \neq 0$.

Conversely, let $Z(A) = 0$. Then $(0 : 1) = 0$, where $1 \in A$.

Next we give a characterization of a ring with essential singular ideal (for example, when A is prime or uniform with $Z(A) \neq 0$).

PROPOSITION 8. *The following conditions are equivalent :*

(1) *$Z(A)$ is essential in A ;*

(2) *For any module M, $Z(M)$ is essential in M ;*

(3) *$Z(M) = M$ if $Z(M)$ is injective ;*

(4) *$Z(M) \neq 0$ for every non-zero module M.*

PROOF. (1) implies (2). If $Z(M) = M$, there is nothing to prove. So let $x \in M$; $x \notin Z(M)$. Then there is a nonzero $b \in A$ such that $Ab \cap (0 : x) = 0$. Since $Z(A)$ is essential in A, there is a $t \in A$ such that $0 \neq tb \in Z(A)$. Now $Atb \cap (0 : x) = 0$ and therefore $(0 : tb) = (0 : tbx)$. Hence $0 \neq tbx \in Z(M)$, which proves that $Z(M)$ is essential in M.

(2) implies (3) obviously.

(3) implies (4). If $Z(M) = 0$, then $M = Z(M) = 0$.

Finally, if $Z(A)$ is not essential in A, there is a nonzero ideal I such that $Z(I) = 0$. Hence (4) implies (1).

REMARK 2. A necessary and sufficient condition for $Z(A)$ to be essential in A is that if N is a submodule of M, $Z(N)$ is essential in $Z(M)$ if, and only if, N is essential in M. (This follows from Lemma 2 and the fact that condition (2) in Proposition 8 implies $Cl(N)$ is essential in M).

It is known that for any ring A, $Z(A) = 0$ if, and only if, A has a regular maximal left quotient ring Q. In that case, ${}_AQ$ is an injective hull of ${}_AA$. Also ${}_QQ$ is injective. (see, for example, [4]).

We conclude this note with a characterization of a self-injective, regular ring.

THEOREM 9. *The following conditions are equivalent :*

(1) *A is a self-injective regular ring*;

(2) *If N is a submodule of M, for any* $x \in M$, $(Cl(N):x)$ *is injective*;

(3) $(0:b)$ *is injective for every* $b \in A$;

(4) *There is a faithful module M such that the annihilator of every element of M is injective.*

PROOF. Let us first remark that for any submodule N of M and any $x \in M$,

$$Cl(N:x) = (Cl(N):x).$$

(1) implies (2). Since $Z(A) = 0$, for a submodule N of M, $ClCl(N) = Cl(N)$ by Theorem 4. By the above remark, if $I = (Cl(N):x)$, $I = Cl_A(I)$.

Since A is self-injective, I is injective.

(2) implies (3). Let $M = A$, $N = 0$. Then $Cl(N) = Z(A)$. Take $I = (Cl(N):1) = Cl(N)$, where $1 \in A$. Then $Z(A)$ is injective and Lemma 3 implies $Z(A) = 0$. Hence $(0:b)$ is injective for every $b \in A$.

(3) implies (4) evidently.

(4) implies (1). Since A annihilates $0 \in M$, A is self-injective.

Let $0 \neq x \in M$. Then $A = (0:x) \oplus J$, where J is a nonzero ideal of A. Therefore $x \notin Z(M)$ which proves $Z(M) = 0$.

Since M is faithful, $Z(A) = 0$. (Otherwise let $0 \neq b \in Z(A)$. There is an $x \in M$ such that $0 \neq bx \in Z(M)$ since $(0:b) \subseteq (0:bx)$.)

This proves that A is self-injective, regular.

REMARK 3. A is a self-injective ring if, and only if, for any module M with $Z(M) = 0$, the annihilator of every element of M is an injective module. (This is because if $Z(M) = 0$, for $x \in M$, $(0:x) = Cl_A(0:x)$ by the remark in the proof of Theorem 9).

REMARK 4. Combining Proposition 7 with Remark 3, we see that A is a self-injective, non-regular ring if, and only if, for any module M with $Z(M) = 0$, $(0:x)$ is non-zero injective for every $x \in M$.

REMARK 5. We would like to thank Dr. G. Renault who pointed out the following:

(a) Proposition 5 holds for M quasi-injective. In fact, if $Z(A) = 0$, $M \xrightarrow{f} Q \longrightarrow 0$ is exact with M quasi-injective, then since $f^{-1}(Z(Q)) = Cl(\mathrm{Ker}\, f)$, by Theorem 4 and [3, prop. 1.5], $M = f^{-1}(Z(Q)) \oplus N$. Hence $Q = f(M) = Z(Q) \oplus f(N)$, where $f(N) \approx N$.

(b) $Z(A) = 0$ if, and only if, $Cl(I) = ClCl(I)$ for every left ideal I of A. The implication in one direction follows from Theorem 4. Conversely, suppose

$ClCl(I)=Cl(I)$ for every I and $Z(A)\neq 0$. Let J be a complement ideal of $Z(A)$. Then since $J\oplus Z(A)$ is essential in A and $J\oplus Z(A)\subseteq Cl(J)$, $Cl(J)=ClCl(J)=A$. This proves that $J=(J:1)$ is essential in A which contradicts $Z(A)\neq 0$.

REMARK 6. We may also add the following characterization : $Z(A)=0$ if, and only if, for every quasi-injective module M, the closure of any submodule N is a direct summand of M.

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