Tôhoku Math. Journ.
22(1970), 212-219.

SOME HYPERSURFACES OF A SPHERE

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(Received Nov. 26, 1969)

1. Introduction. K.Nomizu [2] studied the effect of the condition

(*) $$R(X, Y)\cdot R = 0 \text{ for any tangent vectors } X \text{ and } Y$$

for hypersurfaces M^m of the Euclidean space E^{m+1}, where R denotes the Riemannian curvature tensor and $R(X,Y)$ operates on the tensor algebra at each point as a derivation. P.J.Ryan [4] treated the same condition for hypersurfaces of spaces of non-zero constant curvature. On the other hand, one of the authors [6] discussed the effect of the condition

(**) $$R(X, Y)\cdot R_1 = 0 \text{ for any tangent vectors } X \text{ and } Y$$

for hypersurfaces of the Euclidean space, where R_1 denotes the Ricci curvature tensor.

The condition (*) implies the condition (**).

Recently, P.J.Ryan informed one of the authors that the conditions (*) and (**) are equivalent if the ambient space is of non-zero constant curvature.

In this note we prove

THEOREM. *Let M^m, $m \geqq 4$, be an m-dimensional connected and complete Riemannian manifold which is isometrically immersed in a sphere $S^{m+1}(\bar{c})$ of curvature $\bar{c}$. Then M^m satisfies the condition (**), if and only if M^m is one of the following spaces:*

(i) $M^m = S^m(\bar{c})$; *great sphere*,

(ii) $M^m = S^m(c)$; *small sphere, where* $c > \bar{c}$,

(iii) $M^m = S^p(c_1) \times S^{m-p}(c_2)$, *where* p, $m-p \geqq 2$ *and* $c_1 > \bar{c}$, $c_2 > \bar{c}$ *such that* $c_1^{-1} + c_2^{-1} = \bar{c}^{-1}$,

(iv) $M^m = M^1 \times S^{m-1}(c)$, *where* $c > \tilde{c}$ *and* M^1 *is a covering space* $(E^1/(2\pi rz)$ *for an integer* $z)$ *of a circle of radius* $r = (\tilde{c}^{-1} - c^{-1})^{-1/2}$.

If M^m has the parallel Ricci tensor, then (**) is satisfied. Conversely, if a certain hypersurface M^m in $S^{m+1}(\tilde{c})$ has property (**), then the theorem says that the Ricci tensor is parallel (precisely, M^m is (locally) symmetric).

2. Reduction of the condition (**). Let M be an m-dimensional connected Riemannian manifold which is isometrically immersed in an $(m+1)$-dimensional Riemannian manifold of constant curvature $\tilde{c} \neq 0$, and let g be the Riemannian metric of M. Then the equation of Gauss is

$$R(X, Y) = \tilde{c}X \wedge Y + AX \wedge AY, \tag{2.1}$$

where, in general, $X \wedge Y$ denotes the endomorphism which maps Z upon $g(Z, Y)X - g(Z, X)Y$. The type number $t(x)$ is, by definition, the rank of the second fundamentel form operator A at a point x of M. For a point x of M, take an orthonormal basis $\{e_1, \cdots, e_m\}$ of the tangent space M_x at x such that $Ae_a = \lambda_a e_a$, $a = 1, \cdots, m$, where λ_a's are eigenvalues of A at x. Then (2.1) is equivalent to

$$R(e_a, e_b) = (\tilde{c} + \lambda_a\lambda_b)\, e_a \wedge e_b, \tag{2.2}$$

and the condition (**) is equivalent to

$$(\tilde{c} + \lambda_a\lambda_b)(R_{aa} - R_{bb}) = 0, \tag{2.3}$$

where R_{ab} are the components of the Ricci tensor R_1 with respect to the basis. Taking account of (2.2), we get

$$R_{ab} = (m-1)\,\tilde{c}\delta_{ab} + \lambda_a\delta_{ab}\theta - \lambda_a^2\delta_{ab}, \tag{2.4}$$

where $\theta = \text{trace } A = \Sigma_a \lambda_a$. In particular, we have

$$R_{aa} = (m-1)\tilde{c} + \theta\lambda_a - \lambda_a^2. \tag{2.5}$$

Thus (2.3) becomes

$$(\tilde{c} + \lambda_a\lambda_b)(\lambda_a - \lambda_b)(\theta - \lambda_a - \lambda_b) = 0. \tag{2.6}$$

Now, suppose $\lambda_1, \lambda_2, \cdots, \lambda_r \neq 0$ and $\lambda_{r+1} = \cdots = \lambda_m = 0$ at x of M, and suppose $1 \leqq r \leqq m-1$. Then (2.6) for $b = m$ implies $\tilde{c}\lambda_a(\theta - \lambda_a) = 0$ and hence

$\theta-\lambda_a=0$ for $a=1,\cdots,r$. Thus we have $(r-1)\theta=0$. If $\theta=0$, then $\theta-\lambda_a=0$ implies $\lambda_a=0$. Hence we have $r=1$. Thus

LEMMA 1. *Let M be an m-dimensional connected Riemannian manifold which is isometrically immersed in an $(m+1)$-dimensional Riemannian manifold $\widetilde{M}$ of constant curvature $\tilde{c}\neq 0$ and satisfies the condition (**). Then the type number $t(x)\leqq 1$ or $t(x)=m$ at each point x of M.*

Suppose there are three distinct principal curvatures, say λ_1, λ_2 and λ_3, at a point. Then (2.6) implies

$$\tilde{c}+\lambda_a\lambda_b=0 \text{ or } \theta=\lambda_a+\lambda_b \quad \text{for } (a,b)=(1,2),\ (1,3),\ (2,3).$$

But these three conditions do not hold simultaneously. Hence there are at most two distinct principal curvatures at each point. We put $\lambda=\min\{\lambda_a\}$ and $\mu=\max\{\lambda_a\}$ at each point. λ and μ are locally defined functions with respect to unit normal vector fields. $\lambda\mu$ is globally defined. Now let

$$U=\{x\in M;\ t(x)=m\},$$

and let U_0 be a component of U. Then U_0 is open. Let

$$V=\{x\in U_0;\ \tilde{c}+\lambda\mu\neq 0\},$$

and let V_0 be a component of V. Then V_0 is open. Suppose U_0 and V_0 are non-empty. Then (2.3) and (2.4) imply that V_0 is an Einstein hypersurface of $\widetilde{M}$. On the other hand, we have

LEMMA 2. (A.Fialkow[1]) *Let M^m $(m\geqq 3)$ be an Einstein hypersurface $(R_1=Kg)$ of a Riemannian manifold of constant curvature $\tilde{c}$. Then we have*

(i) *if $K>(m-1)\tilde{c}$, then M^m is totally umbilic, and of constant curvature,*

(ii) *if $K=(m-1)\tilde{c}$, then $t(X)\leqq 1$ on M^m,*

(iii) *if $K<(m-1)\tilde{c}$, then there are exactly two distinct and constant principal curvatures ν and ρ, of multiplicity $\geqq 2$, satisfying $\tilde{c}+\nu\rho=0$.*

Therefore, in our case, if $m\geqq 3$, V_0 is totally umbilic and of constant curvature. Hence $\lambda=\mu$ is constant on V_0 and on the closure of V_0. Consequently, we get $V_0=U_0=M$. Thus, we have

LEMMA 3. *Let M and $\tilde{M}$ be as in Lemma 1. If $m \geqq 3$, and if $\tilde{c} + \lambda\mu \neq 0$ at x_0 where $t(x_0) = m$, then $\tilde{c} + \lambda\mu \neq 0$ and $t(x) = m$ hold on M and M is totally umbilic ($\lambda = \mu$).*

By Lemma 3, if $U \neq \emptyset$ and if $V = \emptyset$, then $\tilde{c} + \lambda\mu = 0$ on U and hence on the closure $\bar{U}$ of U. Since $\tilde{c} \neq 0$ and $t(x) \leqq 1$ imply $\tilde{c} + \lambda\mu \neq 0$, $\tilde{c} + \lambda\mu = 0$ on $\bar{U}$ implies $t(x) = m$ on $\bar{U}$. Thus we get $U = M$ and we have

LEMMA 4. *Let M and $\tilde{M}$ be as in Lemma 1. If $m \geqq 3$ and if $\tilde{c} + \lambda\mu = 0$ at x_0 where $t(x_0) = m$, then $\tilde{c} + \lambda\mu = 0$ and $t(x) = m$ hold on M.*

Combining Lemmas 1, 3, and 4, we get

LEMMA 5. *Let M and $\tilde{M}$ be as in Lemma 1. If $m \geqq 3$, then we have one of the followings:*

(a) $t(x) \leqq 1$ *on* M,

(b) $t(x) = m$ *and* $\tilde{c} + \lambda\mu \neq 0$ *on* M.

(c) $t(x) = m$ *and* $\tilde{c} + \lambda\mu = 0$ *on* M.

3. Local theorems.

THEOREM 1. *Let M be an m-dimensional connected Riemannian manifold which is isometrically immersed in an $(m+1)$-dimensional Riemannian manifold $\tilde{M}$ of constant curvature $\tilde{c}$, where $m \geqq 3$ and $\tilde{c} > 0$. If M satisfies the condition (**), then we have one of the followings:*

(i) $t(x) \leqq 1$ *on M and hence M is of constant curvature $\tilde{c}$,*

(ii) *M is totally umbilic and of constant curvature $> \tilde{c}$,*

(iii) *M is locally a product of two spaces of constant curvature $> \tilde{c}$ and of dimension $\geqq 2$,*

(iv) *M is locally a product of E^1 and an $(m-1)$-dimensional space of constant curvature $> \tilde{c}$,*

(v) *M is a manifold such that the Ricci tensor has two eigenvalues 0 and γ of multiplicity 1 and $m-1$, respectively, where γ is a non-constant positive function.*

PROOF. Lemma 5 says that we have either $t(x)\leqq 1$ on M or $t(x)=m$ on M. If $t(x)\leqq 1$ on M, then (i) holds. In the following we assume $t(x)=m$ on M. If $\tilde{c}+\lambda\mu\neq 0$ on M, then Lemma 3 says that M is of type (ii). If $\tilde{c}+\lambda\mu=0$ on M, then we have $\lambda\mu<0$, since $\tilde{c}>0$. And we have $\lambda<0<\mu$ on M. Thus the multiplicities of λ and μ are constant. If the multiplicities of λ and μ are not smaller than 2, then λ and μ are constant, as is well known (cf. Prop. 2. 3, [4]), and this is of type (iii). Suppose the multiplicity of λ or μ is 1. If λ or μ is constant, then the rest is also constant and this is of type (iv). If λ or μ is not constant, then the rest is neither constant. If, for example, the multiplicity of λ is 1, then (2.5) implies

$$\begin{aligned} R_{11} &= (m-1)\tilde{c}+\lambda\theta-\lambda^2 \\ &= (m-1)(\tilde{c}+\lambda\mu)=0, \\ R_{ii} &= (m-1)\tilde{c}+\mu\theta-\mu^2 \\ &= (m-2)(\tilde{c}+\mu^2), \end{aligned}$$

where $Ae_1=\lambda e_1$ and $Ae_i=\mu e_i$, $i=2,\cdots,m$. This is of type (v).

THEOREM 2. *Let M be an m-dimensional connected Riemannian manifold which is isometrically immersed in an $(m+1)$-dimensional Riemannian manifold of constant curvature $\tilde{c}$, where $m\geqq 3$ and $\tilde{c}<0$. If M satisfies the condition (**), then we have one of the followings*:

(i) *$t(x)\leqq 1$ on M and M is of constant curvature $\tilde{c}$,*

(ii) *M is totally umbilic and of constant curvature $>\tilde{c}$,*

(iii) *M is locally a product of two spaces of constant curvature $>\tilde{c}$ and of dimension $\geqq 2$,*

(iv) *M is locally a product of E^1 and an $(m-1)$-dimensional space of constant curvture $>\tilde{c}$,*

(v) *M is a manifold such that the Ricci tensor has at most two distinct eigenvalues at each point. They are not constant and if there are two distinct eigenvalues at a point, then one of them is 0 with multiplicity 1.*

PROOF. For (i), (ii), the proof is the same as that of (i), (ii) of Theorem 1.

So, in the following, we assume $t(x)=m$ and $\tilde{c}+\lambda\mu=0$ on M. If $\lambda<\mu$ at a point and if the multiplicities of λ and μ are not smaller than 2 at the point, then λ and μ are constant on M and this is of type (iii). If one of the principal curvatures is simple and if λ or μ is constant, then the rest is also constant and this is of type (iv). The remaining possibilities are (a) λ or μ is simple at some point and λ and μ are not constant, and (b) $\lambda=\mu$ on M. The case (a) implies the type (v) as in Theorem 1, and the case (b) implies the type (ii).

4. Conullity operator. We apply A.Rosenthal's method [5]. Let $F(M)$, θ^a, $w_b{}^a$ be the frame bundle, solder forms, and connexion forms. We denote by N_x and C_x the nullity space at x and the conullity space at x:

$$N_x=\{X\in M_x;\ R(A,B)X=0 \text{ for any } A,B\in M_x\},$$

$$C_x=\{Y\in M_x;\ g(X,Y)=0 \text{ for any } X\in N_x\}.$$

Assume $\dim N_x=1$ on an open set U. An orthonormal frame $(e_1,\cdots,e_m)$ at x is called an adapted frame if $e_1\in N_x$ and $e_i\in C_x$ $(i=2,\cdots,m)$. Let $F_0(U)$ be the set of adapted frames over U. We denote θ^a, $w_b{}^a$ restricted on $F_0(U)$ by the same letters. Then

$$w_i{}^1=A^1_{i1}\,\theta^1+B^1_{ij}\,\theta^j,$$

$$w_1{}^i=A^i_{11}\,\theta^1+B^i_{1j}\theta^j,$$

where $i,j\in(2,\cdots,m)$. The conullity operator $T=T_{e_1}:C_x\to C_x$, for $e_1\in N_x$ is defined by $Te_i=B^j_{1i}e_j$. Then we have the followings (Theorem 2.3, Cor.2.4, Theorem 3.1, [5]):

LEMMA 6. (A) $A^j_{11}=-A^1_{j1}=0$ (*the nullity varieties are totally geodesic*).

(B) *If* $\dim N_x\leqq m-3$ *on* U, *then* T *satisfies*

$$R(X,Y)(TZ)+R(Y,Z)(TX)+R(Z,X)(TY)=0 \text{ for } X,Y,Z\in C_x.$$

(C) *If M is complete, then the real eigenvalues of T vanish.*

5. Proof of the main theorem. First we show

LEMMA 7. *In Theorem* 1, *if M is complete and $m\geqq 4$, then the case* (v) *does not occur.*

In Theorem 2, if M is complete, $m \geqq 4$, and the scalar curvature S is positive or negative on M, the case (v) *does not occur.*

PROOF. Let M be a manifold stated in (v). Assume that the multiplicity of λ is 1 and $Ae_1 = \lambda e_1$, $Ae_j = \mu e_j$ $(j = 2, \cdots, m)$. Since $\tilde{c} + \lambda\mu = 0$, by (2.2) we have $R(e_1, e_j)e_1 = 0$. Again by (2.2) we have $R(e_j, e_k)e_1 = 0$. Hence we have $R(X, Y)e_1 = 0$ for any tangent vectors X and Y. Furthermore, we have

$$R(e_j, e_k) = (\tilde{c} + \mu^2)\ e_j \wedge e_k \qquad 2 \leqq j,\ k \leqq m. \tag{5.1}$$

If $\tilde{c} > 0$, then $\tilde{c} + \mu^2 \neq 0$ on M. On the other hand, by (2.5) the scalar curvature S is given by

$$S = \Sigma R_{aa} = (m-1)(m-2)(\tilde{c} + \mu^2),$$

and so $S > 0$ or $S < 0$ implies $\tilde{c} + \mu^2 \neq 0$. Thus M has constant nullity, and by Lemma 6 (B) we have

$$R(e_j, e_k)(Te_i) + R(e_k, e_i)(Te_j) + R(e_i, e_j)(Te_k) = 0.$$

If we put $B^j_{1i} = B_i{}^j$, then $Te_i = B_i{}^h e_h$ and we have

$$(B_i{}^k e_j - B_i{}^j e_k) + (B_j{}^i e_k - B_j{}^k e_i) + (B_k{}^j e_i - B_k{}^i e_j) = 0.$$

Thus we have $B_i{}^k = B_k{}^i$, and T is symmetric. Consequently, all eigenvalues are real. By Lemma 6 (C) we have $T = 0$. $T = 0$ $(B^i_{1j} = -B^1_{ij} = 0)$ together with Lemma 6 (A) implies $w_i{}^1 = -w_1{}^i = 0$. That is locally a product space $E^1 \times M^{m-1} (m-1 \geqq 3)$. By (5.1) M^{m-1} is of constant curvature $\tilde{c} + \mu^2$. In particular, λ and μ are constant on M. This is a contradiction and the case (v) does not occur.

For (i) of the main theorem, we need the following lemma:

LEMMA 8. (B. O'Neill and E. Stiel [3]) *An m-dimensional complete Riemannian manifold of constant curvature $\tilde{c} > 0$ which is isometrically immersed in an $(m+1)$-dimensional Riemannian manifold of constant curvature $\tilde{c}$ is totally geodesic.*

Now (i) follows from Theorem 1 and Lemma 8.
For (ii), (iii) and (iv), we need the following:

LEMMA 9. (P.J.Ryan [4]) *Let f and $\bar{f}$ be isometric immersions of an*

m-dimensional connected Riemannian manifold $^*M$, into an $(m+1)$-dimensional simply connected real space form $\widetilde{M}$. If $t(x)>3$ at each point of $^*M$, then there is an isometry Φ of $\widetilde{M}$ such that $\Phi\cdot f=\bar{f}$.

Let $^*M$ be the universal covering manifold of M ($\pi: {}^*M\rightarrow M$) and let $\widetilde{M}=S^{m+1}(\tilde{c})$. Then for $\varphi: M\rightarrow\widetilde{M}$, we have $f=\varphi\cdot\pi: {}^*M\rightarrow\widetilde{M}$. On the other hand, we have the standard immersions $\bar{f}$ of $S^m(c)$, $S^p(c_1)\times S^{m-p}(c_2)\,(c_1^{-1}+c_2^{-1}=\tilde{c}^{-1})$, and $E^1\times S^{m-1}(c)$ into $S^{m+1}(\tilde{c})$. Thus, we have (ii), (iii) and (iv) from Lemma 9 (f, $\bar{f}$; congruent) and Theorem 1.

REMARK.

(1) This theorem is a generalization of Theorem 4.10 of P.J.Ryan [4].

(2) If $m=3$ and the scalar curvature S is constant, then we have the similar results (i), (ii) and (iv).

(3) $\lambda_a=\lambda$ or μ and the discussion in § 1 imply that condition (**) is equivalent to (*). (In fact, recall that (*) is equivalent to $(\lambda_a\lambda_b+\tilde{c})(\lambda_a-\lambda_b)\lambda_c=0$ for distinct a, b, c, [4]).

REFERENCES

[1] A. FIALKOW, Hypersurfaces of a space of constant curvature, Ann. of Math., 39(1938), 762-785.
[2] K. NOMIZU, On hypersurfaces satisfying a certain condition on the curvature tensor, Tôhoku Math. J., 20(1968), 46-59.
[3] B. O'NEILL AND E. STIEL, Isometric immersions of constant curvature manifolds, Michigan Math J., 10(1963), 335-339.
[4] P. J. RYAN, Homogeneity and some curvature conditions for hypersurfaces, Tôhoku Math. J., 21(1969), 363-388.
[5] A. ROSENTHAL, Riemannian manifolds of constant nullity, Michigan Math. J., 14(1967), 469-480.
[6] S. TANNO, Hypersurfaces satisfying a certain condition on the Ricci tensor, Tôhoku Math. J., 21 (1969), 297-303.

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