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LOCAL CLASSIFICATION OF INVARIANT η-EINSTEIN SUBMANIFOLDS OF CODIMENSION 2 IN A SASAKIAN MANIFOLD WITH CONSTANT ϕ-SECTIONAL CURVATURE

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1. Introduction. In [2] Yano-Ishihara studied the invariant η-Einstein submanifolds of codimension 2 in a Sasakian manifold of constant curvature. The purpose of this note is to give a proof of the following Theorem:

THEOREM. *Let M^{2n-1} be an invariant η-Einstein (for the induced metric) submanifold of codimension 2 in a Sasakian manifold with constant ϕ-sectional curvature c. Then, if $c \leqq -3$, M^{2n-1} is totally geodesic. If $c > -3$, M^{2n-1} is totally geodesic or an η-Einstein manifold with the scalar curvature* $(n-1)\{c(n-1)+3n-5\}$.

2. Preliminaries. Let $\bar{M}^{2n+1}[c]$ be a Sasakian manifold with constant ϕ-sectional curvature c. The notation is that of [2], the structure tensors of $\bar{M}^{2n+1}[c]$ being denoted by $(F^h{}_i, E_i, E^h, G_{ij})$. The following ranges of indices will be used throughout this paper:

$$1 \leqq i, j, k, l, \cdots \leqq 2n+1, \quad 1 \leqq a, b, c, \cdots \leqq 2n-1.$$

The curvature tensor K_{ijkl} of $\bar{M}^{2n+1}[c]$ is of the form

$$(1) \qquad 4K_{ijkl} = (c+3)(G_{jk}G_{il} - G_{jl}G_{ik}) + (c-1)(F_{il}F_{jk} - F_{ik}F_{jl} - 2F_{ij}F_{kl} + E_lE_jG_{ik} - E_kE_jG_{il} + E_iE_kG_{jl} - E_lE_iG_{jk}).$$

As an invariant submanifold in $\bar{M}^{2n+1}[c]$ is Sasakian for the induced metric, we denote the structure tensors by $(f_b{}^a, e_b, e^a, g_{ac})$. Let C^h and $D^h = F^h{}_sC^s$ be a pair of local mutually orthogonal unit vector fields normal to M^{2n-1}. h_{cb} and k_{cb} are the second fundamental tensors and l_c the third fundamental tensor with respect to C^h and D^h. Making use of (1) and (2.7) of [2] we have

$$(2) \quad R_{dcba} = \frac{c+3}{4}(g_{bc}g_{ad} - g_{ac}g_{bd}) + \frac{c-1}{4}(f_{ad}f_{bc} - f_{bd}f_{ac} - 2f_{cd}f_{ab} + e_ae_cg_{bd} - e_be_cg_{ad} + e_be_dg_{ac} - e_ae_dg_{bc}) + h_{ad}h_{bc} - h_{ac}h_{bd} + k_{ad}k_{bc} - k_{ac}k_{bd},$$

where R_{dcba} is the curvature tensor of M^{2n-1}. We omit the proof of the following Proposition since we can prove it by the same method as Yano-Ishihara prove the Proposition 5.2 in [2].

PROPOSITION 1. *Let M^{2n-1} be an invariant submanifold in $\bar{M}^{2n+1}[c]$. If M^{2n-1} is an η-Einstein manifold, then*

$$h_{ba,f} = k_{ba}l_f + k_{bf}e_a + k_{af}e_b + k_{ab}e_f, \tag{3}$$

$$k_{ba,f} = -h_{ba}l_f - h_{bf}e_a - h_{af}e_b - h_{ab}e_f. \tag{4}$$

Throughout this paper we assume that M^{2n-1} is an η-Einstein manifold, i. e.,

$$R_{ab} = ag_{ab} + be_ae_b. \tag{5}$$

Since M^{2n-1} is a minimal submanifold, we then see, from (2), that the Ricci tensor of M^{2n-1} has the form

$$R_{cb} = \frac{(c+3)n-4}{2}g_{cb} - \frac{(c-1)n}{2}e_ce_b - h_c^a h_{ba} - k_c^a k_{ba}. \tag{6}$$

Using (5), (6), $h^2 = k^2$ (see (4.20) of [2]) and $a+b = 2n-2$, we find

$$h_c^a h_a^b = -\mu f_c^a f_a^b, \text{ where } \mu = \frac{(c+3)n-4-2a}{4}. \tag{7}$$

Since we have, by (7), $g_{ad}(h_b^d X^b)(h_c^a X^c) = \mu g_{ad}(f_b^d X^b)(f_c^a X^c)$, we find $\mu \geqq 0$.

LEMMA. *Under the same assumption as Proposition* 1, *we have*

$$l_{f,d} - l_{d,f} = \left(2\mu + \frac{c-1}{2}\right)f_{fd}. \tag{8}$$

PROOF. Making use of (7) and $k_{cb} = -h_{ca}f_b^a$(see 4.14 of [2]), we have

$$h_d^a k_{ea} = \mu f_{de}. \tag{9}$$

By (2.9) of [2], (1) and (9), Lemma follows.

3. The proof of Theorem. We take the covariant differentiation of (3). This gives, using (4),

$$
\begin{aligned}
(10) \qquad h_{ba,f,d} = {} & l_{f,d}k_{ba} - (l_d h_{ba} + h_{bd}e_a + h_{ad}e_b + h_{ab}e_d)l_f - (l_d h_{bf} + h_{bd}e_f + h_{fd}e_b \\
& + h_{bf}e_d)e_a - (l_d h_{fa} + h_{df}e_a + h_{ad}e_f + h_{af}e_d)e_b - (l_d h_{ab} + h_{bd}e_a \\
& + h_{ad}e_b + h_{ab}e_d)e_f + k_{bf}f_{da} + k_{af}f_{db} + k_{ab}f_{df}\,.
\end{aligned}
$$

From (1), (8), (10) and the Ricci's identity,

$$h_{ba,f,d} - h_{ba,d,f} = h_a^c R_{bcfd} - h_b^c R_{cafd}\,,$$

we have, after simplication,

$$
\begin{aligned}
(11) \qquad & \left(\mu - \frac{c+3}{4}\right) \{h_{af}g_{bd} - h_{ad}g_{bf} - h_{bd}g_{af} + h_{bf}g_{ad} - 2f_{df}k_{ab} + (h_{ad}e_f \\
& - h_{af}e_d)e_b + (h_{bd}e_f - h_{bf}e_d)e_a - k_{af}f_{db} + k_{ad}f_{fb} - k_{bf}f_{ad} + k_{bd}f_{fa}\} = 0.
\end{aligned}
$$

If $\mu - (c+3)/4 \neq 0$, we have, transvecting (11) with g^{bd}, $h_{af} = 0$, and, by $k_{ba} = -f_{ac}h_b^c$, M^{2n-1} is totally geodesic. Hence M^{2n-1} is not totally geodesic only if $\mu = (c+3)/4$, which implies $c > -3$. From (6) and (7) the scalar curvature of M^{2n-1} with $\mu = (c+3)/4$ is $(n-1)\{c(n-1)+3n-5\}$. Q. E. D.

REMARK. The global version of the Theorem is seen to [1].

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