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BANACH ALGEBRA RELATED TO THE JACOBI POLYNOMIALS

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1. Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n, order $(\alpha,\beta), \alpha, \beta > -1$, defined by

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}].$$

Define $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$. We expand $R_m^{(\alpha,\beta)}(x)\, R_n^{(\alpha,\beta)}(x)$ by $R_k^{(\alpha,\beta)}(x)$, getting

$$R_m^{(\alpha,\beta)}(x)\, R_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{\infty} c(k\,;m,n)\, l(k)\, R_k^{(\alpha,\beta)}(x),$$

where

$$c(k\,;m,n) = \int_{-1}^{1} R_m^{(\alpha,\beta)}(x)\, R_n^{(\alpha,\beta)}(x)\, R_k^{(\alpha,\beta)}(x)(1-x)^\alpha(1+x)^\beta\, dx$$

and

$$l(k) = \frac{[P_k^{(\alpha,\beta)}(1)]^2}{\int_{-1}^{1}[P_k^{(\alpha,\beta)}(x)]^2(1-x)^\alpha(1+x)^\beta\, dx}.$$

If there is a constant C independent on m and n such that

$$\sum_{k=0}^{\infty} |c(k; m,n)|\, l(k) \leqq C,$$

then we have a dual convolution structure to expansions in Jacobi polynomials. This is proved for $\alpha \geqq \beta \geqq -1/2$ by Askey and Wainger [2] and for the larger

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region $\alpha \geqq \beta \geqq -1$, $\alpha+\beta+1>0$ with $C=1$ by G. Gasper.[1]

For a convenience we mention some special values of $c(k; m, n)\ l(k)$.

$$(1) \qquad c(k\,; m, n) = 0 \quad \text{if} \quad |m-n| > k \quad \text{or} \quad m+n < k\,.$$

$$(2) \qquad c(k-1\,;1,k)\ l(k-1) = \frac{(k+\alpha)(k+\beta)(\alpha+\beta+2)}{(2k+\alpha+\beta)(2k+\alpha+\beta)}\ \frac{P_{k-1}^{(\alpha,\beta)}(1)}{P_1^{(\alpha,\beta)}(1)\,P_k^{(\alpha,\beta)}(1)}\,.$$

$$(3) \qquad c(k\,;1,k)\ l(k) = \frac{2k(\alpha-\beta)(k+\alpha+\beta+1)}{(2k+\alpha+\beta+2)(2k+\alpha+\beta)}\ \frac{1}{P_1^{(\alpha,\beta)}(1)}\,.$$

$$(4) \qquad c(k+1\,;1,k)\ l(k+1) = \frac{(\alpha+\beta+2)(k+1)(k+\alpha+\beta+1)}{(2k+\alpha+\beta+2)(2k+\alpha+\beta+1)}\ \frac{P_{k+1}^{(\alpha,\beta)}(1)}{P_1^{(\alpha,\beta)}(1)\,P_k^{(\alpha,\beta)}(1)}\,.$$

These formulae follow from the orthogonality of $P_n^{(\alpha,\beta)}(x)$ and the recurrence formula for the Jacobi polynomials (see for example Szegö [4; p. 71]).

Let $A^{(\alpha,\beta)}$ be the space of absolute convergent sequences $a=\{a_m\}_{m=0}^{\infty}$ whose norm is defined by

$$\|a\| = \sum_{m=0}^{\infty} |a_m|\,.$$

For $a=\{a_m\}$ and $b=\{b_n\}$ of $A^{(\alpha,\beta)}$ define the product by

$$a\cdot b = \left\{ \sum_{m,n=0}^{\infty} c(k;\, m, n)\ l(k)\, a_m\, b_n \right\}_{k=0}^{\infty}\,.$$

Then $A^{(\alpha,\beta)}$ is a commutative Banach algebra with identity $u = \{\delta_{m0}\}_{m=0}^{\infty}$, where δ_{mn} indicates the Kronecker's symbol.

The authors are informed that I. I. Hirschman, Jr. has shown the following theorem for the ultraspherical polynomials after they proved it. For the sake of convenience we give a complete proof.

THEOREM 1. *Suppose $\alpha \geqq \beta > -1$, $\alpha+\beta+1 \geqq 0$ and $\alpha \geqq -1/2$. Then the maximal ideal space $\mathfrak{M}$ of the algebra $A^{(\alpha,\beta)}$ is homeomorphic to the closed interval $[-1,1]$ and the Fourier-Gelfand transform of $a=\{a_m\}$ in $A^{(\alpha,\beta)}$ is given by the Jacobi polynomial series*

1) This is communicated by Professor Askey and will appear in "Linearization of the product of Jacobi polynomials I, Canad. J. Math." Prof. Askey also pointed out that the Pollaczeck polynomials have the dual convolution structure and that the similar method to the proof of Theorem 1 below is applicable in this case too.

$$\hat{a}(M_x) = \sum_{k=0}^{\infty} a_m R_m^{(\alpha,\beta)}(x),$$

where M_x *denotes the maximal ideal correspondings to* x, $-1 \leqq x \leqq 1$.

PROOF. Since the homomorphisms of $A^{(\alpha,\beta)}$ are bounded, there is, for each $M \in \mathfrak{M}$, a bounded sequence $\{\alpha_m(M)\}_{m=0}^{\infty}$ such that

$$\hat{a}(M) = \sum_{k=0}^{\infty} a_m \alpha_m(M).$$

Since $\hat{u}(M) = \alpha_0(M)$ and $\hat{u}(M) = 1$, we get

$$\alpha_0(M) = 1 \qquad \text{for all } M \in \mathfrak{M}.$$

We have, by the multiplicative relation for $\{\delta_{m1}\}$ and $\{\delta_{nk}\}$,

$$\alpha_1(M)\,\alpha_k(M) = c(k-1;1,k)\,l(k-1)\,\alpha_{k-1}(M) + c(k;1,k)\,l(k)\,\alpha_k(M) + c(k+1;1,k)\,l(k+1)\,\alpha_{k+1}(M) \tag{5}$$

for $k = 0, 1, 2, \cdots$.

We shall show that the mapping $\alpha_1(M)$ is a homeomorphism of $\mathfrak{M}$ onto $\left[-\frac{\beta+1}{\alpha+1}, 1\right]$. If $\alpha_1(M) = \alpha_1(M')$, where $M, M' \in \mathfrak{M}$, then (5) implies that $\alpha_k(M) = \alpha_k(M')$ for all $k = 0, 1, 2, \cdots$. Thus $\hat{a}(M) = \hat{a}(M')$ for all $a \in A^{(\alpha,\beta)}$. Therefore α_1 is one to one mapping. Since $\mathfrak{M}$ is compact, it suffices to prove that this mapping is onto.

Let correspond x to M as follows:

$$P_1^{(\alpha,\beta)}(1)\,\alpha_1(M) = \frac{\alpha+\beta+2}{2}x + \frac{\alpha-\beta}{2}. \tag{6}$$

In this case we denote M by M_x. Substituting (2), (3), (4) and (6) in (5), we get

$$\begin{aligned}(2k+\alpha+\beta+2)(2k+a+\beta+1)(2k+\alpha+\beta)\,x\,P_k^{(\alpha,\beta)}(1)\,\alpha_k(M_x) \\ = 2(k+1)(2k+\alpha+\beta)(k+\alpha+\beta+1)\,P_{k+1}^{(\alpha,\beta)}(1)\,\alpha_{k+1}(M_x) \\ - (\alpha^2-\beta^2)(2k+\alpha+\beta+1)\,P_k^{(\alpha,\beta)}(1)\,\alpha_k(M_x) \\ + 2(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)\,P_{k-1}^{(\alpha,\beta)}(1)\,\alpha_{k+1}(M_x)\end{aligned} \tag{7}$$

which is nothing but the recurrence formula for the Jacobi polynomials. Thus

$P_k^{(\alpha,\beta)}(1)a_k(M_x) = P_k^{(\alpha,\beta)}(x)$ or $\alpha_k(M_x) = R_k^{(\alpha,\beta)}(x)$.

Let α_1 be any number in $\left[-\frac{\beta+1}{\alpha+1}, 1\right]$ and x be the point corresponding to α_1 by the relation (6). Then we have $-1 \leqq x \leqq 1$ and the inequalities

$$|R_k^{(\alpha,\beta)}(x)| \leqq 1, \qquad k = 0, 1, 2, \cdots$$

show that $|a_k| \leqq 1$, $k=0, 1, 2, \cdots$ when α_k are defined by (7). The multiplicative property of α_k holds good, since it does for the Jacobi polynomials. Thus we have a maximal ideal M such that $M=M_x$. On the other hand if α_1 is outside of $\left[-\frac{\beta+1}{\alpha+1}, 1\right]$, that is, $x \notin [-1, 1]$,then $\{\alpha_k\}$ is unbounded as $k \to \infty$, since

$$R_k^{(\alpha,\beta)}(x) = \frac{1}{P_k^{(\alpha,\beta)}(1)}(x-1)^{-\alpha/2}(x+1)^{-\beta/2}[(x+1)^{1/2}+(x-1)^{1/2}]^{\alpha+\beta}$$

$$\cdot \frac{1}{\sqrt{2\pi k}}(x^2-1)^{-1/4}[x+(x^2-1)^{1/2}]^{k+(1/2)}$$

(see Szegö [2; p. 196]). This shows that there is no maximal ideal M such that $\alpha_1(M) = \alpha_1$.

2. A closed set E in $[-1, 1]$ will be calld Helson set with respect to the algebra $A^{(\alpha,\beta)}$, if every continuous function is the restriction of a function in $A^{(\alpha,\beta)}$ to E.

A theorem in [2] suggests the following :

THEOREM 2. *Suppose* $\alpha \geqq \beta \geqq -1/2$ *and* $\alpha > -1/2$. *A closed set in* $[-1, 1]$ *is Helson set with respect to* $A^{(\alpha,\beta)}$ *if and only if it is finite.*

PROOF. If E is a finite set $\{x_1, x_2, \cdots, x_n\}$ and $a_1, a_2, \cdots, a_n$ are any complex numbers, then we have a polynomial $p(x)$ such that $p(x_k) = a_k$, $k=1, 2, \cdots, n$. Hence E is a Helson set.

Conversely, suppose that E is an infinite Helson set. Since any closed subset of E is also Helson set, we may assume that E is countable and has only one accumulation point x_∞, say, $E=\{x_1, x_2, \cdots, x_\infty\}$.

By definition there exists an absolute convergent sequence $\{a_m\}_{m=0}^{\infty}$ such that

$$\sum_{m=0}^{\infty} a_m R_m^{(\alpha,\beta)}(x_n) = \frac{(-1)^n}{n}, \qquad n=1, 2, \cdots, \infty. \tag{8}$$

First assume $x_\infty=1$. We may assume $1-x_n < 2^{-2n}$. Since

$$\sum_{m=0}^{\infty} a_m(-1)^n[R_m^{(\alpha,\beta)}(x_n) - R_m^{(\alpha,\beta)}(1)] = \frac{1}{n},$$

we have

$$(9) \qquad \sum_{m=0}^{\infty} a_m \sum_{n=1}^{N}(-1)^n[R_m^{(\alpha,\beta)}(x_n) - R_m^{(\alpha,\beta)}(1)] = \sum_{n=1}^{N} \frac{1}{n},$$

where N is an arbitrary positive integer.

We shall show that the inner sum is uniformly bounded. In fact

$$R_m^{(\alpha,\beta)}(x_n) - R_m^{(\alpha,\beta)}(1) = \frac{P_m^{(\alpha,\beta)}(x_n) - P_m^{(\alpha,\beta)}(1)}{P_m^{(\alpha,\beta)}(1)}$$

$$= O\left(\frac{m^{2+\alpha}(1-x_n)}{m^\alpha}\right) = O(m^2 2^{-2n}).$$

Thus

$$\sum_{n=[\log m]+1}^{N} |R_m^{(\alpha,\beta)}(x_n) - R_m^{(\alpha,\beta)}(1)| = O(1),$$

uniformly in N and m. On the other hand

$$\sum_{n=1}^{[\log m]} |R_m^{(\alpha,\beta)}(x_n)| = \sum_{n=1}^{[\log m]} O(m^{-\alpha} m^{-1/2}(1-x_n)^{(\alpha+1/2)/2})$$

$$= \sum_{n=1}^{[\log m]} O(m^{-(\alpha+1/2)} 2^{(\alpha+1/2)n}) = O(1)$$

uniformly in N and m. Therefore the inner sum of (9) is uniformly bounded. This implies that the left hand side of (9) is bounded, which is impossible.

If $x_\infty = -1$, we get from (8)

$$\sum_{m=0}^{\infty} (-1)^m a_m \frac{P_m^{(\beta,\alpha)}(-x_n)}{P_m^{(\alpha,\beta)}(1)} = \frac{(-1)^n}{n}.$$

Thus the proof proceeds as above.

Next we assume $\{x_n\}$ is bounded away from 1 and -1. In this case the similar method will be applicable if $\beta \geqq 1/2$ but we shall use the dual argument to get the complete proof.

If E were a Helson set, we would have $\delta > 0$ such that for any absolute convergent sequence $\{b_n\}_{n=1}^{\infty}$

$$\sup_m \left| \sum_{n=1}^{\infty} b_n R_m^{(\alpha,\beta)}(x_n) \right| \geqq \delta \sum_{n=1}^{\infty} |b_n| \,. \tag{10}$$

This follows from the similar argument to the Fourier series case (cf. for example [3: p. 142]) and considering the measure $\sum_{n=1} b_n \delta_{x_n}$ where δ_x is the Dirac measure concentrated at x.

Since $R_m^{(\alpha,\beta)}(x_n) = O(m^{-\alpha-1/2})$, there exists $N > 0$ not depending on $\{b_n\}$ such that

$$\left| \sum_{n=1}^{\infty} b_n R_m^{(\alpha,\beta)}(x_n) \right| < \frac{\delta}{2} \sum_{n=1}^{\infty} |b_n|$$

for $m > N$. On the other hand there exists obviously a non-zero sequence $\{b_n\}$ such that

$$\sum_{n=1}^{\infty} b_n R_m^{(\alpha,\beta)}(x_n) = 0 \,, \qquad m = 1, 2, \cdots, N.$$

Therefore

$$\sup_m \left| \sum_{n=1}^{\infty} b_n R_m^{(\alpha,\beta)}(x_n) \right| \leqq \frac{\delta}{2} \sum_{n=1}^{\infty} |b_n| \,,$$

which contradicts (10).

REMARK. If $\alpha = \beta = -1/2$, $R_m^{(\alpha,\beta)}(\cos\theta) = \cos m\theta$ and it is well-known that there exists an infinite Helson set with respect to $A^{(\alpha,\beta)}$.

REFERENCES

[1] R. ASKEY AND S. WAINGER, A dual convolution structure for Jacobi polynomials, Proc. Conference, Orthogonal Expansions and their Continuous Analogues, (1967), 25–36.

[2] C. DUNKL, Operators and harmonic analysis on the sphere, Trans. Amer. Math. Soc., 125(1967), 250-263.

[3] J.-P. KAHANE AND R. SALEM, Ensembles Parfaits et Séries Trigonométriques, Hermann, 1963.

[4] G. SZEGÖ, Orthogonal Polynomials, Amer. Math. Soc. Colloquium, 1959.

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