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EXTREME POINTS AND OUTER FUNCTIONS IN $H^1(U^n)$

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It is well known that every extreme point of the unit ball of $H^1(U)$ is an outer function with norm 1 and vice versa [1]. In this note we shall point out that for $n \geqq 2$, every outer function (a generalized notion of 1-dimensional outer function, introduced by W.Rudin [2],) with norm 1 is also an extreme point of the unit ball S of $H^1(U^n)$, but there exists an extreme point of S which has so many zeros and is consequently not outer. We shall show Theorem 5 for such an example. We state first some facts about extreme points of S and outer functions, which can be shown by a little modification of de Leeuw-Rudin's method [1].

Let m_n denote the Haar measure of the torus T^n, the distiguished boundary of the unit polydisc U^n in the space of n complex variables. If f is holomorphic in U^n, define

$$f^*(w) = \lim_{r \to 1} f(rw)$$

for those $w \in T^n$ for which this radial limit exists. A holomorphic function $f \in H^1(U^n)$ is said to be outer if

$$\log|f(0)| = \int_{T^n} \log|f^*(w)|\, dm_n(w)$$

THEOREM 1. *Every outer function with norm* 1 *is an extreme point of* S, $(n \geqq 1)$.

THEOREM 2. *If* $f = gh$, *for some non-constant inner function* g *and* $h \in H^1(U^n)$, $\|h\|_1 = 1$, *then* f *is not an extreme point of* S, $(n \geqq 1)$.

THEOREM 3. *A function* $f \in H^1(U^n)$ *lies in the norm closure of the set of all outer functions with norm* 1 *if and only if* $\|f\|_1 = 1$ *and* $f(z) \neq 0$ *for all* $z \in U^n$, $(n \geqq 1)$.

THEOREM 4. *A function* $f \in H^1(U^n)$ *lies in the weak*-closure of the set*

We use systematically the notations of [2].

of all outer functions with norm 1 *if and only if* $f \in S$ *and* $f(z) \neq 0$ *for all* $z \in U^n$, *or if* f *is identically* 0, $(n \geqq 1)$.

Now we give an example stated above.

THEOREM 5. $\frac{\pi}{4}(z_1 + z_2)$ *is an extreme point of* S, $(n \geqq 2)$.

PROOF. We shall show in the case $n = 2$ for simplicity of notation, but our proof is general. Assume

$$z_1 + z_2 = \frac{f_1(z_1, z_2) + f_2(z_1, z_2)}{2}$$

where $f_j \in H^1(U^2)$, $\|f_j\|_1 = \|z_1 + z_2\|_1 = \frac{4}{\pi} (j = 1, 2)$. Then we have

$$(1) \qquad \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} (|f_1^*(e^{i\theta_1}, e^{i\theta_2})| + |f_2^*(e^{i\theta_1}, e^{i\theta_2})|)d\theta_1 d\theta_2 = \frac{8}{\pi}$$

and

$$(2) \qquad \frac{1}{2\pi}\int_0^{2\pi} |f_1^*(e^{i\theta_1}, e^{i\theta_2}) + f_2^*(e^{i\theta_1}, e^{i\theta_2})| d\theta_1 = \frac{1}{2\pi}\int_0^{2\pi} |2(e^{i\theta_1} + e^{i\theta_2})| d\theta_1 = \frac{8}{\pi} \qquad \text{a. e. } \theta_2 \in (0, 2\pi).$$

Hence there is a measurable set E_1 of $mE_1 = 1$ such that

$$(3) \qquad \frac{1}{2\pi}\int_0^{2\pi} (|f_1^*(e^{i\theta_1}, e^{i\theta_2})| + |f_2^*(e^{i\theta_1}, e^{i\theta_2})|)d\theta_1 = \frac{8}{\pi} \qquad (\theta_2 \in E_1).$$

On the other hand, there is a measurable set E_2 of $mE_2 = 1$ such that

$$f_j(z_1, e^{i\theta_2}) \in H^1(U) \qquad (j = 1, 2, \theta_2 \in E_2)$$

and

$$\lim_{r_1 \to 1} f_j(r_1 e^{i\theta_1}, e^{i\theta_2}) = f_j^*(e^{i\theta_1}, e^{i\theta_2}) \quad \text{a.e. } \theta_1 \ (\theta_2 \in E_2, j = 1, 2).$$

Set $E = E_1 \cap E_2$ and $\|f_j(z_1, e^{i\theta_2})\|_1 = \alpha_j(\theta_2)$, $j = 1, 2$, $\theta_2 \in E$.
Then, by (3), we have $\alpha_1(\theta_2) + \alpha_2(\theta_2) = \frac{8}{\pi}$.
Fix $\theta_2 \in E$. Assume $\alpha_1(\theta_2) \leqq \alpha_2(\theta_2)$ and put

$$g_1(z_1) = f_1(z_1, e^{i\theta_2}) + \frac{\alpha_2 - \alpha_1}{2\alpha_2} f_2(z_1, e^{i\theta_2})$$

$$g_2(z_1) = f_2(z_1, e^{i\theta_2}) \frac{\alpha_1 + \alpha_2}{2\alpha_2}.$$

Then we have

$$\|g_j(z_1)\|_1 \leqq \frac{\alpha_1 + \alpha_2}{2} = \frac{4}{\pi} \qquad (j = 1, 2).$$

By assumption we have

$$\frac{g_1(z_1) + g_2(z_1)}{2} = \frac{f_1(z_1, e^{i\theta_2}) + f_2(z_1, e^{i\theta_2})}{2}$$

$$= z_1 + e^{i\theta_2}.$$

Since $\|z_1 + e^{i\theta_2}\|_1 = \frac{4}{\pi}$, and $z_1 + e^{i\theta_2}$ is an outer function in $H^1(U)$ and thus an extreme point of $H^1(U)$, we must have

$$g_j(z_1) = z_1 + e^{i\theta_2} \qquad (j = 1, 2).$$

We have, therefore,

$$(4) \qquad f_j(z_1, e^{i\theta_2}) = \frac{\pi}{4}\alpha_j(\theta_2)(z_1 + e^{i\theta_2}) \qquad (j = 1, 2).$$

These expressions hold also for $\alpha_1 > \alpha_2$, and so for all $\theta_2 \in E$.

Now there is a measurable set E_3 of $mE_3 = 1$ and $\theta_1 \in (0, 2\pi)$ such that

$$f_1(e^{i\theta_1}, z_2) \in H^1(U)$$

and

$$\begin{aligned}&\lim_{r_1\to 1}\lim_{r_2\to 1} f_1(r_1e^{i\theta_1}, r_2e^{i\theta_2})\\ &= \lim_{r_2\to 1}\lim_{r_1\to 1} f_1(r_1e^{i\theta_1}, r_2e^{i\theta_2})\\ &= f_1^*(e^{i\theta_1}, e^{i\theta_2}) \qquad (\theta_2 \in E_3).\end{aligned}$$

Using this fact and (4), we extend $\alpha_1(\theta_2)$ holomorphically into $U = \{z_2 : |z_2| < 1\}$ by

$$f_1(e^{i\theta_1}, z_2) = \frac{\pi}{4}\alpha_1(z_2)(e^{i\theta_1} + z_2) \ \ (z_2 \in U).$$

Since $e^{i\theta_1} + z_2$ is an outer function and $f(e^{i\theta_1}, z_2) \in H^1(U)$, $\alpha_1(z)$ lies in $N_*(U)$. And as $0 \leqq \alpha_1(\theta_2) \leqq 8/\pi, \alpha_1(z)$ lies in $H^\infty(U)$ and hence must be constant. By definition of $\alpha_1(\theta_2)$ and $\alpha_2(\theta_2)$, we must have

$$\alpha_1 = \alpha_2 = \frac{4}{\pi},$$

which shows via (4) that $f_1 = f_2 = z_1 + z_2$. This proves that $\frac{\pi}{4}(z_1 + z_2)$ is an extreme point of the unit ball of $H^1(U^2)$. Q. E. D.

References

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