Tôhoku Math. Journ.
22(1970), 184-190.

ϕ-HOLOMORPHIC SPECIAL BISECTIONAL CURVATURE

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(Received June 12, 1969)

1. Introduction. S. I. Goldberg and S. Kobayashi [2] studied holomorphic bisectional curvatures on Kählerian manifolds, and they generalized results on Kählerian manifolds with positive curvature to results on Kählerian manifolds with positive holomorphic bisectional curvature. Let M be a Kählerian manifold with complex structure J and metric G. For two holomorphic planes σ and σ' in $T_x(M)$, $x \in M$, the holomorphic bisectional curvature $H(\sigma, \sigma')$ is defined by

$$H(\sigma, \sigma') = H(X, Y) = R(X, JX, Y, JY), \tag{1.1}$$

where R is the Riemannian curvature tensor, X is a unit tangent vector in σ and Y is a unit tangent vector in σ'. Denote by $K(X, Y)$ the sectional curvature for (X, Y)-plane. If σ and σ' are perpendicular (in other words, $G(X, Y) = G(X, JY) = 0$), then we have

$$H(X, Y) = K(X, Y) + K(X, JY), \tag{1.2}$$

and we call such $H(X, Y)$ holomorphic special bisectional curvature. Then some results in [2] are valid even if we replace the condition "positive holomorphic bisectional curvature" by "positive holomorphic *special* bisectional curvature". Utilizing these results we get some corresponding results on Sasakian manifolds. All manifolds are assumed to be connected (and without boundary).

2. Two results on Kählerian manifolds. We do not restate theorems in [2], but we state essential parts of two theorems so that we can apply them to Sasakian manifolds.

PROPOSITION A. *Let N be a Kählerian manifold with positive holomorphic special bisectional curvature. If V and W are complex submanifolds of N such that*

$$\dim V + \dim W \geqq \dim N,$$

then there is no (*non-trivial*) *geodesic which is the shortest one from* V *to* W.

PROPOSITION B. *Let* N *be an Einstein Kählerian manifold with positive holomorphic special bisectional curvature. If the maximum value of the holomorphic sectional curvature is attained at some point of* N, *then* N *is of constant holomorphic sectional curvature* $k>0$.

3. ϕ-holomorphic special bisectional curvature of Sasakian manifolds and local fiberings. Let M be a Sasakian manifold with structure tensors (ϕ, ξ, η, g), the notations being the same as in [7]. For unit vectors X and Y such that $\eta(X)=\eta(Y)=0$, and $g(X, Y)=g(X, \phi Y)=0$, we define the ϕ-holomorphic special bisectional curvature $H^*(X, Y)=H^*(\sigma, \sigma')$ *by*

$$H^*(X, Y) = K(X, Y) + K(X, \phi Y), \tag{3.1}$$

where $\sigma=(X, \phi X)$-plane and $\sigma'=(Y, \phi Y)$-plane.

On the other hand, for a vector X such that $\eta(X)=0$, $H^*(\sigma)=H^*(X)=K(X, \phi X)$ is called the ϕ-holomorphic sectional curvature for $\sigma=(X, \phi X)$-plane.

Let x be an arbitrary point of M. Then we have a sufficiently small coordinate neighborhood U of x, which is cubical and flat with respect to ξ (cf. [5]). That is, U is a regular Sasakian manifold and has a fibering:

$$\pi : U \longrightarrow V = U/\xi . \tag{3.2}$$

Since U is Sasakian, V is Kählerian (cf. [3]). We denote by J and G the structure tensors on V. Then we have

$$\phi u^* = (Ju)^*, \qquad g = \pi^*G + \eta\otimes\eta , \tag{3.3}$$

where u^* on U is the horizontal lift of a vector field u on V with respect to the contact form η, which acts like an infinitesimal connection form, although U is not a principal fibre bundle. The sectional curvatures on U and V are related by

$$K(u^*, v^*) = K(u, v)\cdot\pi - 3[g(u^*, \phi v^*)]^2 \tag{3.4}$$

for every orthonormal vectors u and v on V (cf. (5.8) in [6], etc.). Assume that $H(u, v)$ is holomorphic special bisectional curvature. Then, since $g(u^*, \phi v^*)=0$, we see that the ϕ-holomorphic special bisectional curvature $H^*(u^*, v^*)$ is given by

(3.5) $$H^*(u^*, v^*) = H(u, v) \cdot \pi$$

by virtue of (3.1) and (3.4). In particular, this means that U has positive ϕ-holomorphic special bisectional curvature if and only if V has positive holomorphic special bisectional curvature.

By (3.4) the relation between the holomorphic sectional curvature $H(u)$ and the ϕ-holomorphic sectional curvature $H^*(u^*)$ is given by

(3.6) $$H^*(u^*) = H(u) \cdot \pi - 3 .$$

4. **Submanifolds of Sasakian manifolds.** A submanifold E of a Sasakian manifold M is called invariant, if ξ of M is tangent to E on E and, for any tangent vector X to E, ϕX is tangent to E. In a Sasakian case, a theorem analogous to that of T. Frankel [1] is as follows:

THEOREM 4.1. *Let M be a compact Sasakian manifold with positive ϕ-holomorphic special bisectional curvature and let E and F be compact and invariant submanifolds of M. If* $\dim E + \dim F \geqq 1 + \dim M$, *then* $\dim(E \cap F) \geqq 1$.

PROOF. Assume that $E \cap F$ is empty. Let $l = \{l(t), 0 \leqq t \leqq \alpha\}$ be one of the shortest geodesics from E to F, where t is the arc-length parameter and α is the length of l. Since the tangent vector T_0 to l at $l(0)$ is orthogonal to E and since ξ is tangent to E, T_0 and $(\xi)_{l(0)}$ are perpendicular. Because ξ is a Killing vector fields, ξ is perpendicular to the geodesic l at $l(t)$ for each t. That is, l is a horizontal geodesic in the sense that $\eta(T_t) = 0$, where $T_t = dl(t)/dt$. We cover l by open sets $U_i (i = 1, \cdots, s)$ stated in §3 such that $\pi : U_i \longrightarrow U_i/\xi$ is a fibering. Then $U = \cup U_i$ is a regular Sasakian manifold with respect to the induced structure and we have the fibering of U:

$$\pi : U = \cup U_i \longrightarrow V = (\cup U_i)/\xi .$$

V is a Kählerian manifold which contains πl. Moreover, $\pi(E \cap U)$ and $\pi(F \cap U)$ are complex submanifolds of V, since E and F are invariant in U. Since $\dim V = \dim U - 1$, we have

$$\dim \pi(E \cap U) + \dim \pi(F \cap U) = \dim E + \dim F - 2 \geqq \dim V .$$

Next we show that πl is a geodesic from $\pi(E \cap U)$ to $\pi(F \cap U)$. Denote by u a vector field on V such that u is tangent to πl at $\pi l(t)$ for each t and of unit length on πl. Generally we have

(4.1) $$\nabla^*_{X^*}Y^* = (\nabla_X Y)^* + (1/2)d\eta([X^*, Y^*])\xi$$

for any vector fields X and Y on V, where ∇ and ∇^* are the Riemannian connections defined by G and g, respectively (cf. [6]). In particular, we have $\nabla^*_{X^*}X^* = (\nabla_X X)^*$. Since $(u^*)_{l(t)}$ and T_t coincide at $l(t)$ for each t, we have $\nabla^*_{u^*}u^* = 0$ on l and $\nabla_u u = 0$ on $\pi(l)$. Consequently, πl is a geodesic on V with the same length α as l on U (cf. (3.3)). Since any curve τ in V from $\pi(E\cap U)$ to $\pi(F\cap U)$ near πl is lifted as a horizontal curve with the same length as τ near l, and since l is the shortest, πl is the shortest one. This is a contradiction to Proposition A. Therefore $E\cap F$ contains at least one trajectory of ξ.

5. The first Betti number.

THEOREM 5.1. *Let M be a compact Sasakian manifold. Assume that*

(i) *every ϕ-holomorphic special bisectional curvature $H^*(\sigma, \sigma') > 0$,*

(ii) *every ϕ-holomorphic sectional curvature $H^*(\sigma) > -3$.*

Then the first Betti number $b_1(M) = 0$.

PROOF. In the same notation as in [7], (i) implies that

$$K_{\lambda\mu} + K_{\lambda\mu^*} > 0 \quad \text{for} \quad \lambda \neq \mu$$

and (ii) implies $K_{\lambda\lambda^*} > -3$. Hence we have

$$\Sigma_\mu (K_{\lambda\mu} + K_{\lambda\mu^*}) > -3\,,$$

and Theorem 4.1 in [7] completes the proof.

6. The second Betti number. By Theorem 5.1 it is clear that the second Betti number $b_2(M)$ of a compact Sasakian manifold of 3-dimension is zero if $H^*(\sigma) > -3$.

THEOREM 6.1. *Let M be a compact Sasakian manifold. Assume that*

(i) *every ϕ-holomorphic special bisectional curvature $H^*(\sigma, \sigma') > 0$,*

(ii) *every ϕ-holomorphic sectional curvature $H^*(\sigma) > -3$.*

Then we have $b_2(M)=0$.

PROOF. Similarly as in the proof of Theorem 5.1, Theorem 6.1 follows from Theorem 5.7 and Theorem 5.10 in [7].

7. Einstein Sasakian manifolds. The proof for the next Proposition given by E. M. Moskal [4] is rather lengthy and so we give here a simple proof by reducing the discussion to the Kählerian case.

PROPOSITION 7.1. (E. M. Moskal) *Let* M *be a compact simply connected Einstein Sasakian manifold with positive curvature (more precisely, positive* ϕ*-holomorphic special bisectional curvature). Then* M *is isometric to a unit sphere.*

To prove this it is enough to show the following

PROPOSITION 7.2. *A compact Einstein Sasakian manifold with positive* ϕ*-holomorphic special bisectional curvature is of constant curvature* 1.

PROOF. Let x be a point where the maximum value of the ϕ-holomorphic sectional curvature is attained, and let $\pi: U\to V=U/\xi$ be a local fibering, U being a neighborhood of x. Then V is an Einstein Kählerian manifold ([6]) and the maximum value of the holomorphic sectional curvature of V is attained at πx by (3.6). So we can apply Proposition B, which tells us that V is of constant holomorphic sectional curvature $k>0$. By the way, the scalar curvature S of an Einstein Sasakian manifold of m-dimension is given by $S=m(m-1)$ (cf. (2.7) in [7]). By (5.12) in [6], the scalar curvature S' of V is given by

$$S'=S+m-1=m^2-1\,. \tag{7.1}$$

On the other hand, we have $S'=(n^2+n)k$ where $2n+1=m$. Then by (7.1) we have $k=4$. By (3.6) U has constant ϕ-holomorphic sectional curvature $H^*=k-3=1$. Next by (12.1) and Lemma 6.4 in [7], U is of constant curvature 1. Therefore M is of constant curvature 1.

8. η**-Einstein Sasakian manifolds.** A Sasakian manifold is called an η-Eienstein manifold, if the Ricci tensor R_1 is of the form: $R_1=ag+b\eta\otimes\eta$ for some functions a and b on M. If $m\geqq 5$, then a and b are constant. A deformation $(\phi,\xi,\eta,g)\longrightarrow({}^*\phi,{}^*\xi,{}^*\eta,{}^*g)$ such that

$${}^*g=\alpha g+(\alpha^2-\alpha)\eta\otimes\eta\,, \tag{8.1}$$

(8.2) $$*\phi = \phi, \quad *\eta = \alpha\eta, \quad *\xi = \alpha^{-1}\xi$$

for some positive constant α is called D-homothetic (cf. [7]). If (ϕ, ξ, η, g) is Sasakian then $(*\phi, *\xi, *\eta, *g)$ is also Sasakian.

PROPOSITION 8.1. *In a compact η-Einstein Sasakian manifold M of m-dimension, assume that*

(i) *every ϕ-holomorphic special bisectional curvature $H^*(\sigma, \sigma') > 0$,*

(ii) *every ϕ-holomorphic sectional curvature $H^*(\sigma) > -3$, and*

(iii) *either $m \geqq 5$, or $m = 3$ and a, b are constant.*

Then the structure is D-homothetic to a Sasakian structure with constant curvature 1.

PROOF. The scalar curvatures $*S$ and S are related by

(8.3) $$\alpha *S = S - (\alpha - 1)(m - 1)$$

(cf. [7]). For a suitable ϕ-basis the non-vanishing components of the Ricci curvature tensor are given by (cf. §4 in [7])

$$R_{00} = R_1(\xi, \xi) = m - 1,$$

$$R_{\lambda\lambda} = R_{\lambda^*\lambda^*} = 1 + \Sigma_{\mu\neq\lambda}(K_{\lambda\mu} + K_{\lambda\mu^*}) + K_{\lambda\lambda^*}.$$

Then under the assumptions (i) and (ii) we have

(8.4) $$S = R_{00} + 2\Sigma_\lambda R_{\lambda\lambda} > m - 5.$$

Solving α from (8.3) putting $*S = m(m-1)$, we have

(8.5) $$\alpha = (S + m - 1)/(m^2 - 1).$$

By (8.4) and (8.5) we have

$$\alpha > 2(m - 3)/(m^2 - 1) \geqq 0.$$

Condition (iii) implies that α defined by (8.5) is constant. Therefore, by the D-homothetic deformation for such α, $*g$ has the scalar curvature $m(m-1)$. Then $*g$ is an Einstein metric. If we notice that (i), (ii) and (iii) are invariant by a D-homothety, then Proposition 8.1 follows from Proposition 7.2.

THEOREM 8.2. *In a compact Sasakian manifold of m-dimension, assume that*

(i) *every ϕ-holomorphic special bisectional curvature $H^*(\sigma, \sigma') > 0$,*
(ii) *every ϕ-holomorphic sectional curvature $H^*(\sigma) > -3$, and*
(iii) *the scalar curvatrue is constant.*

Then the structure is D-homothetic to a Sasakian structure with constant curvature 1.

PROOF. By Theorem 6.1, the second Betti number of M is zero. Then M is an η-Einstein manifold by Corollary 5.7 in [6] and hence by Proposition 8.1, the structure is D-homothetic to a Sasakian structure with constant curvature 1.

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