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SATURATION OF THE APPROXIMATION BY EIGENFUNCTION EXPANSIONS ASSOCIATED WITH THE LAPLACE OPERATOR

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1. Let $n \geqq 2$ and Ω be an open domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$. Suppose that $u_k(x)$, $k = 1, 2, 3, \cdots$ are eigenfunctions of the Laplace operator Δ and λ_k are the corresponding eigenvalues, that is,

$$\Delta u_k(x) + \lambda_k u_k(x) = 0 \text{ in } \Omega .$$

We assume that $\{u_k(x)\}_{k=1}^{\infty}$ is a complete orthonormal system in $L^2(\Omega)$, and furthermore λ_k are non-decreasing and tend to infinity. These assumptions will be satisfied if we impose some boundary conditions on eigenfunctions and Ω.

For a function f in $L^2(\Omega)$ let

$$f \sim \sum_{k=1}^{\infty} f_k u_k(x)$$

be the Fourier expansion, where

$$f_k = \int_{\Omega} f(x) \overline{u_k(x)}\, dx .$$

We denote the λ-th $R(\lambda_k, \delta)$ mean by

$$s_\lambda^\delta(f, x) = \sum_{\lambda_k < \lambda} \left(1 - \frac{\lambda_k}{\lambda}\right)^{\delta} f_k u_k(x).$$

$f(x)$ is said to be regulated at x if there exists an approximate identity $\{\varphi_\varepsilon(x)\}$ of infinitely differentiable functions with supports contained in $\{x;\ |x| \leqq \varepsilon\}$ such that $f * \varphi_\varepsilon(x)$ tends to $f(x)$.

Let $\alpha = (n-1)/2$ be the critical index and denote by $\|\ \|_K$ the supremum

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norm on the set K. Our aim is to prove the following.

THEOREM. *Let D be an open subdomain of Ω, f be a function in $L^2(\Omega)$ regulated in D and $\delta \geqq \alpha+2$.*

(i) *It holds that*

$$\|s_\lambda^\delta(f)-f\|_K = o(1/\lambda)$$

as $\lambda\to\infty$ for every compact set K in D, if and only if f is harmonic in D.

(ii) *It holds that*

$$\|s_\lambda^\delta(f)-f\|_K = O(1/\lambda)$$

as $\lambda\to\infty$ for every compact set K in D, if and only if Δf in the sense of distribution is bounded in every compact set of D.

REMARK. Let $\delta\geqq\alpha$ and assume the condition of (i). Then we have

$$\|s_\lambda^\delta(f)-f\|_K = o(\sqrt{\lambda}^{\,\alpha-\delta})$$

as $\lambda\to\infty$ for every compact set K in D. This inequality is valid under the hypothesis of (ii) if $2>\delta-\alpha\geqq 0$

2. The local saturation problem for trigonometric expansions of a variable is studied by [3], for example, but for our case a difficulty arises mainly from the fact that we fail to find any (quasi-) positive summability kernels like the Cesàro or the Poisson kernels, and some different devices will be needed.

Let $\delta>-1$ and x be any point in Ω. If $R>0$ is so small that the sphere $S(x,R)$ of radius R with the center at x is contained in Ω, then we have

$$s_\lambda^\delta(f,x) = v_\lambda^{\delta,R}(f,x) + w_\lambda^{\delta,R}(f,x),$$

where

$$v_\lambda^{\delta,R}(f,x) = \frac{2^\delta\Gamma(\delta+1)}{(2\pi)^{n/2}}\sqrt{\lambda}^{\,\frac{n}{2}-\delta}\int_{S(0,R)}\frac{J_{\frac{n}{2}+\delta}(\sqrt{\lambda}y)}{|y|^{\frac{n}{2}+\delta}}f(x-y)dy$$

and

$$w_\lambda^{\delta,R}(f,x) = 2^\delta\Gamma(\delta+1)\sqrt{\lambda}^{\,\frac{n}{2}-\delta}\sum_{k=1}^{\infty}f_k u_k(x)\frac{1}{\sqrt{\lambda_k}^{\,\frac{n}{2}-1}}\int_R^\infty J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)J_{\frac{n}{2}-1}(\sqrt{\lambda_k}r)r^{-\delta}dr$$

(see [4; p. 205]).

The order of $w_\lambda^{\delta,R}(f,x)$ is given in [2] and [4], but we shall need more accurate estimation.

LEMMA 1. *If $f \in L^2(\Omega)$, $\delta > 0$ and K is a compact set in Ω, then*

$$\|w_\lambda^{\delta,R}(f)\|_K = o(\sqrt{\lambda}^{-\alpha-\delta})$$

as $\lambda \to \infty$ for every R such that $0 < R < \mathrm{dis}(K, \Omega^c)$.

PROOF. Put

$$I_k^\lambda = \int_R^\infty J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)J_{\frac{n}{2}-1}(\sqrt{\lambda_k}\,r)r^{-\delta}\,dr.$$

By integration by parts and an asymptotic formula for the Bessel function we have

$$|I_k^\lambda| < A\lambda^{-\frac{1}{4}}\lambda_k^{-\frac{1}{4}}$$

for all positive λ and λ_k,

$$|I_k^\lambda| < A\frac{\lambda^{-\frac{3}{4}}\lambda_k^{\frac{1}{4}}}{\sqrt{\lambda}-\sqrt{\lambda_k}} + A\lambda^{-\frac{3}{4}}\lambda_k^{-\frac{1}{4}} \quad (\lambda_k < \lambda),$$

and

$$|I_k^\lambda| < A\frac{\lambda^{\frac{1}{4}}\lambda_k^{-\frac{3}{4}}}{\sqrt{\lambda_k}-\sqrt{\lambda}} + A\lambda^{-\frac{1}{4}}\lambda_k^{-\frac{3}{4}} \quad (\lambda_k > \lambda),$$

where A denotes a constant and may be different in each occasion (see [4; p. 202]).

Divide the summation $w_\lambda^{\delta,R}(f,x)$ into three sums; $\sqrt{\lambda_k} < \sqrt{\lambda}-1$, $|\sqrt{\lambda_k}-\sqrt{\lambda}| \leqq 1$ and $\sqrt{\lambda}+1 < \sqrt{\lambda_k}$, and denote by Σ_1, Σ_2, Σ_3 the corresponding terms respectively.

In Σ_2 we have $|I_k^\lambda| < A\lambda^{-\frac{1}{2}}$. By Schwarz' inequality

$$|\Sigma_2| \leqq A\sqrt{\lambda}^{-\delta} \sum_{|\sqrt{\lambda_k}-\sqrt{\lambda}|\leqq 1} |f_k u_k(x)|$$

$$\leqq A\sqrt{\lambda}^{-} \left(\sum |f_k|^2\right)^{1/2}\left(\sum |u_k(x)|^2\right)^{1/2}.$$

But

$$\sum_{|\sqrt{\lambda_k}-M|\leqq 1} |u_k(x)|^2 = O(M^{n-1})$$

uniformly on every compact set (see [1]). Thus

$$\Sigma_2 = o(\sqrt{\lambda}^{\,\alpha-\delta}).$$

For Σ_1 we have

$$|\Sigma_1| \leqq A\sqrt{\lambda}^{\,\alpha-\delta-1} \sum_{\sqrt{\lambda_k}<\sqrt{\lambda}-1} \left(\frac{1}{\lambda_k^{\frac{n}{4}-\frac{3}{4}}(\sqrt{\lambda}-\sqrt{\lambda_k})} + \frac{1}{\lambda_k^{\frac{n}{4}-\frac{1}{4}}} \right) |f_k u_k(x)| .$$

The first term on the right hand side is dominated by

$$A\sqrt{\lambda}^{\,\alpha-\delta-1} + A\sqrt{\lambda}^{\,\alpha-\delta-1}\left(\sum_{k>N}|f_k|^2\right)^{1/2}\left(\sum_{\sqrt{\lambda_k}<\sqrt{\lambda}-1}\frac{|u_k(x)|^2}{\lambda_k^{\frac{n}{2}-\frac{3}{2}}(\sqrt{\lambda}-\sqrt{\lambda_k})^2}\right)^{1/2}$$

$$= o(\sqrt{\lambda}^{\,\alpha-\delta}) + \sqrt{\lambda}^{\,\alpha-\delta-1}\varepsilon_N \sum_{1\leqq M\leqq\sqrt{\lambda}-1}\left(\frac{1}{M^{n-3}(\sqrt{\lambda}-M)^2}\sum_{|\sqrt{\lambda_k}-M|\leqq 1}|u_k(x)|^2\right)^{1/2}$$

$$= o(\sqrt{\lambda}^{\,\alpha-\delta}) + \sqrt{\lambda}^{\,\alpha-\delta}\varepsilon_N',$$

where N is an arbitrarily fixed number and ε_N, $\varepsilon_N \to 0$ as $N\to\infty$. The second term is $o(\sqrt{\lambda}^{\,\alpha-\delta})$ in the similar way. Hence $\Sigma_1 = o(\sqrt{\lambda}^{\,\alpha-\delta})$.

Σ_3 is bounded by

$$|\Sigma_3| \leqq A\sqrt{\lambda}^{\,\frac{n}{2}-\delta} \sum_{\sqrt{\lambda_k}>\sqrt{\lambda}+1}\left(\frac{\lambda^{\frac{1}{4}}}{\lambda_k^{\frac{n}{4}+\frac{1}{4}}(\sqrt{\lambda_k}-\sqrt{\lambda})} + \frac{\lambda^{-\frac{1}{4}}}{\lambda_k^{\frac{n}{4}+\frac{1}{4}}}\right)|f_k u_k(x)| .$$

The first term on the right hand side is dominated by

$$A\sqrt{\lambda}^{\,\frac{n}{2}-\delta+\frac{1}{2}}\left(\sum|f_k|^2\right)^{1/2}\left(\sum\frac{|u_k(x)|^2}{\lambda_k^{\frac{n+1}{2}}(\sqrt{\lambda_k}-\sqrt{\lambda})^2}\right)^{1/2},$$

which is $o(\sqrt{\lambda}^{\,\alpha-\delta})$ by the same way as in the Σ_1 case. The order of the second term is rather easily estimated and $o(\sqrt{\lambda}^{\,\alpha-\delta})$. Thus the lemma is proved.

3. PROOF of (i). First assume that f is harmonic in D. Let K be a compact set contained in D. If $x\in K$ and $0<R<\mathrm{dis}(K,\Omega^c)$, then

$$v_\lambda^{\delta,R}(f,x) = c\sqrt{\lambda}^{\,\frac{n}{2}-\delta}\int_0^R J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)\, r^{\frac{n}{2}-\delta-1}\left(\int_{|\omega|=1} f(x-r\omega)d\omega\right)dr$$

$$=c\sqrt{\lambda}^{\frac{n}{2}-\delta}\omega_n f(x)\int_0^R J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)\, r^{\frac{n}{2}-\delta-1}dr,$$

where $c=2^\delta\Gamma(\delta+1)/\sqrt{2\pi}^n$ and ω_n is the surface area $\sqrt{2\pi}^n/\Gamma(n/2)$ of the unit ball in $\boldsymbol{R}^n$. If $\delta>\alpha-1$,

$$c\omega_n\sqrt{\lambda}^{\frac{n}{2}-\delta}\int_0^\infty J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)\, r^{\frac{n}{2}-\delta-1}dr=1.$$

Thus

$$v_\lambda^{\delta,R}(f,x)-f(x)=-c\ \omega_n\sqrt{\lambda}^{\frac{n}{2}-\delta}f(x)\int_R^\infty J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)\, r^{\frac{n}{2}-\delta-1}dr.$$

By the asymptotic formula (see [5; p.199])

$$J_\nu(s)=\left(\frac{2}{\pi s}\right)^{1/2}\cos\left[s-(2\nu+1)\frac{\pi}{4}\right]+O\left(\frac{1}{s^{3/2}}\right),$$

$$v_\lambda^{\delta,R}(f,x)-f(x)=f(x)O(\sqrt{\lambda}^{\alpha-\delta-1})$$

as $\lambda\to\infty$. Therefore $\|s_\lambda^\delta(f)-f\|_K=o(\sqrt{\lambda}^{\alpha-\delta})$ for $\delta>\alpha-1$.

Next we assume that $\|s_\lambda^\delta(f)-f\|_K=o(1/\lambda)$ for a compact set K in D. Let φ be an infinitely differentiable function whose support is contained in K. Then the integral

$$\int_\Omega \lambda[s_\lambda^\delta(f,x)-f(x)]\ \varphi(x)dx$$

tends to zero. But the last integral equals

$$\sum_{k=1}^\infty\frac{\lambda}{\lambda_k}\left[\left(1-\frac{\lambda_k}{\lambda}\right)^{+\delta}-1\right]f_k\lambda_k\varphi_k$$

where $(1-t)^+=\max(1-t,0)$. Since $\Sigma\lambda_k|f_k\varphi_k|<\infty$ and the function $[(1-t)^{+\delta}-1]/t$ is bounded, the above sum tends to

$$-\delta\sum_{k=1}^\infty f_k\lambda_k\varphi_k=-\delta\int_\Omega f(x)\Delta\varphi(x)dx=0.$$

By the arbitrariness of φ, we conclude that f is almost everywhere equal to a harmonic function in K. Thus f is harmonic in K or more strongly in D.

4. To treat (ii) we shall use the following lemma. We give a proof of it passing the Fourier transformation.

LEMMA 2. *Let f be a function in $L^2(\boldsymbol{R}^n)$ and D be an open domain in $\boldsymbol{R}^n$. Suppose that f is regulated in D and that the Laplacian Δf of f in the sense of distribution is bounded in D. If the sphere of radius r with the center at x is contained in D, then we have*

$$\frac{1}{\omega_n}\int_{|\omega|=1} f(x-r\omega)d\omega - f(x) = 2^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)\int^{r} ds\int_{|y|\leq s}\Delta f(x-y)s^{-n+1}dy\,.$$

PROOF. We may assume that f is infinitely differentiable and rapidly decreasing approximating f by such functions. Then the interchanges of integrations in the following calculations are legitimate.

Let $\hat{f}(\xi)$ be the Fourier transform of f, i. e.,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n}\int_{R^n} f(x)e^{-i\xi x}dx\,.$$

By Fourier inversion formula

$$f(x-r\omega) - f(x) = \frac{1}{\sqrt{2\pi}^n}\int_{R^n}\hat{f}(\xi)[e^{-ir\omega\xi}-1]e^{i\xi x}d\xi\,.$$

Integrating on the unit sphere we get

$$\frac{1}{\omega_n}\int_{|\omega|=1} f(x-r\omega)d\omega - f(x) = \frac{1}{\sqrt{2\pi}^n}\int_{R^n}\hat{f}(\xi)\left[\frac{1}{\omega_n}\int_{|\omega|=1} e^{-ir\omega\xi}d\omega - 1\right]e^{i\xi x}d\xi$$

$$= \frac{1}{\sqrt{2\pi}^n}\int_{R^n}\hat{f}(\xi)\left[2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}(r|\xi|)}{(r|\xi|)^{\frac{n-2}{2}}} - 1\right]e^{i\xi x}d\xi\,.$$

Now by the Lommel's formula ([5 ; p.45])

$$\int^{r} \frac{J_{\nu+1}(\mu s)}{s^{\nu}} ds = \frac{-1}{\mu} \frac{J_{\nu}(\mu r)}{r^{\nu}}$$

we have

$$2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{1}{r^2}\int_0^r \frac{J_{\frac{n}{2}}(s|\xi|)}{(s|\xi|)^{\frac{n}{2}}} s ds = -\frac{1}{(r|\xi|)^2}\left[2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}(r|\xi|)}{(r|\xi|)^{\frac{n-2}{2}}} - 1\right].$$

Its Fourier transform is, by Bochner's formula,

$$2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{1}{r^2}\int_0^r s\, ds \frac{1}{\sqrt{2\pi}^n}\int_{R^n}\frac{J_{\frac{n}{2}}(s|\xi|)}{(s|\xi|)^{\frac{n}{2}}} e^{i\xi x} d\xi$$

$$= 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{1}{r^2}\int_0^r \chi_s(x) s^{-n+1} ds,$$

where $\chi_s(x)$ is the characteristic function of the ball $\{x : |x| \leqq s\}$.

Since $\frac{1}{\sqrt{2\pi}^n}\int_{R^n}|\xi|^2\hat{f}(\xi)e^{i\xi x}d\xi = -\Delta f(x)$, by the convolution relation

$$\frac{1}{\omega_n}\int_{|\omega|=1} f(x-r\omega)d\omega - f(x)$$

$$= 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\int_0^r\left(\int_{R^n}\Delta f(x-y)\chi_s(y)dy\right)s^{-n+1}ds.$$

If the sphere of radius r with the center at x is contained in D, then the last term is dominated in absolute value by

$$r^2\|\Delta f\|_D\frac{2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)}{\sqrt{2\pi}^n}\frac{1}{r^2}\int_0^r\left(\int_{R^n}\chi_s(y)dy\right)s^{-n+1}ds = \frac{1}{2n}r^2\|\Delta f\|_D.$$

PROOF of (ii). Suppose $f \in L^2(\Omega)$ and Δf in the sense of distribution is bounded in a compact set K of Ω. We prove that

$$\|s_\lambda^\delta(f) - f\|_{K'} = O(1/\lambda)$$

for a closed subset K' strictly contained in K.

By Lemma 1 it suffices to see that

$$\|v_\lambda^{\delta,R}(f)-f\|_{K'}\leqq A/\lambda$$

for $\lambda>1$. To prove this inequality we chose R so small that $0<R<\mathrm{dis}(K', K^c)$. By the similar way to the case (i) we have

$$\begin{aligned}
&v_\lambda^{\delta,R}(f,x)-f(x)\\
&=\frac{2^\delta\Gamma(\delta+1)}{\sqrt{2\pi}^n}\omega_n\sqrt{\lambda}^{-\frac{n}{2}-\delta}\int_0^R J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)r^{\frac{n}{2}-\delta-1}\left[\frac{1}{\omega_n}\int_{|\omega|=1}f(x-r\omega)d\omega-f(x)\right]dr\\
&\quad-\frac{2^\delta\Gamma(\delta+1)}{\sqrt{2\pi}^n}\omega_n\sqrt{\lambda}^{-\frac{n}{2}-\delta}f(x)\int_R^\infty J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)r^{\frac{n}{2}-\delta-1}dr.
\end{aligned}$$

Since Δf is bounded in K, so is f in K'. Thus the second term on the right hand side is $O(\sqrt{\lambda}^{\alpha-\delta-1})$ by the same method as in (i).

The first term is, up to a constant multiple, equal to

$$\begin{aligned}
&\sqrt{\lambda}^{-\frac{n}{2}-\delta}\int_0^R J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)r^{\frac{n}{2}-\delta-1}dr\int_0^r s^{-n+1}ds\int_{R^n}\Delta f(x-y)\chi_s(y)dy\\
&=\sqrt{\lambda}^{-\frac{n}{2}-\delta}\left(\int_0^{1/\sqrt{\lambda}}+\int_{1/\sqrt{\lambda}}^R\right)s^{-n+1}ds\int_{R^n}\Delta f(x-y)\chi_s(y)dy\int_s^R J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)r^{\frac{n}{2}-\delta-1}dr.\ (*)
\end{aligned}$$

Changing a variable we get

$$\int_s^R J_{\frac{n}{2}+\delta}(\sqrt{\lambda}r)r^{\frac{n}{2}-\delta-1}dr=\left(\frac{1}{\sqrt{\lambda}}\right)^{\frac{n}{2}-\delta}\int_{s\sqrt{\lambda}}^{R\sqrt{\lambda}}J_{\frac{n}{2}+\delta}(t)t^{\frac{n}{2}-\delta-1}dt.$$

By the asymptotic formula before-mentioned the last term is $O(\sqrt{\lambda}^{-\frac{3}{2}}s^{\alpha-\delta-1})$ if $\delta>\alpha-1$ and $s\sqrt{\lambda}>1$. Since $J_\mu(t)=O(t^\mu)$ as $t\to 0$, it is also $O(\sqrt{\lambda}^{\delta-\frac{n}{2}})$ if $\delta>\alpha-1$ and $s\sqrt{\lambda}\leqq 1$. Thus (*) is dominated in K' by

$$A\sqrt{\lambda}^{-\frac{n}{2}-\delta}\|\Delta f\|_K\left(\int_0^{1/\sqrt{\lambda}}s^{-n+1}s^n\sqrt{\lambda}^{\delta-\frac{n}{2}}ds+\int_{1/\sqrt{\lambda}}^R s^{-n+1}s^n s^{\alpha-\delta-1}\sqrt{\lambda}^{-\frac{3}{2}}ds\right)$$

which is not greater than $A\|\Delta f\|_K\sqrt{\lambda}^{\alpha-\delta-1}$ if $\alpha+1>\delta>\alpha-1$, $A\|\Delta f\|_K\lambda^{-1}\log\lambda$ if $\delta=\alpha+1$ and $A\|\Delta f\|_K\lambda^{-1}$ if $\delta>\alpha+1$ respectively.

Next we assume $\|s_\lambda^\delta(f)-f\|_K=O(1/\lambda)$. For an infinitely differentiable function φ whose support is contained in K, we have

$$-\delta \sum_{k=1}^{\infty} \lambda_k f_k \varphi_k \leqq A \|\varphi\|_{L^1(K)},$$

which is proved similarly to the case (i). Thus

$$| < f, \Delta\varphi > | = | < \Delta f, \varphi > | \leqq A\|\varphi\|_{L^1(K)}.$$

Therefore Δf is (essentially) bounded in K.

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