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ENTIRE FUNCTIONS DEFINED BY GAP POWER SERIES AND SATISFYING A DIFFERENTIAL EQUATION

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1. Introduction. Let $f(z)$ be a transcendental entire function. Then $f(z)$ is said to be of bounded index if there exists a non-negative integer N such that

$$\max_{0\leqq n\leqq N}\left\{\frac{|f^{(n)}(z)|}{n!}\right\}\geqq\left\{\frac{|f^{(k)}(z)|}{k!}\right\},\quad k=0,1,\cdots \tag{1.1}$$

for every complex number z. The index of $f(z)$ is then defined to be the smallest integer N such that (1.1) holds for every z (see [1], [2]).

It is known [3] that if $f(z)$ is of index N then

$$\limsup_{r\to\infty}\frac{\log M(r,f)}{r}\leqq N+1.$$

It is also known [4] that any transcendental entire function $f(z)$ satisfying the differential equation

$$P_0(z)f^{(k)}(z)+P_1(z)f^{(k-1)}(z)+\cdots+P_k(z)f(z)=Q(z), \tag{1.2}$$

where $P_j(z)$, $j=0,1,2,\cdots,k$, $Q(z)$ are polynomials and $P_0(z)$ ($\not\equiv 0$) is of degree not less than of any $P_j(z)$, is of bounded index.

We consider here functions of bounded index satisfying (1.2) and given by the power series expansion

$$f(z)=\sum_{\nu=0}^{\infty}a_\nu z^{m\nu},\quad m \text{ positive integer}. \tag{1.3}$$

Our aim is to give a method for estimating the index of the given function $f(z)$ for which we need an additional hypothesis concerning the coefficients a_ν of its power series representation.

As an application of the procedure the index of the entire functon

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$f(z)=z^{-k}J_k(z)$ for $0 \leqq k \leqq 0.21$ has been calculated, and for $k>0.21$ an upper bound for the index has been determined. (See Theorem 2.)

2. Gap power series.

We prove the following

THEOREM 1. *Let* $F(z)=\sum_{\nu=0}^{\infty} a_\nu z^{m\nu}$, *where* $m \geqq 1$ *is an integer, be an entire function.*

Suppose that $F(z)$ *satisfies a differential equation of the form*

$$z^l y^{(k)}+P_1(z)y^{(k-1)}+\cdots+P_k(z)y=0, \quad l \geqq 1, \tag{2.1}$$

where

$$\deg P_i(z)=\lambda_i \quad \text{and} \quad \lambda_i \leqq l, \quad i=1,2,\cdots,k. \tag{2.2}$$

If

$$\frac{(m(n+1))!}{(mn)!}\left|\frac{a_{n+1}}{a_n}\right| \leqq \frac{1}{c^m}, \quad n=0,1,2,\cdots, \tag{2.3}$$

where $c>1$ *and*

$$l \leqq c(\log 2)\left\{\log\left(2-\frac{2}{1+c^m(m+1)^m}\right)\right\}, \tag{2.4}$$

then $F(z)$ *is of bounded index* N *and*

$$N<\max(m, n_0) \tag{2.5}$$

where $n_0=n_0(l,k,c,m,\lambda_i)$ *can be determined.*

PROOF. Since $F(z)$ satisfies (2.1) it is of bounded index by Theorem 1 of [4] and we have to prove (2.5).

Let

$$T=c\log\left\{2-\frac{2}{1+(m+1)^m c^m}\right\}; \tag{2.6}$$

(i) We first consider the case $|z| \leqq T$. By differentiating $F(z)$ $\{(n-1)m+p\}$ times we obtain

$$(2.7)\quad \frac{F^{((n-1)m+p)}(z)}{((n-1)m+p)!} = a_n\beta_{n,p}z^{m-p}\sum_{\nu=0}^{\infty}\left(\frac{a_{n+\nu}}{a_n}\right)\frac{(m(\nu+n))!(m-p)!}{(mn)!(m(\nu+1)-p)!}\; z^{\nu m}$$

where

$$(2.8)\qquad \beta_{n,p} = \frac{(mn)!}{(m(n-1)+p)!(m-p)!}\,, \quad n = 1, 2, \cdots,\ p = 1, 2, \cdots, m.$$

Now

$$(2.9)\qquad \frac{(m-p)!\,(\nu m)!}{(m(\nu+1)-p)!} = \frac{(m-p)(m-p-1)\cdots 1}{(m\nu+m-p)\cdots(m\nu+1)} \leqq 1\,,$$

and we deduce from (2.3) that

$$(2.10)\qquad \left|\frac{a_{n+\nu}}{a_n}\right| \leqq \frac{1}{c^{m\nu}}\,\frac{(mn)!}{(mn+m\nu)!}\,, \quad n = 1, 2, \cdots, \nu = 0, 1, \cdots.$$

(2.7), (2.9) and (2.10) yield

$$(2.11)\qquad \frac{|F^{((n-1)m+p)}(z)|}{(m(n-1)+p)!} \leqq |a_n|\beta_{n,p}|z|^{m-p}\sum_{\nu=0}^{\infty}\frac{1}{(m\nu)!}\left(\frac{|z|}{c}\right)^{m\nu}$$
$$\leqq |a_n|\beta_{n,p}|z|^{m-p}e^{|z|/c}\,.$$

On the other hand

$$(2.12)\qquad \frac{|F^{((n-1)m+p)}(z)|}{(m(n-1)+p)!} \geqq |a_n|\beta_{n,p}|z|^{m-p}(2-e^{|z|/c})$$

and both relations (2.11), (2.12) hold for $n = 1, 2, \cdots$, $p = 1, 2, \cdots, m$. Write

$$(2.13)\qquad \gamma_n = \frac{(m(n+1))!}{(mn)!}\,c^m\,\frac{\beta_{n,p}}{\beta_{n+1,p}}\,.$$

Then from (2.3), (2.11), (2.12) and (2.13) we obtain

$$(2.14)\quad \frac{|F^{(mn+p)}(z)|}{(mn+p)!} \leqq \frac{|F^{((n-1)m+p)}(z)|}{(m(n-1)+p)!}\,, \quad n = 1, 2, \cdots,\quad p = 1, 2, \cdots, m,$$

(which gives (1.1) with $N = m$), provided that

$$e^{|z|/c} \leqq \gamma_n(2-e^{|z|/c}),\quad \text{that is}$$

$$(2.15)\qquad |z| \leqq c\log\frac{2\gamma_n}{1+\gamma_n}\ .$$

Since $\gamma_n = c^m(mn+p)(mn+p-1)\cdots(m(n-1)+p+1)$, $1+\gamma_n \geqq 1+(m+1)^m c^m$, $n\geqq 2$, $p=1,2,\cdots,m$ and we deduce that

$$(2.16)\qquad \frac{2\gamma_n}{1+\gamma_n} \geqq 2\left(1-\frac{1}{1+(m+1)^m c^m}\right),$$

so that (2.14) holds for $n\geqq 2$, $p=1,\cdots,m$ and $|z|\leqq T$.

In order to conclude that (2.14) holds with the same T for $n=1$ also, we estimate $\frac{|F^{(p)}(z)|}{p!}$ more precisely by

$$(2.17)\qquad \frac{|F^{(p)}(z)|}{p!} \geqq |a_1|\beta_{1,p}|z|^{m-p}\left\{1-\frac{(m-p)!m!}{(2m-p)!}\left(\exp\frac{|z|}{c}-1\right)\right\}$$

and $\frac{|F^{(p+m)}(z)|}{(p+m)!}$ by

$$(2.18)\qquad \frac{|F^{(p+m)}(z)|}{(p+m)!} \leqq |a_2|\beta_{2,p}|z|^{m-p}\left\{1+\frac{(m-p)!m!}{(2m-p)!}\left(\exp\frac{|z|}{c}-1\right)\right\}.$$

Let first $p=m$. Then (2.17) and (2.18) become by (2.3) and (2.8),

$$\frac{|F^{(m)}(z)|}{m!} \geqq \frac{|F^{(2m)}(z)|}{(2m)!}$$

for any z such that $|z|\leqq T$.

Next, let $1\leqq p\leqq m-1$ and put

$$X=\frac{(m-p)!m!}{(2m-p)!}\left(\exp\frac{|z|}{c}-1\right);$$

then (2.17), (2.18) yield (2.14) with $n=1$, provided that

$$\frac{1}{p!}(1-X) \geqq \frac{1}{c^m(m+p)!}(1+X)$$

or that

$$|z| \leqq c\log\left\{1+\frac{c^m(m+p)!-p!}{c^m(m+p)!+p!}\,\frac{(2m-p)!}{m!(m-p)!}\right\}.$$

We now use the following

LEMMA. *Let* $c>1$, $p=1,2,\cdots m$. *Then*

$$2-\frac{2}{1+(m+1)^m c^m} \leqq 1+\frac{(c^m(m+p)!-p!)((2m-p)!)}{(c^m(m+p)!+p!)(m!(m-p)!)}. \tag{2.19}$$

The lemma shows that (2. 14) holds for $|z|\leqq T$ and $n=1$ also.

We are left therefore with the

PROOF OF LEMMA. If $m=1=p$, (2. 19) reduces to an equality. Let $m=p>1$. Since $(2m)!\geqq(m+1)^m(m)!$, we have

$$\frac{1}{1+c^m(m+1)^m} \geqq \frac{m!}{c^m(2m)!+m!}$$

and so

$$1-\frac{2}{1+c^m(m+1)^m} \leqq 1-\frac{2(m!)}{c^m(2m)!+m!}$$

and (2. 19) follows in this case also.

Let $m\geqq 2$, $p\leqq m-1$. Then observe that

$$\frac{(2m-p)!}{(m-p)!m!} \geqq (m+1) \geqq 3 \tag{2.20}$$

and that

$$3c^m(m+p)!-3p! \geqq c^m(m+p)!+p! \tag{2.21}$$

which is true for, $\frac{c^m}{2}(m+p)(m+p-1)\cdots(p+1)>1$.

Therefore by (2. 20) and (2. 21)

$$1+\frac{c^m(m+p)!-p!}{c^m(m+p)!+p!}\ \frac{(2m-p)!}{m!(m-p)!}\geqq 2,$$

and the lemma is proved.

(ii) Let us next consider the case $|z|\geqq T$. By differentiating (2. 1) $(n-k)$ times, where $n\geqq k+l$, we obtain

$$z^l F^{(n)}+F^{(n-1)}\{(n-k)lz^{l-1}+P_1\}+\cdots$$

$$F^{(n-k)}\left\{\binom{n-k}{k}l(l-1)\cdots(l-k+1)z^{l-k}+\binom{n-k}{k-1}P_1^{(k-1)}+\cdots\right.$$

$$\left.+\binom{n-k}{0}\right\}P_k+\cdots+F^{(n-k-l)}\left\{\binom{n-k}{l}P_1^{(l)}\right\}=0,$$

and so

$$\frac{|F^{(n)}|}{n!}\leqq\frac{|F^{(n-1)}|}{(n-1)!}\alpha_1+\frac{|F^{(n-2)}|}{(n-2)!}\alpha_2+\cdots+\frac{|F^{(n-k-l)}|}{(n-k-l)!}\alpha_{k+l} \tag{2. 22}$$

where

$$\alpha_j=\frac{1}{n(n-1)\cdots(n-j+1)}\left\{\binom{n-k}{j}\frac{l(l-1)\cdots(l-j+1)}{|z|^j}+\binom{n-k}{j-1}\frac{|P_1^{(j-1)}|}{|z|^l}+\cdots\right.$$

$$\left.\cdots+\binom{n-k}{0}\frac{|P_j|}{|z|^l}\right\},\ 1\leqq j\leqq l,$$

$$\alpha_{k+l}=\frac{1}{n(n-1)\cdots(n-k-l+1)}\left\{\frac{(n-k)\cdots(n-k-l+1)}{l!}\frac{|P_k^{(l)}(z)|}{|z|^l}\right\}.$$

Now we have

$$\alpha_1+\alpha_2+\cdots+\alpha_{k+l}\leqq\left(1+\frac{1}{T}\right)^l-1+\frac{1}{n}\sum_{j=0}^{l}\sum_{i=1}^{k}\max_{|z|=T}|P_i^{(j)}(z)/z^l|,$$

and so from $P_i(z)=\sum_{\nu=0}^{\lambda_i}A_\nu^{(i)}z^{\lambda_i-\nu}$, and

$$P_i^{(j)}(z)=\sum_{\nu=0}^{\lambda_i-j}A_\nu^{(i)}(\lambda_i-\nu)(\lambda_i-\nu-1)\cdots(\lambda_i-\nu-j+1)z^{\lambda_i-\nu-j},\ j=1,2,\cdots,$$

we get

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{k+l} \leqq \left(1 + \frac{1}{T}\right) - 1 + S \tag{2.23}$$

where

$$S = \frac{1}{n} \sum_{j=0}^{l} \sum_{i=1}^{k} \sum_{\nu=0}^{\lambda_i - j} B_{\nu,j}^{(i)} T^{\nu - l + \lambda_i - \nu - j}$$

and

$$B_{\nu,j}^{(i)} = \begin{cases} |A_\nu^{(i)}| \text{ for } j = 0, \quad 0 \leqq \nu \leqq \lambda_i, \quad 1 \leqq i \leqq k, \\ |A_\nu^{(i)}|(\lambda_i - \nu) \cdots (\lambda_i - \nu - j + 1) \text{ otherwise}; \end{cases}$$

k is a positive integer, and $\lambda_i \leqq l$.

Now choose n_0 as the smallest integer $n \geqq k+l$ such that for $n \geqq n_0$, $S + \left(1 + \frac{1}{T}\right)^l \leqq 2$; then we have from (2.22) and (2.23) that

$$\max \left\{ |F(z)|, \frac{|F^{(1)}(z)|}{1!}, \cdots, \frac{|F^{(n-1)}(z)|}{(n-1)!} \right\} \geqq \frac{|F^{(j)}(z)|}{j!} \tag{2.24}$$

and all $j = 1, 2, \cdots$, $|z| \geqq T$, and $n \geqq n_0$.

Formulas (2.14), (2.24) together prove Theorem 1.

3. **Bessel functions**. As an application of a slight refinement of the above procedure we prove here the following

THEOREM 2. *Let N denote the index of the entire function*

$$f(z) = f_k(z) = z^{-k} J_k(z), \quad k \geqq 0.$$

Then

(a) $N = 1$ *if* $0 \leqq k \leqq 0.21$.

(*b*) $1 \leqq N \leqq 3$ *if* $0.21 < k \leqq 2.31$.

(c) $1 \leqq N \leqq \max \left\{ 4, \left[\frac{2k}{1.17} \right] \right\}$ *otherwise.*

REMARK. The function $f(z)$ satisfies a differential equation

$$zy'' + (1 + 2k)y' + zy = 0$$

of the form (1.2) and is therefore of bounded index.

PROOF. Since

$$f(z)=\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}z^{2\nu}}{2^{2\nu+k}\nu!\,\Gamma(k+\nu+1)}\overset{\text{def}}{=}\sum_{\nu=0}^{\infty}a_{\nu}z^{2\nu}, \tag{3.1}$$

the condition (2.3) of Theorem 1 holds with $c=1$ so that the theorem cannot be applied directly.

Therefore we improve the estimates (2.11) and (2.12) to

$$\begin{aligned}\frac{|f^{(2n)}(z)|}{(2n)!}&<|a_n|\left(1+\sum_{\nu=1}^{\infty}\frac{|z|^{2\nu}}{(2\nu)!}\right)\\ &=|a_n|\cosh r,\ n=1,2,\cdots,\end{aligned} \tag{3.2}$$

$$\frac{|f^{(2n-1)}(z)|}{(2n-1)!}\leqq 2n|a_n|\sinh r,\ n=1,2,\cdots, \tag{3.3}$$

and

$$|f(z)|\geqq\frac{1}{2^k\Gamma(k+1)}\left\{2-\exp\left(\frac{r^2}{4(k+1)}\right)\right\}, \tag{3.4}$$

$$|f'(z)|\geqq 2|a_1|(2r-\sinh r) \tag{3.5}$$

where, and in what follows, we write $|z|=r$.

Hence, from (3.2) and (3.4) we get

$$\frac{|f^{(2)}(z)|}{2!}\leqq|f(z)| \tag{3.6}$$

provided that

$$\exp\left(\frac{r^2}{4(k+1)}\right)+\frac{1}{4(k+1)}\cosh r\leqq 2 \tag{3.7}$$

and

$$\frac{|f^{(2n)}(z)|}{(2n)!}\leqq|f(z)|;\quad n=2,3,\cdots \tag{3.8}$$

provided that

$$\exp\left(\frac{r^2}{4(k+1)}\right)+\frac{1}{32(k+1)(k+2)}\cosh r\leqq 2 \tag{3.9}$$

and

$$\frac{|f^{(3)}(z)|}{3!} \leqq \frac{|f'(z)|}{1!} \tag{3.10}$$

provided that

$$\sinh r \leqq 4(k+2)(2r - \sinh r) \tag{3.11}$$

and

$$\frac{|f^{(2n-1)}(z)|}{(2n-1)!} \leqq \frac{|f'(z)|}{1!}, \quad n = 3, 4, \cdots \tag{3.12}$$

provided that

$$\frac{\sinh r}{r} + \frac{2}{1+2^5(k+2)(k+3)} \leqq 2. \tag{3.13}$$

To prove (a) we calculate that (3.6)-(3.13) hold for $r \leqq r_1(k)$ if $k \leqq 0.21$ where

$$\begin{aligned} r_1(k) &= 1.28 \quad \text{when} \quad 0 \leqq k \leqq 0.14 \\ &= 1.35 \quad \text{when} \quad 0.14 < k \leqq 0.175 \\ &= 1.40 \quad \text{when} \quad 0.175 < k \leqq 0.2 \\ &= 1.42 \quad \text{when} \quad 0.2 < k \leqq 0.21. \end{aligned}$$

Hence for $|z| \leqq r_1(k)$, $k \leqq 0.21$ we have

$$\max\left(|f(z)|, \frac{|f'(z)|}{1!}\right) \geqq \frac{|f^{(n)}(z)|}{n!}, \quad n = 1, 2, \cdots. \tag{3.14}$$

Let next $|z| \geqq r_1$. By differentiating $(n-2)$ times the equation satisfied by $f(z)$, we get by the argument of Theorem 1,

$$\frac{|f^{(n)}(z)|}{n!} \leqq \frac{|f^{(n-2)}(z)|}{(n-1)!}\alpha_1 + \frac{|f^{(n-2)}(z)|}{(n-2)!}\alpha_2 + \frac{|f^{(n-3)}(z)|}{(n-3)!}\alpha_3 \tag{3.15}$$

where

$$\alpha_1 = \frac{n-1+2k}{nr}, \quad \alpha_2 = \frac{1}{n(n-1)}, \quad \alpha_3 = \frac{1}{n(n-1)r}, \quad n=2, 3, \cdots \tag{3.16}$$

and $\alpha_3 = 0$ when $n = 2$.

Now for $k \leqq 0.21$, $n=2$ and $r \geqq r_1$, $\alpha_1 + \alpha_2 \leqq 1$ and for $n \geqq 3$, $\alpha_1 + \alpha_2 + \alpha_3 \leqq 1$. Hence the inequality (3.14) holds for all z, when $k \leqq 0.21$ and so $N \leqq 1$. Since $f(z)$ has simple zeroes, $N \geqq 1$ and the part (a) is proved.

To prove (b), again we note that the relations (3.8) - (3.13) hold for $r \leqq r_2 = 1.65$ when $k \geqq 0.21$, and so we have for $|z| \leqq r_2$, $k \geqq 0.21$

$$\left(\max|f(z)|, \frac{|f^{(1)}(z)|}{1!}, \frac{|f^{(2)}(z)|}{2!}\right) \geqq \frac{|f^{(n)}(z)|}{n!}, \quad n = 1, 2, \cdots. \tag{3.17}$$

For $|z| \geqq r_2$ we note that

(i)

$$\frac{n-1+2k}{n} + \frac{1}{n(n-1)} \leqq r_2 \left(1 - \frac{1}{n(n-1)}\right) \tag{3.18}$$

provided $n \geqq 3$ and $k \leqq 0.8125$. The formulae (3.15), (3.16) and (3.17) show that $N \leqq 2$ if $k \leqq 0.8125$.

(ii) If $k > 0.8125$ then (3.17) holds. Also (3.18) is satisfied if $n \geqq 4$ and $k \leqq 8.15/6$. Hence $N \leqq 3$ if $k \leqq 8.15/6$.

(iii) If $k > 8.15/6$ then (3.8) - (3.13) are satisfied for $r \leqq r_3 = 2.02$ and so (3.17) holds when $|z| \leqq r_3$. For $|z| > r_3$ (3.18), with r_2 replaced by r_3, is satisfied provided $n \geqq 4$ and $k \leqq 12.22/6$.

Hence in this case $N \leqq 3$.

(iv) For $k > 12.22/6$ again (3.8), (3.9), (3.12) and (3.13) are satisfied for $|z| \leqq r_4 = 2.17$. Consequently we have for $|z| \leqq r_4$

$$\left(\max|f(z)|, \frac{|f'(z)|}{1!}, \frac{|f^{(2)}(z)|}{2!}, \frac{|f^{(3)}(z)|}{3!}\right) \geqq \frac{|f^{(n)}(z)|}{n!}, \quad n = 1, 2, \cdots. \tag{3.19}$$

The inequality (3.18), with r_2 replaced by r_4, is satisfied if $k \leqq 13.87/6 = 2.311\cdots$ and $n \geqq 4$. Hence in this case also $N \leqq 3$, and (b) is proved.

(c) Let $k > 13.87/6$, $n \geqq 5$, $n \geqq (2k)/(r_4 - 1)$. Then (3.19) holds for $|z| \leqq r_4$ and (3.18), with r_2 replaced by r_4 is also satisfied for

$$\left(1 + \frac{2k}{n} - r_4\right) - \frac{1}{n}\left(1 - \frac{1+r_4}{n-1}\right) \leqq 0.$$

Hence
$$N \leqq \max\left\{4, \left[\frac{2k}{r_4 - 1}\right]\right\}.$$

and (c) is proved.

4. Remarks and Examples.

(i) If the relation (2. 3) in Theorem 1 holds for $n \geqq n_1$ only then also the same procedure is valid but the index N will now depend on n_1 also.

(ii) Write the equation (2. 1) as $Ly=0$. If F does not satisfy this equation but satisfies the equation $Ly=f(z)$ where $f(z)$ is an entire function satisfying an equation of the form (1. 2) and hence of bounded index N_f, then also our argument gives an upper bound for N_F which will now depend on N_f also.

(iii) Example. Let

$$f(z) = \cos\frac{z}{c} = 1 - \left(\frac{z}{c}\right)^2 \frac{1}{2!} + \cdots.$$

Then $m=2$ and f satisfies the equation

$$zf'' + \frac{zf}{c^2} = 0.$$

The condititions (2. 3) and (2. 4) are satisfied if we choose $c \geqq 3$. Thus there exist entire functions satisfying the conditions of Theorem 1.

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