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ORDER OF COMPOSITE FUNCTIONS OF INTEGRAL FUNCTIONS

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1. Let $f(z)$ be a meromorphic function in the plane $|z|<+\infty$, $T(r,f)$ Nevanlinna's characteristic function of $f(z)$ and

$$M(r,f)=\max_{|z|=r}|f(z)|.$$

We define the order μ and the lower order λ of $f(z)$ as follows:

$$\mu=\overline{\lim_{r\to\infty}}\frac{\log T(r,f)}{\log r},\qquad \lambda=\underline{\lim_{r\to\infty}}\frac{\log T(r,f)}{\log r}.$$

If $f(z)$ is an integral function, we may define the order μ and the lower order λ of $f(z)$ by using $\log M(r,f)$ instead of $T(r,f)$. About a composite function of integral functions $f(z)$ and $g(z)$, Pólya [4] proved the following theorem.

PÓLYA'S THEOREM. *If $f(z)$ and $g(z)$ are integral functions and if $g(f(z))$ is an integral function of finite order, then there are only two possible cases: either*

(a) *the internal function $f(z)$ is a polynomial and the external function $g(z)$ is of finite order; or else*

(b) *the internal function $f(z)$ is not a polynomial but a function of finite order, and the external function $g(z)$ is of zero order.*

Ozawa [3] treated this theorem in detail and Baker [1] discussed the order of an integral function $f(f(z))$ in the case when $f(z)$ is of order zero. In this paper we consider the converse problem of Pólya's theorem, that is, the problem to investigate the order of $g(f(z))$ under the condition (a) or (b). The main part will be the problem corresponding to the condition (b).

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2. About the case (a) of Pólya's theorem, it is obvious from the maximum modulus principle that if $g(z)$ is a transcendental integral function of finite order μ and if $f(z)$ is a polynomial of degree n, then $g(f(z))$ is of order μn.

3. From now on, we shall investigate the case (b). We start from the following.

LEMMA 1 (Valiron [5]). *Suppose that $f(z)$ is a transcendental integral function of finite order. Let $r = l(u)$ be the inverse function of $u = M(r, f)$. Then, given $\varepsilon > 0$, there exists a constant $A(\varepsilon)$ such that the equation $f(z) = a$ has a root in the annulus*

$$l(|a|) \leqq |z| \leqq l(|a|)^{1+\varepsilon}$$

provided that $|a| > A(\varepsilon)$.

Using this lemma we can prove the following lemma which will be used in the proof of Theorem 1.

LEMMA 2. *Suppose that $f(z)$ and $g(z)$ are integral functions of finite order and put*

$$\log M(r, g(z)) \equiv (\log r)^{\varphi(r)}. \tag{1}$$

Then, for any $\varepsilon > 0$,

$$\log \log M(r^{1+\varepsilon}, g(f(z))) \geqq \varphi(M(r, f(z))) \log \log M(r, f(z)) \tag{2}$$

and

$$\log \log M(r, g(f(z))) \leqq \varphi(M(r, f(z))) \log \log M(r, f(z)) \tag{3}$$

for all sufficiently large values of r.

PROOF. By Lemma 1, given $\varepsilon > 0$, there exists a constant $A(\varepsilon)$ such that the equation $f(z) = a$ has a root in the annulus

$$l(|a|) \leqq |z| \leqq l(|a|)^{1+\varepsilon}$$

provided that $|a| > A(\varepsilon)$. Now we choose a number r_0 such that

$$M(r_0, f) > A(\varepsilon),$$

and we take $\rho=\rho(r,f)=M(r,f)$ for any $r\geqq r_0$. Then, there exists an a_ρ such that $|a_\rho|=\rho$ and

$$\max_{|w|=\rho}|g(w)| = |g(a_\rho)|$$

and such that the equation $f(z)=a_\rho$ has a root in the annulus

$$r = (|a_\rho|)\leqq|z|\leqq(|a_\rho|)^{1+\varepsilon}=r^{1+\varepsilon}.$$

Thus, there exists a z_0 such that

$$|z_0|\leqq r^{1+\varepsilon} \quad\text{and}\quad f(z_0)=a_\rho.$$

Therefore, we have

$$M(r^{1+\varepsilon}, g(f(z)))\geqq|g(f(z_0))| = |g(a_\rho)| = M(\rho, g(w))$$

for all $r\geqq r_0$, where $\rho=M(r,f(z))$. Hence we see

$$\begin{aligned}\log M(r^{1+\varepsilon}, g(f(z)))&\geqq \log M(\rho, g(w))\\ &= (\log\rho)^{\varphi(\rho)} = \{\log M(r,f(z))\}^{\varphi(M(r,f(z)))}\end{aligned}$$

and

$$\log\log M(r^{1+\varepsilon}, g(f(z)))\geqq\varphi(M(r,f(z)))\log\log M(r,f(z))$$

for all $r\geqq r_0$.

On the other hand, by the maximum modulus principle, clearly we have

$$M(r, g(f(z)))\leqq M(M(r,f(z)), g(w)).$$

Thus we obtain

$$\log\log M(r, g(f(z)))\leqq\varphi(M(r,f(z)))\log\log M(r,f(z)).$$

This proves Lemma 2.

We note that it is possible to show a similar result to Lemma 2 by Pólya's method (by using theorems of Schottky and Bohr).

Now we can prove the following theorem.

THEOREM 1. *Suppose that $f(z)$ is an integral function of positive and* *finite order μ and that $g(z)$ is a transcendental integral function of zero order. Then $g(f(z))$ is of infinite order if $f(z)$ has the positive lower order λ and*

$$\overline{\lim_{r\to\infty}}\varphi(r) = \infty ,$$

or if $f(z)$ is of zero lower order and

$$\lim_{r\to\infty} \varphi(r) = \infty .$$

PROOF. By (2), for any $\varepsilon > 0$, we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r^{1+\varepsilon}, g(f))}{\log r^{1+\varepsilon}} \geqq \overline{\lim_{r\to\infty}} \frac{\varphi(M(r,f)) \log\log M(r,f)}{(1+\varepsilon)\log r} .$$

If $f(z)$ has the positive lower order λ and $\overline{\lim\limits_{r\to\infty}}\varphi(r) = \infty$, then taking $\varepsilon = \dfrac{\lambda}{2}$, we see

$$\log M(r,f) > r^{\lambda-\varepsilon} = r^{\lambda/2}$$

for all sufficiently large values of r. Thus

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r^{1+\varepsilon}, g(f))}{\log r^{1+\varepsilon}} \geqq \overline{\lim_{r\to\infty}} \frac{\varphi(M(r,f))\log r^{\lambda/2}}{(1+(\lambda/2))\log r}$$

$$= \frac{\lambda/2}{1+(\lambda/2)} \overline{\lim_{r\to\infty}} \varphi(M(r,f)) = \infty ,$$

since $M(r,f)$ is increasing, continuous and unbounded in r. If $f(z)$ is of finite positive order μ and of zero lower order and if

$$\lim_{r\to\infty} \varphi(r) = \infty ,$$

then for any $\varepsilon > 0$, it holds that

$$\overline{\lim_{r\to\infty}} \frac{\log\log M(r^{1+\varepsilon'}, g(f))}{\log r^{1+\varepsilon}} \geqq \overline{\lim_{r\to\infty}} \frac{\varphi(M(r,f)) \log\log M(r,f)}{\log r^{1+\varepsilon'}}$$

$$\geqq \underline{\lim_{r\to\infty}}\, \varphi(M(r,f)) \cdot \overline{\lim_{r\to\infty}} \frac{\log\log M(r,f)}{(1+\varepsilon')\log r}$$

$$= \frac{\mu}{1+\varepsilon'} \varliminf_{r\to\infty} \varphi(M(r,f)) = \infty .$$

This proves Theorem 1.

4. In this section we shall show that conditions in Theorem 1 can not be weakend, that is, even if $\varlimsup_{r\to\infty} \varphi(r) = \infty$, the function $g(f(z))$ is not always of infinite order if $f(z)$ is of zero lower order and if $\varliminf_{r\to\infty} \varphi(r)$ is finite. (In this case, $g(f(z))$ is of order greater than or equal to μ. This is easily seen from the proof of Theorem 1.)

For that purpose, we have only to give an example.

First we construct two functions $Y(R)$ and $y(r)$ defined on $(0, \infty)$ which satisfy the following conditions, respectively:

$Y(R)$ has the properties that $\varlimsup_{R\to\infty} Y(R) < +\infty$ and that $(\log R)^{Y(\log R)}$ is increasing, convex of $\log R$ and is of zero order and $y(\log r)$ is increasing, convex in $\log r$ and is of the order 1 and of the lower order zero.

(i) We put

$$Y_1(R) = 2, \qquad (0 < R \leqq R_1),$$

where R_1 is a fixed and sufficiently large value, for example, $R_1 = e^4 + 1$, and further we put

$$y_1(r) = r, \qquad (0 < r \leqq r_1)$$

where $r_1 = R_1$. Next we put

$$y_2(r) = e^r - A_2, \qquad (r_1 \leqq r \leqq r_2),$$

where A_2 is determined by the equation $y_1(r_1) = y_2(r_1)$, and r_2 is determined by the equation $y_2(r_2) = \dfrac{1}{2} e^{r_1}$, whence

$$A_2 = e^{r_1} - r_1 \quad \text{and} \quad r_2 = \log 2(e^{r_1} - r_1) > r_1 .$$

For this r_2, we take $R_2 = \dfrac{1}{2} e^{r_2}$ and put

$$Y_2(R) = 2, \qquad (R_1 \leqq R \leqq R_2).$$

We put

$$y_3(r) = e^{r_2}r - A_3 \qquad (r_2 \leqq r \leqq r_3),$$

where r_3 will be determined after R_3 is determined and A_3 is determined by the equation $y_3(r_2) = y_2(r_2)$, so

$$A_3 = e^{r_2} \cdot r_2 - \frac{1}{2} e^{r_2} = e^{r_2}\left(r_2 - \frac{1}{2}\right).$$

We choose $r_{2,1}$ such that $y_3(r_{2,1}) = (r_{2,1})^2$. Hence

$$(r_{2,1})^2 = e^{r_2} \cdot r_{2,1} - e^{r_2}\left(r_2 - \frac{1}{2}\right),$$

$$(r_{2,1})^2 - e^{r_2} \cdot r_{2,1} + e^{r_2}\left(r_2 - \frac{1}{2}\right) = 0,$$

$$r_{2,1} = \frac{e^{r_2} + \sqrt{(e^{r_2})^2 - 4e^{r_2}(r_2 - (1/2))}}{2} \ (> r_2).$$

For this $r_{2,1}$, we choose $R_{2,2}$ such that for any fixed $\varepsilon > 0$,

$$R_{2,2} = (1 + \varepsilon)R_{2,1},$$

where $R_{2,1} = y_3(r_{2,1}) = (r_{2,1})^2$. We put

$$Y_{3,1}(R) = 2, \qquad (R_2 \leqq R \leqq R_{2,1}),$$

and

$$Y_{3,2}(R) = \begin{cases} 2, & (R_{2,1} \leqq R \leqq R_{2,2}), \\ \log R - B_2, & (R_{2,2} \leqq R \leqq R_{2,3}), \end{cases}$$

where B_2 is determined by the equation

$$Y_{3,2}(R_{2,2}) = 2,$$

so $B_2 = \log R_{2,2} - 2 > 2$. Next we choose $R_{2,3}$ such that

$$Y_{3,2}(R_{2,3}) = \frac{1}{2} \log R_{2,3}.$$

Thus

$$\log R_{2,3} - \log R_{2,2} + 2 = \frac{1}{2} \log R_{2,3}$$

and

$$R_{2,3}=\left(\frac{R_{2,2}}{e^2}\right)^2>R_{2,2}\,.$$

We next determine $Y_{3,3}(R)$, $(R_{2,3}\leqq R\leqq R_3)$ such that

$$R^{Y_{3,3}(R)}=k_3R-B_3\,,\qquad (R_{2,3}\leqq R\leqq R_{3,1})\,,$$

where $k_3=\left(\frac{2\log R_{2,3}-B_2}{R_{2,3}}\right)R_{2,3}{}^{(\log R_{2,3}-B_2)}$ is the derivative of $R^{Y_{3,2}(R)}$ at the point $R_{2,3}$, and B_3 is determined by the equation

$$R_{2,3}^{Y_{3,3}(R_{2,3})}=k_3R_{2,3}-B_3=R_{2,3}{}^{(1/2)\log R_{2,3}},$$

so

$$B_3=k_3\,R_{2,3}-R_{2,3}{}^{(1/2)\log R_{2,3}}>0\,.$$

Thus

$$Y_{3,3}(R)=\frac{\log\{k_3R-B_3\}}{\log R}\,,\quad (R_{2,2}\leqq R\leqq R_{3,1})\,.$$

Here $R_{3,1}$ is determined as follows: $Y_{3,3}(R)$ is monotone decreasing in R $(\geqq R_{3,0})$ for some $R_{3,0}(\geqq R_{2,3})$ and tends to 1 as $R\to\infty$. Thus there exists an $R_{3,1}$ such that

$$Y_{3,3}(R_{3,1})=2\,.$$

For this $R_{3,1}$ we choose $R_3=(1+\varepsilon)R_{3,1}$ and put

$$Y_{3,3}(R)=2\,,\qquad (R_{3,1}\leqq R\leqq R_3)\,.$$

We choose an r_3 such that $y_3(r_3)=R_3$, so

$$e^{r_2}r_3-e^{r_2}\left(r_2-\frac{1}{2}\right)=R_3\,,$$

and

$$r_3=\frac{e^{r_2}(r_2-(1/2))+R_3}{e^{r_2}}>r_{2,1}\,.$$

Now we put

$$y_4(r) = e^r - A_4, \qquad (r_3 \leqq r \leqq r_4),$$

where A_4 and r_4 are determined by the equations $y_4(r_3)=y_3(r_3)$ and $y_4(r_4)=\frac{1}{2}e^{r_4}$, respectively, so

$$A_4 = e^{r_3} - R_3 > 0, \qquad r_4 = \log 2A_4 > r_3.$$

We take $R_4 = \frac{1}{2}e^{r_4}$ and put

$$Y_4(R) = 2, \qquad (R_3 \leqq R \leqq R_4).$$

(ii) Similarly, we define $y_{2n}(r)$, $y_{2n+1}(r)$, $Y_{2n}(R)$, $Y_{2n+1,i}(R)$ $(i=1,2,3)$, as follows.

Assume that $y_{2n-1}(r)$ has been defined. Then clearly

$$r_{2n-1} < y_{2n-1}(r_{2n-1}) < r_{2n-1}^2.$$

We put

$$y_{2n}(r) = e^r - A_{2n}, \qquad (r_{2n-1} \leqq r \leqq r_{2n}),$$

where A_{2n} and r_{2n} are determined by the equations

$$y_{2n}(r_{2n-1}) = y_{2n-1}(r_{2n-1})$$

and

$$y_{2n}(r_{2n}) = \frac{1}{2}e^{r_{2n}},$$

respectively. Hence

$$A_{2n} = e^{r_{2n-1}} - y_{2n-1}(r_{2n-1}) > 0,$$

$$\begin{aligned} r_{2n} &= \log 2A_{2n} > \log 2(e^{r_{2n-1}} - r_{2n-1}^2) \\ &> \log 2\left(\frac{1}{2}e^{r_{2n-1}}\right) = r_{2n-1}. \end{aligned}$$

Next we put

$$Y_{2n}(R) = 2, \qquad (R_{2n-1} \leqq R \leqq R_{2n}),$$

where $R_{2n} = y_{2n}(r_{2n})$, and put

$$y_{2n+1}(r) = e^{r_{2n}} \cdot r - A_{2n+1} \qquad (r_{2n} \leqq r \leqq r_{2n+1}).$$

Here r_{2n+1} will be determined after R_{2n+1} is determined and A_{2n+1} is determined by the equation

$$y_{2n+1}(r_{2n}) = y_{2n}(r_{2n}).$$

Hence it holds that

$$e^{r_{2n}} \cdot r_{2n} - A_{2n+1} = \frac{1}{2} e^{r_{2n}}$$

and

$$A_{2n+1} = e^{r_{2n}} \left(r_{2n} - \frac{1}{2} \right).$$

We choose next $r_{2n,1}$ such that $y_{2n+1}(r_{2n,1}) = (r_{2n,1})^2$ or

$$(r_{2n,1})^2 = e^{r_{2n}} r_{2n,1} - e^{r_{2n}} \left(r_{2n} - \frac{1}{2} \right),$$

$$r_{2n,1} = \frac{e^{r_{2n}} + \sqrt{(e^{r_{2n}})^2 - 4e^{r_{2n}}(r_{2n} - (1/2))}}{2} \, (> r_{2n}).$$

For this $r_{2n,1}$, we take $R_{2n,2} = (1+\varepsilon) R_{2n,1}$, where

$$R_{2n,1} = (r_{2n,1})^2 .$$

We put

$$Y_{2n+1,1}(R) = 2, \qquad (R_{2n} \leqq R \leqq R_{2n,1}),$$

and

$$Y_{2n+1,2}(R) = \begin{cases} 2, & (R_{2n,1} \leqq R \leqq R_{2n,2}) \\ \log R - B_{2n}, & (R_{2n,2} \leqq R \leqq R_{2n,3}), \end{cases}$$

where B_{2n} and $R_{2n,3}$ are determined by the equations

$$Y_{2n+1,2}(R_{2n,2}) = 2$$

and

$$Y_{2n+1,2}(R_{2n,3}) = \frac{1}{2} \log R_{2n,3},$$

respectively. Hence

$$B_{2n} = \log \ R_{2n,2} - 2 \quad \text{and} \quad R_{2n,3} = \left(\frac{R_{2n,2}}{e^2}\right)^2 > R_{2n,2}\,.$$

We next determine $Y_{2n+1,3}(R)$ in $R_{2n,3} \leqq R \leqq R_{2n+1}$ as follows. First we put

$$R^{Y_{2n+1,3}(R)} = k_{2n+1}R - B_{2n+1}, (R_{2n,3} \leqq R \leqq R_{2n+1,1}),$$

where

$$k_{2n+1} = \left(\frac{2\log R_{2n,3} - B_{2n}}{R_{2n,3}}\right) R_{2n,3}{}^{(\log R_{2n,3} - B_{2n})}$$

and B_{2n+1} is determined by the equation

$$R_{2n,3}^{Y_{2n+1,3}(R_{2n,3})} = k_{2n+1}R_{2n,3} - B_{2n+1} = R_{2n,3}^{(1/2)\log R_{2n,3}},$$

so

$$B_{2n+1} = k_{2n+1}R_{2n,3} - R_{2n,3}^{(1/2)\log R_{2n,3}} > 0\,.$$

Thus

$$Y_{2n+1,3}(R) = \frac{\log\{k_{2n+1}R - B_{2n+1}\}}{\log R}, (R_{2n,3} \leqq R \leqq R_{2n+1,1})\,.$$

Here $R_{2n+1,1}$ is uniquely determined in the following way; $Y_{2n+1,3}(R)$ is monotone decreasing in $R(\geqq R_{2n+1,0})$ for some $R_{2n+1,0}(\geqq R_{2n,3})$, and tends to 1 as $R\to\infty$. Thus there exists an $R_{2n+1,1}$ such that

$$Y_{2n+1,3}(R_{2n+1,1}) = 2\,.$$

We take $R_{2n+1} = (1+\varepsilon)R_{2n+1,1}$ and put

$$Y_{2n+1,3}(R) = 2\,, \qquad (R_{2n+1,1} \leqq R \leqq R_{2n+1})\,.$$

For this R_{2n+1}, we choose r_{2n+1} such that

$$y_{2n+1}(r_{2n+1}) = R_{2n+1}\,.$$

Hence

$$r_{2n+1} = \frac{e^{r_{2n}}(r_{2n} - (1/2)) + R_{2n+1}}{e^{r_{2n}}} > r_{2n,1} > r_{2n}\,.$$

Thus, we obtain

$$Y(R)\cdot\log y(r)$$

$$\equiv\begin{cases} Y_{2n}(R)\log y_{2n}(r)=A,\ (r_{2n-1}\leqq r\leqq r_{2n}) \text{ and } (R_{2n-1}\leqq R\leqq R_{2n}) \\ Y_{2n+1,1}(R)\log y_{2n+1}(r)=B,\ (r_{2n}\leqq r\leqq r_{2n,1}) \text{ and } (R_{2n}\leqq R\leqq R_{2n,1}) \\ Y_{2n+1,2}(R)\log\ y_{2n+1}(r)=C,\ (r_{2n,1}\leqq r\leqq r_{2n,3}) \text{ and } (R_{2n,1}\leqq R\leqq R_{2n,3}) \\ Y_{2n+1,3}(R)\log y_{2n+1}(r)=D,\ (r_{2n,3}\leqq r\leqq r_{2n+1}) \text{ and } (R_{2n,3}\leqq R\leqq R_{2n+1}). \end{cases}$$

From the above construction, we have

$$A\leqq 2\log\frac{1}{2}e^{r}<2r,$$

$$B\leqq 2\log e^{r}=2r,$$

$$C\leqq\frac{1}{2}\log y_{2n+1}(r)\cdot\log y_{2n+1}(r)=\frac{1}{2}(\log y_{2n+1}(r))^{2}$$

$$\leqq\frac{1}{2}(\log r^{2})^{2}=2(\log r)^{2}$$

and

$$D\leqq\log y_{2n+1}(r)\log y_{2n+1}(r)\leqq 4(\log r)^{2}.$$

Now, by using these functions $Y(R)$ and $y(r)$, we show the existence of $f(z)$ and $g(z)$ which satisfy the property stated at the beginning of this section. By a result of Edrei and Fuchs [2], there exist integral functions $f(z)$ and $g(z)$ such that, given any $\varepsilon>0$, estimates

$$\left(\frac{1}{1+\varepsilon}\right)y(r)\leqq\log M(e^{r},f)\leqq(1+\varepsilon)y(r)$$

and

$$\left(\frac{1}{1+\varepsilon}\right)R^{Y(R)}\leqq\log M(e^{R},g)\leqq(1+\varepsilon)R^{Y(R)}$$

hold for all sufficiently large values of r and R, since $y(r)$ and $R^{Y(R)}$ are positive, increasing, unbounded and convex functions of r and R, respectively. Here $R=y(r)$ and $\log M(e^{R},g)=R^{\varphi(e^{R})}$. Thus we see

$$\frac{Y(\log M(e^r, f))}{Y(R)} \begin{cases} = 1, (r_{2n-1} \leqq r \leqq r_{2n,1}) \text{ and } (R_{2n-1} \leqq R \leqq R_{2n,1}) \\ \leqq \dfrac{\log y(r)}{2} \leqq \dfrac{\log r^2}{2} = \log r, \quad (r_{2n,1} \leqq r \leqq r_{2n+1}) \text{ and} \\ \qquad\qquad\qquad\qquad\qquad\qquad (R_{2n,1} \leqq R \leqq R_{2n+1}) \end{cases}$$

for all sufficiently large values of r.

Further, we have

$$\lim_{R\to\infty} \frac{\varphi(e^R)}{Y(R)} = 1 \quad \text{and} \quad \lim_{r\to\infty} \frac{\log\log M(e^r, f)}{\log y(r)} = 1,$$

since

$$\lim_{R\to\infty} \frac{\log M(e^R, g)}{R^{Y(R)}} = \lim_{R\to\infty} \frac{R^{\varphi(e^R)}}{R^{Y(R)}} = 1$$

and

$$\lim_{r\to\infty} \frac{\log M(e^r, f)}{y(r)} = 1.$$

We also have

$$\varphi(M(e^r, f)) \log\log M(e^r, f)$$

$$= \frac{\varphi(M(e^r, f))}{Y(\log M(e^r, f))} \frac{Y(\log M(e^r, f))}{Y(R)} \frac{\log\log M(e^r, f)}{\log y(r)} Y(R)\cdot\log y(r)$$

and

$$\frac{Y(\log M(e^r, f))}{Y(R)} Y(R) \log y(r)$$

$$\begin{cases} \leqq 1\cdot 2r, (r_{2n-1} \leqq r \leqq r_{2n,1}) \text{ and } (R_{2n-1} \leqq R \leqq R_{2n,1}) \\ \leqq \log r\cdot 4(\log r)^2 = 4(\log r)^3, \ (r_{2n,1} \leqq r \leqq r_{2n+1}) \text{ and } (R_{2n,1} \leqq R \leqq R_{2n+1}) \end{cases}$$

for all sufficiently large values of r. Therefore

$$\varlimsup_{r\to\infty} \frac{\log\log M(e^r, g(f))}{r} \leqq \varlimsup_{r\to\infty} \frac{\varphi(M(e^r, f)) \log\log M(e^r, f)}{r}$$

$$= \varlimsup_{r\to\infty} \frac{\dfrac{\varphi(M(e^r, f))}{Y(\log M(e^r f))} \dfrac{Y(\log M(e^r, f))}{Y(y(r))} \dfrac{\log\log M(e^r, f)}{\log y(r)} Y(y(r)) \log y(r)}{r}$$

$$\leqq 2.$$

Further we have

$$\varlimsup_{r\to\infty}\frac{\log\log M(e^r,f)}{r}\leqq\varlimsup_{r\to\infty}\frac{\log(1+\varepsilon)\,y(r)}{r}\leqq\varlimsup_{r\to\infty}\frac{\log e^r+\log(1+\varepsilon)}{r}=1,$$

$$\varlimsup_{r\to\infty}\frac{\log\log M(e^r,f)}{r}\geqq\varlimsup_{r\to\infty}\frac{\log(1/1+\varepsilon)\,y(r)}{r}\geqq\varlimsup_{r_{2n}\to\infty}\frac{\log(1/2)\,e^{r_{2n}}+\log(1/1+\varepsilon)}{r_{2n}}$$

$$=1,$$

$$\varliminf_{r\to\infty}\frac{\log\log M(e^r,f)}{r}\leqq\varliminf_{r\to\infty}\frac{\log(1+\varepsilon)y(r)}{r}\leqq\varliminf_{r_{2n+1}\to\infty}\frac{\log\{(1+\varepsilon)r_{2n+1}^2\}}{r_{2n+1}}=0,$$

$$\varlimsup_{R\to\infty}\frac{\log\log M(e^R,g)}{R}\leqq\varlimsup_{R\to\infty}\frac{\log\{(1+\varepsilon)R^{Y(R)}\}}{R}\leqq\varlimsup_{R\to\infty}\frac{\log\{(1+\varepsilon)\,R^{\log R}\}}{R}=0,$$

$$\varlimsup_{R\to\infty}\varphi(e^R)\geqq\varlimsup_{R\to\infty}Y(R)\varliminf_{R\to\infty}\frac{\varphi(e^R)}{Y(R)}=\infty,$$

and

$$\varliminf_{R\to\infty}\varphi(e^R)\leqq\varliminf_{R\to\infty}Y(R)\varlimsup_{R\to\infty}\frac{\varphi(e^R)}{Y(R)}=2.$$

Therefore we can see that there exist integral functions $f(z)$ and $g(z)$ such that $f(z)$ is of order 1 and of lower order zero and such that $g(z)$ is of zero order and $\varlimsup_{R\to\infty}\varphi(R)=\infty$ and $\varliminf_{R\to\infty}\varphi(R)<\infty$ and further such that $g(f(z))$ is of finite order.

However, we note that, by Lemma 2, the order of $g(f(z))$ is greater than or equal to that of $f(z)$.

5. Now we shall deal with the case (b) where $\varlimsup_{r\to\infty}\varphi(r)$ is finite.

THEOREM 2. *Suppose that $f(z)$ is an integral function of the positive and finite order μ and that $g(z)$ is a transcendental integral function of zero order. If $\varlimsup_{r\to\infty}\varphi(r)=M$ is finite, then the order of $g(f(z))$ is finite and less than or equal to μM, where $\varphi(r)$ is defined as in* (1).

PROOF. If $f(z)$ is of order μ, then for any $\varepsilon>0$, we have

$$\varphi(r)<M+\varepsilon$$

and

$$\log M(r,f) < r^{\mu+\varepsilon}$$

for all sufficiently large values of r. Hence (3) implies

$$\varlimsup_{r\to\infty} \frac{\log\log M(r,g(f))}{\log r} \leqq \varlimsup_{r\to\infty} \frac{\varphi(M(r,f))\log\log M(r,f)}{\log r}$$

$$\leqq \varlimsup_{r\to\infty} \frac{(M+\varepsilon)\log r^{\mu+\varepsilon}}{\log r} = (M+\varepsilon)(\mu+\varepsilon).$$

Since ε is arbitrary, the order of $g(f)$ is less than or equal to μM. This proves Theorem 2.

We note that the order of $g(f(z))$ is equal to μM if the limit

$$\lim_{r\to\infty} \varphi(r) = M$$

exists.

When $f(z)$ and $g(z)$ are both of zero order, we can prove the following.

THEOREM 3. *Suppose that $f(z)$ and $g(z)$ are transcendental integral functions of zero order and satisfy the following condition* (I) *or* (II):

$$\text{(I)}\qquad \begin{cases} \varliminf_{r\to\infty} \dfrac{\log\log M(r,f)}{(\log r)^{c_1}} = A_1 > 0, \\ \varlimsup_{r\to\infty} \dfrac{\varphi(r)}{(\log\log r)^{d_1}} = B_1 > 0 \end{cases}$$

for any c_1 and d_1 satisfying $0<c_1<1, d_1>0$ *and* $c_1(d_1+1)>1$;

$$\text{(II)}\qquad \begin{aligned} &\varliminf_{r\to\infty} \frac{\log\log M(r,f)}{(\log\log r)^{c_2}} = A_2 > 0, \\ &\varlimsup_{r\to\infty} \frac{\log\varphi(r)}{(\log\log r)^{d_2}} = B_2 > 0, \end{aligned}$$

for any c_2 and d_2 satisfying $c_2>1$, $0<d_2<1$ *and* $c_2d_2>1$. *Then $g(f(z))$ is of infinite order. Here $\varphi(r)$ is defined as in* (1).

PROOF. Suppose that (I) holds. Then we see

$$\log\log M(r,f) > \frac{1}{2} A_1 (\log r)^{c_1}$$

for all sufficiently large values of r and there also exists a sequence $\{r_n\}$ such that $r_n \to \infty$ as $n \to \infty$ and such that

$$\varphi(M(r_n,f)) > \frac{1}{2} B_1 (\log\log M(r_n,f))^{d_1},$$

since $M(r,f)$ is continuous, increasing and unbounded of r. Thus, by (2), we have for any $\varepsilon > 0$,

$$\begin{aligned}\overline{\lim_{r\to\infty}} \frac{\log\log M(r, g(f))}{\log r} &\geqq \overline{\lim_{r_n\to\infty}} \frac{\varphi(M(r_n,f)) \log\log M(r_n,f)}{(1+\varepsilon)\log r_n} \\ &\geqq \overline{\lim_{r_n\to\infty}} \frac{(1/2)B_1 \cdot \{\log\log M(r_n,f)\}^{d_1} \log\log M(r_n,f)}{(1+\varepsilon)\log r_n} \\ &\geqq \overline{\lim_{r_n\to\infty}} \frac{(1/2)B_1 \{(1/2)A_1(\log r_n)^{c_1}\}^{d_1+1}}{(1+\varepsilon)\log r_n} \\ &= \overline{\lim_{r_n\to\infty}} \frac{(1/2)^{d_1+2} B_1 \cdot A_1{}^{d_1+1} (\log r_n)^{c_1(d_1+1)}}{(1+\varepsilon)\log r_n} = \infty ,\end{aligned}$$

since $c_1(d_1+1) > 1$ by our hypothesis. In particular, if $A_1B_1 = \infty$, we may replace a part of the condition (I) by $c_1(d_1+1) \geqq 1$ instead of $c_1(d_1+1) > 1$.

Suppose that (II) holds. Then for any sufficiently small $\varepsilon > 0$, we have

$$\log\log M(r,f) > (A_2 - \varepsilon)(\log\ \log\ r)^{c_2}$$

for all sufficiently large values of r and there also exists a sequence $\{r_n\}$ such that $r_n \to \infty$ as $n \to \infty$ and such that

$$\varphi(M(r_n,f)) > \exp\{(B_2 - \varepsilon)(\log\log M(r_n,f))^{d_2}\}.$$

Thus by (2), we have for any sufficiently small $\varepsilon > 0$,

$$\begin{aligned}\overline{\lim_{r\to\infty}} \frac{\log\log M(r, g(f))}{\log r} &\geqq \overline{\lim_{r_n\to\infty}} \frac{\varphi(M(r_n,f)) \log\log M(r_n,f)}{(1+\varepsilon)\log r_n} \\ &\geqq \overline{\lim_{r_n\to\infty}} \frac{\exp[(B_2-\varepsilon)\{\log\log M(r_n,f)\}^{d_2}] \log\log M(r_n,f)}{(1+\varepsilon)\log r_n}\end{aligned}$$

$$\geqq \varlimsup_{r_n\to\infty} \frac{\exp[(B_2-\varepsilon)\{(A_2-\varepsilon)(\log\log r_n)^{c_2}\}^{d_2}](A_2-\varepsilon)(\log\log r_n)^{c_2}}{(1+\varepsilon)\log r_u}$$

$$= \varlimsup_{r_n\to\infty} \frac{\{\exp(\log\log r_n)\}^{(B_2-\varepsilon)(A_2-\varepsilon)^{d_2}(\log\log r_n)^{c_2d_2-1}}(A_2-\varepsilon)(\log\log r_n)^{c_2}}{(1+\varepsilon)\log r_n}$$

$$= \varlimsup_{r_n\to\infty} \frac{(\log r_n)^{(B_2-\varepsilon)(A_2-\varepsilon)^{d_2}(\log\log r_n)^{c_2d_2-1}}(A_2-\varepsilon)(\log\log r_n)^{c_2}}{(1+\varepsilon)\log r_n}$$

$$= \infty\,,$$

since $c_2d_2-1>0$ by our hypothesis. In particular, if $A_2^{d_2}B_2>1$, we may replace a part of the condition (II) by $c_2d_2\geqq 1$ instead of $c_2d_2>1$. This proves Theorem 3.

REMARK 1. In (I) or (II), it can be shown in the similar way that we can replace $\varlimsup$ and $\varliminf$ by each other.

REMARK 2. If the conditions (I) and (II) do not hold, then the assertion of Theorem 3 is not valid. In fact, about the condition (I), for any $x_1, y_1 (0<x_1<1, y_1>0$ and $x_1(y_1+1)<1)$, we can find integral functions $f_{1,i}(z)$ and $g_{1,i}(z)$ such that these functions satisfy

$$\varliminf_{r\to\infty} \frac{\log\log M(r,f_{1,i})}{(\log r)^{x_1}} > 0 \text{ and } \varlimsup_{r\to\infty} \frac{\varphi_{1,i}(r)}{(\log\log r)^{y_1}} > 0,\ (i=1,2,3)$$

and that $g_{1,1}(f_{1,1})$, $g_{1,2}(f_{1,2})$ and $g_{1,3}(f_{1,3})$ are of order infinity, finite and zero respectively, and we can also find integral functions $f_{2,i}(z)$ and $g_{2,i}(z)$ such that these functions satisfy

$$\varlimsup_{r\to\infty} \frac{\log\log M(r,f_{2,i})}{(\log r)^{x_1}} > 0 \text{ and } \varliminf_{r\to\infty} \frac{\varphi_{2,i}(r)}{(\log\log r)^{y_1}} > 0,\ (i=1,2,3)$$

and that $g_{2,1}(f_{2,1})$, $g_{2,2}(f_{2,2})$ and $g_{2,3}(f_{2,3})$ are of order infinity, finite and zero, respectively. We can have the similar result about the condition (II).

The following can be easily proved by the same argument as that in the proof of Theorem 3.

THEOREM 4. *Suppose that $f(z)$ and $g(z)$ are transcendental integral functions of zero order and satisfy the following condition* (I)′ *or* (II)′ :

$$
\text{(I)}' \qquad \begin{cases} \overline{\lim\limits_{r\to\infty}} \dfrac{\log\log M(r,f)}{(\log r)^{c'_1}} < \infty \\ \overline{\lim\limits_{r\to\infty}} \dfrac{\varphi(r)}{(\log\log r)^{d'_1}} < \infty \end{cases}
$$

for any c'_1 *and* d'_1 *satisfying* $0 < c'_1 < 1, d'_1 > 0$ *and* $c'_1(d'_1+1) < 1$;

$$
\text{(II)}' \qquad \begin{cases} \overline{\lim\limits_{r\to\infty}} \dfrac{\log\log M(r,f)}{(\log\log r)^{c'_2}} < \infty \\ \overline{\lim\limits_{r\to\infty}} \dfrac{\log\varphi(r)}{(\log\log r)^{d'_2}} < \infty \end{cases}
$$

for any c_2' *and* d_2' *satisfying* $c_2' > 1, 0 < d_2' < 1$ *and* $c_2'd_2' < 1$. *Then* $g(f(z))$ *is of zero order.*

The proof may be omitted.

REMARK 3. For any $x_2, y_2 (0 < x_2 < 1, y_2 > 0, x_2(y_2+1) > 1)$, we can find integral functions $f_{3,i}(z)$ and $g_{3,i}(z)$ $(i = 1, 2, 3)$ such that these functions satisfy

$$
\overline{\lim_{r\to\infty}} \frac{\log\log M(r, f_{3,i})}{(\log r)^{x_2}} < \infty \text{ and } \overline{\lim_{r\to\infty}} \frac{\varphi_{3,i}(r)}{(\log\log r)^{y_2}} < \infty, (i = 1, 2, 3)
$$

and such that $g_{3,1}(f_{3,1})$, $g_{3,2}(f_{3,2})$ and $g_{3,3}(f_{3,3})$ are of order infinity, finite and zero, respectively. Further for any $x_3, y_3 (x_3 > 1, 0 < y_3 < 1$ and $x_3y_3 > 1)$, we can also find integral functions $f_{4,i}(z)$ and $g_{4,i}(z)$, $(i=1,2,3)$ such that these functions satisfy

$$
\overline{\lim_{r\to\infty}} \frac{\log\log M(r, f_{4,i})}{(\log\log r)^{x_3}} < \infty \text{ and } \overline{\lim_{r\to\infty}} \frac{\log\varphi_{4,i}(r)}{(\log\log r)^{y_3}} < \infty, (i = 1,2,3)
$$

and that $g_{4,1}(f_{4,1})$, $g_{4,2}(f_{4,2})$ and $g_{4,3}(f_{4,3})$ are of order infinity, finite and zero, respectively. Here $\varphi_{k,i}(r)$ $(k = 3,4, i = 1,2,3)$ are functions corresponding to $g_{k,i}(z)$ defined as in (1).

REFERENCES

[1] I. N. BAKER, The iteration of entire transcendental and the solution of $f(f(z)) = F(z)$, Math. Ann., 129(1955), 174–180.

[2] A. EDREI AND W. H. J. FUCHS, Entire and meromorphic functions with asymptotically prescribed characteristic, Canad. J. Math., 17(1965), 385–395.

[3] M. OZAWA, On the solution of the functional equation $f \circ g(z) = F(z)$, I-V, Kōdai Math. Sem. Rep., 20(1968), 159-162, 163-169, 257-263, 272-278, 305-313.
[4] G. PÓLYA, On an integral function of an integral function, J. London Math. Soc., 1(1926), 12-15.
[5] G. VALIRON, Sur un théorème de M. Fatou, Bull. Sci. Math., (2) 46(1922), 200-208.

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