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SPACES OF MAPPINGS AS SEQUENCE SPACES

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1. Let E be a Banach space whose dual is denoted by E'. A sequence $\{b_i\}$ of elements of E is called a Schauder basis of E if each $x \in E$ can be written in a unique way as $x = \sum t_i b_i$. Our summations are always from 1 to ∞ unless other limits of summation are expressly indicated. If $\{f_i\}$ is the sequence of elements of E' biorthogonal to $\{b_i\}$, i.e., $<b_j, f_i> = \delta_{ij}$ with $\delta_{ij}=0$ if $i \neq j$ and $\delta_{ii}=1$, then $x = \Sigma <x, f_i> b_i$. The basis $\{b_i\}$ is called shrinking if $\{f_i\}$ is a basis of E'.

Let F be a Fréchet space whose topology is determined by a countable system of seminorms $\{p_k\}_{k=1}^{\infty}$. We denote by $\mathcal{L}(E, F)$ the space of all continuous linear mappings from E into F. Equipped with the topology determined by the seminorms

$$q_k(T) = \sup\{p_k(Tx) : x \in E, \|x\| \leqq 1\};\ T \in \mathcal{L}(E, F);\ k=1, 2, \cdots,$$

$\mathcal{L}(E, F)$ is a Fréchet space. When E has a Schauder basis $\{b_i\}$, an element $T \in \mathcal{L}(E, F)$ is completely determined by the F-valued sequence $\{Tb_i\}$. We denote by $\mathcal{L}$ the space of F-valued sequences $\{\{Tb_i\} : T \in \mathcal{L}(E, F)\}$. Let $C(E, F)$ be the closed subspace of $\mathcal{L}(E, F)$ consisting of compact mappings and let $C = \{Tb_i\} : T \in C(E, F)\}$.

Dixmier [1] (see also [9], [10]) proves that if E and F are Hilbert spaces then $\mathcal{L}(E, F)$ is the bidual of $C(E, F)$. The dual of $C(E, F)$ is the space of all nuclear mappings from E into F equipped with the trace norm. He points out that this relationship is similar to that between c_0, l^1 and l^∞.

The relationship between these sequence spaces can also be described in a very simple way by means of Köthe spaces: l^1 is the space of all sequence $\{\beta_i\}$ such that for each $\{\alpha_i\} \in c_0$, $\sum |\alpha_i| |\beta_i| < \infty$, i.e., in the notation of Köthe $l^1 = (c_0)^\times$; l^∞ is the space of all sequences $\{\gamma_i\}$ such that for each $\{\beta_i\} \in l^1$, $\sum |\beta_i| |\gamma_i| < \infty$, i.e., $l^\infty = (l^1)^\times = (c_0)^{\times\times}$ (see [4], Chapter 6, §30). On the other hand a compact mapping defined on a Hilbert space vanishes outside a separable subspace ([7], page 202) and a separable Hilbert space has a shrinking Schauder basis.

In this note we prove among other things the existence of a relationship between C and $\mathcal{L}$ similar to that between c_0 and l^∞ when the spaces E and F have suitable properties. We have tried to make this note self contained. But a familiarity with the theory of spaces of vector-valued sequences as described in [12] is still helpful.

In the sequel we shall always assume that the Banach space E has a shrinking Schauder basis $\{b_i\}$. We regard Proposition 8 and Proposition 10 together with its Corollary as the main results of this note.

2. For each $T \in \mathcal{L}(E, F)$ and each $n=1, 2, \cdots$, we define the following mappings from E into F:

$$U_n : x \to \sum_{i=1}^{n} <x, f_i> Tb_i , \quad U_n \in C(E, F),$$

$$V_n : x \to \sum_{i=n+1}^{\infty} <x, f_i> Tb_i , \text{ i.e., } V_n = T - U_n .$$

PROPOSITION 1. *Suppose that the Banach space E has a shrinking Schauder basis* $\{b_i\}$. *Then for each* $T \in C(E, F)$, $U_n \to T$ *in the metrizable topology of* $C(E, F)$ *as* $n \to \infty$.

PROOF. We first observe that the set $\left\{ \sum_{i=n+1}^{\infty} <x, f_i> b_i : x \in E, \|x\| \leqq 1, n=1,2,\cdots \right\}$ is bounded in E. In fact, for each $x' \in E'$, $\sup\left\{ \left| \sum_{i=n+1}^{\infty} <x, f_i><b_i, x'> \right| : x \in E, \|x\| \leqq 1 \right\}$ approaches 0 as $n \to \infty$ because $\{f_i\}$ is a basis in E'. Thus our set is weakly bounded and hence bounded. Let

$$\sup\left\{ \left\| \sum_{i=n+1}^{\infty} <x, f_i> b_i \right\| : x \in E, \|x\| \leqq 1; n=1, 2, \cdots \right\} = A .$$

For each seminorm p_k on F, let $\mathfrak{A}_k = \{y : y \in F, p_k(y) \leqq 1\}$ and denote by $\mathfrak{A}_k^0$ the polar set of $\mathfrak{A}_k$ in F'. We have to show that the following quantity approaches 0 as $n \to \infty$.

$$q_k(V_n) = \sup\left\{ \left| \sum_{i=n+1}^{\infty} <x, f_i><Tb_i, y'> \right| : x \in E, \|x\| \leqq 1, y' \in \mathfrak{A}_k^0 \right\}$$

$$= \sup\left\{ \left| \sum_{i=n+1}^{\infty} <x, f_i><b_i, T^* y'> \right| : x \in E, \|x\| \leqq 1, y' \in \mathfrak{A}_k^0 \right\},$$

where T^* is the transpose of T. Since T is compact, the image of $\mathfrak{A}_k^0$ by T^* is a compact set of E' ([8], page 152). Therefore, for each $\varepsilon > 0$, we can find a finite number of elements of $\mathfrak{A}_k^0$, y'_γ, $\gamma = 1, 2, \cdots, \nu$, such that for each $y' \in \mathfrak{A}_k^0$, $\|T^*y' - T^*y'_\gamma\| \leqq \varepsilon/2A$ for a suitable γ. Choose n so large that

$$\sup\left\{\left|\sum_{i=n+1}^{\infty} <x, f_i><b_i, T^*y'_\gamma>\right| : x \in E, \|x\| \leqq 1\,;\ \gamma = 1, 2, \cdots, \nu\right\} \leqq \varepsilon/2\,,$$

this is possible because $\{f_i\}$ is a basis of E'. We have then

$$\left|\sum_{i=n+1}^{\infty} <x, f_i><b_i, T^*y'>\right|$$
$$\leqq \left|\sum_{i=n+1}^{\infty} <x, f_i><b_i, T^*y_\gamma>\right| + \left|\sum_{i=n+1}^{\infty} <x, f_i><b_i, T^*[y' - y'_\gamma]>\right|,$$

and hence

$$\sup\left\{\left|\sum_{i=n+1}^{\infty} <x, f_i><b_i, T^*y'>\right| : x \in E, \|x\| \leqq 1, y' \in \mathfrak{A}_k^0\right\}$$
$$\leqq \varepsilon/2 + A \cdot \varepsilon/2A = \varepsilon\,.$$

The proof is complete.

We transfer to the sequence spaces C and $\mathcal{L}$, the metrizable topologies of $C(E, F)$ and $\mathcal{L}(E, F)$. We have

PROPOSITION 2. *The dual of C can be identified with the space $C^\times$ of all F'-valued sequences $\{y'_i\}$ such that for each $\{y_i\} \in C$, the series $\sum <y_i, y'_i>$ converges.*

PROOF. Let φ be a continuous linear form on C. Let C_i be the subspace of C consisting of all sequences having an arbitrary element y of F as its i^{th}-coordinate and 0 elsewhere. For such an element of C_i we have

$$q_k(\{0, \cdots, 0, y, 0 \cdots\}) = \sup\{p_k(<x, f_i>y) : x \in E, \|x\| \leqq 1\} = \|f_i\| p_k(y)\,.$$

Consequently, for each $i = 1, 2, \cdots$, the restriction of φ to C_i detemines an element $y'_i \in F'$. Since for each $\{y_i\} \in C$, $\{y_1, y_2, \cdots, y_n, 0, 0, \cdots\} \to \{y_i\}_{i=1}^{\infty}$ as $n \to \infty$ by Proposition 1, we see that

$$\varphi(\{y_i\}) = \sum <y_i, y'_i>$$

and the series converges.

Conversely, let $\{y_i'\} \in C^\times$. For each $n = 1, 2, \cdots, \varphi_n: \{y_i\} \to \sum_{i=1}^{n} <y_i, y_i'>$ is a continuous linear form on C as can be seen easily. Since the series $\sum <y_i, y_i'>$ converges, by the Banach-Steinhaus theorem ([8], page 69), $\varphi: \{y_i\} \to \sum <y_i, y_i'>$ is a continuous linear form on C.

Since $\{f_i\}$ is a basis of E', a continuous linear mapping $\mathcal{A}$ from E' into F' is completely determined by the sequence $\{\mathcal{A} f_i\}$.

PROPOSITION 3. *Let F' be provided with the strong topology $T_b(F', F)$. Each $\{y_i'\} \in C^\times$ determines a continuous linear mapping $\mathcal{A}$ from E' into F' with $\mathcal{A} f_i = y_i'$, $i=1, 2, \cdots$.*

PROOF. For each $x' \in E'$, each $y \in F$, the mapping $x \to [\sum <x, f_i><b_i, x'>]y$ is a compact linear mapping from E into F. Therefore the series $\sum <b_i, x'><y, y_i'>$ converges. The series $\sum <b_i, x'> y_i'$ is a Cauchy series in F' with respect to the weak topology $T_s(F', F)$ and therefore converges in F'. put

$$\mathcal{A} x' = \sum <b_i, x'> y_i'$$

The mapping $\mathcal{A}$ from E' into F' so defined is linear. To show that it is continuous we shall show that for each bounded subset B of F, there exists a constant $M > 0$ such that

$$\sup \{ | \sum <b_i, x'><y, y_i'> | : y \in B \} \leqq M \|x'\|.$$

In fact, we can find a positive integer k and a constant $M_1 > 0$ such that

$$| \sum <b_i, x'><y, y_i'> | \leqq M_1 q_k(\{<b_i, x'> y\}) \leqq M_1 \|x'\| p_k(y)$$

because $\{y_i'\}$ generates a continuous linear form on C and

$$q_k(\{<b_i, x'> y\}) = \sup \{ p_k(\sum <x, f_i><b_i, x'> y) : x \in E, \|x\| \leqq 1\} = \|x'\| p_k(y).$$

Let G be a locally convex space, B a convex, balanced, bounded subset of G. Let G_B be the subspace of B spanned by B equipped with the norm $p_B(x) = \inf \{\rho : \rho > 0, x \in \rho B\}$ $(x \in G_B)$. B is called infracomplete if G_B is a Banach space. We recall that a mapping T from a locally convex space H into G is called nuclear if there is an equicontinuous sequence $\{x_i'\}$ in H', a sequence $\{y_i\}$ contained in a convex, balanced, infracomplete, bounded subset B of G and a complex sequence $\{\lambda_i\}$ with $\sum|\lambda_i|<\infty$ such that T is equal to the mapping $x\rightarrow\sum\lambda_i<x, x_i'>y_i$ ([11], page 481). We now prove

PROPOSITION 4. *Let $\mathcal{A}$ be a nuclear mapping from E' into F', F' equipped with the strong topology $T_b(F', F)$. Then $\{\mathcal{A}f_i\}\in C^{\times}$.*

PROOF. Let $\mathcal{A}=\sum\lambda_i x_i''\otimes y_i'$ where $\sum|\lambda_i|<\infty$, $x_i''\in E''\ \forall_i$, $\sup_i\|x_i\|<\infty$ and the sequence $\{y_i'\}$ is contained in a bounded, convex, balanced and infra-complete subset of F'. We have to show that for each $T\in C(E, F)$ the series $\sum<Tb_i, \mathcal{A}f_i>$ converges. For arbitrary positive integers n, m we have

$$\left|\sum_{i=n}^{i=n+m}<Tb_i, \mathcal{A}f_i>\right|=\left|\sum_{i=n}^{i=n+m}<Tb_i, \sum\lambda_j<x_j'', f_i>y_j'>\right|$$

$$=\left|\sum_j\sum_{i=n}^{i=n+m}\lambda_j<x_j'', f_i><Tb_i, y_j'>\right|$$

$$=\left|\sum_j\sum_i\lambda_j<x_j'', f_i><[U_{n+m}-U_{n-1}]\, b_i, y_i'>\right|$$

$$=\left|\sum_j\sum_i\lambda_j<x_j'', f_i><b_i, V_{nm}^* y_j'>\right|,$$

where $V_{nm}=U_{n+m}-U_{n-1}$ and V_{nm}^* is the transpose of V_{nm}. Since $\{f_i\}$ is a basis of E', we have

$$\left|\sum_i\lambda_j<x_j'', f_i><b_i, V_{nm}^* y_j'>\right|=|\lambda_j|\,|<x_j'', V_{nm}^* y_j'>|\leqq|\lambda_j|\,\|x_j''\|\,\|V_{nm}^* y_j'\|$$

On the other hand

$$\|V_{nm}^* y_j'\|=\sup\{|<x, V_{nm}^* y_j'>|: x\in E, \|x\|\leqq 1\}$$

$$=\sup\{|<V_{nm}x, y_j'>|: x\in E, \|x\|\leqq 1\}.$$

Since the sequence $\{y_j'\}$ is contained in a bounded subset of F' there exist a positive integer k and a constant $M>0$ such that

$$\|V_{nm}^* y_j'\|=\sup\{|<V_{nm}x, y_j'>|: x\in E, \|x\|\leqq 1\}\leqq Mq_k(V_{nm})\quad(j=1, 2, \cdots).$$

Thus

$$\left|\sum_{i=n}^{i=n+m}<Tb_i, \mathcal{A}f_i>\right|\leqq\sum_j\left|\sum_i\lambda_j<x_j'', f_i><b_i, V_{nm}^* y_j'>\right|$$

$$\leqq M(\sum_j |\lambda_j| \, \|x_j''\|) q_k([U_{n+m} - U_{n-1}]) .$$

Since $\sum |\lambda_j| \, \|x_j''\| > \infty$ and the sequence $\{U_n\}$ converges in C, we see that the series $\sum <Tb_i, \mathcal{A}f_i>$ converges.

Let Λ be a linear space of F-valued sequences containing all sequences having only a finite number of coordinates different from 0. Λ is called normal if $\{y_i\} \in \Lambda$ and the sequence of scalars $\{\alpha_i\}$ is such that $|\alpha_i| \leqq 1$ $(i=1, 2, \cdots)$ then $\{\alpha_i y_i\} \in \Lambda$. We denote by $\Lambda^\times$ the associate of Λ, i e., the space of all F'-valued sequences $\{y_i'\}$ such that for each $\{y_i'\} \in \Lambda$ the series $\sum <y_i, y_i'>$ converges. If Λ is normal, then $\Lambda^\times$ is also normal and the series $\sum <y_i, y_i'>$ is even absolutely convergent. $\Lambda^{\times\times}$ is the associate of Λ consisting of F-valued sequences. Λ is called perfect if $\Lambda = \Lambda^{\times\times}$. When Λ is normal, the properties of the dual system $<\Lambda, \Lambda^\times>$ with the canonical bilinear form $(\{y_i\}, \{y_i'\}) \rightarrow \sum <y_i, y_i'>$, $\{y_i\} \in \Lambda$, $\{y_i'\} \in \Lambda^\times$ have been investigated in [12].

Since the Banach space E has a Schauder basis $\{b_i\}$, it can be identified with a space λ of scalar-valued sequences $\lambda = \{\{<x, f_i>\} : x \in E\}$. It is not difficult to see that E' can be identified with $\lambda^\times$ and since the basis $\{b_i\}$ is shrinking, $\lambda^{\times\times}$ represents E''. We transfer to λ the topology of E, λ is called the BK-space isometric to E.

Let $\mathcal{L}^1(E', F')$ be the space of all nuclear mappings from E' into F', F' equipped with the strong topology $T_b(F', F)$, and let us denote by $\mathcal{L}^1$ the space of F'-valued sequences $\{\{\mathcal{A}f_i\} : \mathcal{A} \in \mathcal{L}^1(E', F')\}$. We prove

PROPOSITION 5. *Suppose that the Fréchet space F is reflexive. Then* $(\mathcal{L}^1)^\times \subset \mathcal{L}$, *i.e., if* $\{y_i\} \in (\mathcal{L}^1)^\times$ *then* $x \rightarrow \sum <x, f_i><b_i, x'> y_i$ *is a continuous linear mapping from E into F.*

PROOF. For each $x \in E$, each $y' \in F'$ the mapping $x' \rightarrow [\sum <x, f_i><b_i, x'>] y'$ is a nuclear mappingfrom E' into F'. Therefore, $\{<x, f_i> y'\} \in \mathcal{L}^1$. If $\{y_i\} \in (\mathcal{L}^1)^\times$ then $\sum <x, f_i><y_i, y'>$ converges. Since F is reflexive, it is weakly quasicomplete and the series $\sum <x, f_i> y_i$ converges weakly in F. Define a mapping T from E into by

$$T : x \rightarrow \sum <x, f_i> y_i .$$

This mapping T is clearly linear. For each $y' \in F$, the convergence of $\sum <x, f_i><y_i, y'>$ for each $x \in E$ shows that $\{<y_i, y'>\} \in \lambda^\times$. Because

$$<Tx, y'> = \sum <x, f_i><y_i, y'> \qquad x \in E, y' \in F,$$

the mapping T is weakly continuous and hence continuous when E is equipped with the norm topology and F is equipped with the metrizable topology.

Since by Proposition 4, $\mathcal{L}^1 \subset \mathcal{C}^\times$, it follows that $(\mathcal{L}^1)^\times \supset \mathcal{C}^{\times\times}$ and we have

COROLLARY. *Suppose that the Fréchet space F is reflexive. Then $\mathcal{C}^{\times\times} \subset \mathcal{L}$, i.e., $\{y_i\} \in \mathcal{C}^{\times\times}$ then $x \to \sum <x, f_i> y_i$ is a continuous linear mapping from E into F.*

PROPOSITION 6. *Suppose that the BK-space λ isometric to the Banach space E is normal. Then $\mathcal{L} \subset \mathcal{C}^{\times\times}$, i.e., for each $T \in \mathcal{L}(E, F)$ and each $\{y_i'\} \in \mathcal{C}^\times$, the series $\sum <Tb_i, y_i'>$ converges.*

PROOF. It is not difficult to see that the normality of λ implies that the spaces of F-valued sequences $\mathcal{C}$ and $\mathcal{L}$ are also normal. Let $T \in \mathcal{L}(E,F)$. As we have seen in the proof of Proposition 1, the set $\left\{\sum_{i=1}^{n} <x, f_i> b_i : x \in E, \|x\| \leqq 1;\ n=1, 2, \cdots\right\}$ is bounded in E, hence the set

$$\left\{U_n x = \sum_{i=1}^{n} <x, f_i> Tb_i : x \in E, \|x\| \leqq 1;\ n=1, 2, \cdots\right\}$$

is bounded in F or, in other words, the set $\{U_n : n=1, 2, \cdots\}$ is bounded in the Fréchet space $\mathcal{C}(E, F)$. For each $\{y_i'\} \in \mathcal{C}^\times$, consider the sums $\sum_{i=1}^{n} |<Tb_i, y_i'>|$; $n=1, 2, \cdots$. Since $\mathcal{C}^\times$ is normal, we can find a sequence of scalars $\{\delta_i\}$ such that $|<Tb_i, y_i'>| = <Tb_i, \delta_i y_i'>$, $i=1, 2, \cdots$ and $\{\delta_i y_i'\} \in \mathcal{C}^\times$. Because $\mathcal{C}^\times$ is the dual of $\mathcal{C}$, there exists a constant $M > 0$ such that

$$\sum_{i=1}^{n} |<Tb_i, y_i'>| = \sum_{i=1}^{n} <Tb_i, \delta_i y_i'> \leqq M q_k(U_n) < \infty, \qquad n=1, 2, \cdots$$

for a suitable smeinorm q_k on $\mathcal{C}(E, F)$. This shows that $\sum |<Tb_i, y_i'>| < \infty$ and the proof is complete.

COROLLARY 1. *Suppose that the BK-space λ isometric to the Banach space E is normal. Suppose also that the Fréchet space F is reflexive. Then $\mathcal{L} = \mathcal{C}^{\times\times}$.*

This is an immediate consequence of Proposition 6 and the Corollary of Proposition 5.

COROLLARY 2. *Suppose that E and F are as in Corollary 1. Then* $\mathcal{L}=(\mathcal{L}^1)^\times$.

PROOF. By Proposition 5, $(\mathcal{L}^1)^\times \subset \mathcal{L}$. On the other hand, from Proposition 4 we know that $\mathcal{L}^1 \subset \mathcal{C}^\times$. This implies $(\mathcal{L}^1)^\times \supset \mathcal{C}^{\times\times}=\mathcal{L}$. Hence the result.

PROPOSITION 7. *Suppose that both the Banach space E and the Fréchet space F are reflexive. Suppose also that the BK-space λ isometric to E is normal. Let $\mathcal{A}$ be a nuclear mapping from E' into F', $\mathcal{A}=\sum \lambda_j x_j \otimes y_j'$, where $\sum|\lambda_j|<\infty$, $\{x_j\}$ is bounded in E and $\{y_j'\}$ is contained in a bounded, convex balanced, infracomplete subset of F'. Then for each $T \in \mathcal{L}(E, F)$ we have*

$$\sum_i <Tb_i, \mathcal{A} f_i> = \sum_j \lambda_j <Tx_j, y_j'> .$$

PROOF. We have

$$\mathcal{A} f_i = \sum_j \lambda_j <x_j, f_i> y_j', \qquad i=1, 2, \cdots .$$

Therefore

$$\sum_j <Tb_i, \mathcal{A} f_i> = \sum_i \sum_j \lambda_j <x_j, f_i><Tb_i, y_j'> .$$

Since

$$\sum_i \lambda_j <x_j, f_i> Tb_i = \lambda_j T x_j ,$$

it is sufficient to show that we can change the order of summation of the double series. For that purpose we prove that $\sum_j \sum_i |\lambda_j|\,|<x_j, f_i>|\,|<Tb_i, y_i'>|$ is convergent. Since the BK-space λ isometric to E is normal, for each $j=1, 2, \cdots$ we can find $\bar{x}_j \in E$, $\|\bar{x}_j\|=\|x_j\|$ such that

$$|<x_j, f_i>|\,|<Tb_i, y_j'>| = <\bar{x}_j, f_i><Tb_i, y_j'> \; ; \; i=1, 2, \cdots .$$

Then

$$\sum_j \sum_i |\lambda_j|\,|<x_j, f_i>|\,|<Tb_i, y_j'>| = \sum_j \sum_i |\lambda_j| <\bar{x}_j, f_i><Tb_i, y_j'>$$

$$= \sum_j |\lambda_j| <T\bar{x}_j, y_j'> \leqq M \sum_j |\lambda_j| < \infty ,$$

for a certain constant M.

PROPOSITION 8. *Suppose that the BK-space* λ *isometric to the Banach space E is normal. Then* $\mathcal{L}^1 = C^\times$, *i.e., for each* $\{y'_i\} \in C^\times$ *the linear mapping* $\mathcal{A}$ *from* E' *into* F' *defined by* $\mathcal{A}x' = \sum \langle b_i, x' \rangle y'_i$ $(x' \in E')$ *is nuclear.*

PROOF. For $x \in E$, $y' \in F'$ we denote by $x \otimes y'$ the nuclear mapping $x' \to \langle x, x' \rangle y'$ $(x' \in E')$ from E' into F'. Then for each $T \in C(E, F)$ we have

$$\langle T, x \otimes y' \rangle = \sum \langle Tb_i, (x \otimes y') f_i \rangle = \sum \langle x, f_i \rangle \langle Tb_i, y' \rangle$$
$$= \langle Tx, y' \rangle = \langle x, T^* y' \rangle .$$

Hence

$$q_k(T) = \sup \{ | \sum \langle Tb_i, (x \otimes y') f_i \rangle | : x \in E, \|x\| \leqq 1, y' \in \mathfrak{A}_k^0 \} .$$

Since $\mathcal{L}^1 \subset C^\times$, it is clear that the metrizable topology of $C(E, F)$ determined by the sequence of seminorms $\{q_k\}$ is finer than the weak topology $T_s(C, \mathcal{L}^1)$ and since the dual of C equipped with this metrizable topology is $C^\times$, to show that $C^\times = \mathcal{L}^1$, it is sufficient to show that the metrizable topology of C is not stronger than the Mackey topology $T_k(C, \mathcal{L}^1)$. For this purpose we shall show that the sets

$$\Lambda_k = \{ \{ \langle x, f_i \rangle y' \} : x \in E, \|x\| \leqq 1, \ y' \in \mathfrak{A}_k^0 \}, \qquad k = 1, 2, \cdots,$$

are contained in absolutely convex compact subsets of $\mathcal{L}^1$ equipped with the weak topology $T_s(\mathcal{L}^1, C)$. We recall that the normal topology $T_n(\mathcal{L}^1, C)$ of $\mathcal{L}^1$ is the locally convex topology determined by the system of seminorms

$$p_T(\{y'_i\}) = \sum | \langle Tb_i, y'_i \rangle |, \qquad \{y'_i\} \in \mathcal{L}^1, \ T \text{ runs through } C(E, F).$$

This topology is finer than the weak topology $T_s(\mathcal{L}^1, C)$ and it can be shown (cf. [12], proof of Proposition 5) that $\mathcal{L}^1$ equipped with it is quasicomplete. We know that the absolutely convex closed envelope of a compact set is still compact if that absolutely convex closed envelope is complete. Hence it is sufficient to show that Λ_k is relatively $T_n(\mathcal{L}^1, C)$-compact. Since $\mathcal{L}^1$ is quasicomplete in the normal topology $T_n(\mathcal{L}^1, C)$ Λ_k is relatively $T_n(\mathcal{L}^1, C)$-compact if and only if it is relatively countably $T_n(\mathcal{L}^1, C)$-compact (Eberlein's theorem, cf., e.g., [4], page 316). It is sufficient to show that Λ_k is relatively sequentially $T_s(\mathcal{L}^1, C)$-compact. In fact, it is not difficult [5] to show that a sequence of $\mathcal{L}^1$ is $T_n(\mathcal{L}^1, C)$-convergent if and only if it is $T_s(\mathcal{L}^1, C)$-convergent. Let $\{u^{(n)}\}$ be a sequence of the unit ball of E and let $\{v^{(n)}\}$ be a sequence of $\mathfrak{A}_k^0$. From $\{u^{(n)}\}$ we can extract a subsequence, still denoted by $\{u^{(n)}\}$, converging to $u \in E''$ in the weak topology $T_s(E'', E')$. Similarly we can extract from $\{v^{(n)}\}$ a subsequence converging to $v \in \mathfrak{A}_k^0$ in the weak topology $T_s(F', F)$. Putting $\mathcal{A}=u\otimes v$, $\mathcal{A}^{(n)}=u^{(n)}\otimes v^{(n)}$, $n=1,2,\cdots$, for each $T\in C(E,F)$ we have

$$\begin{aligned}
&|\sum_i <Tb_i, [\mathcal{A}^{(n)}-\mathcal{A}]f_i>| \\
&\quad = |\sum_i [<u^{(n)}, f_i><Tb_i, v^{(n)}> - <u, f_i><Tb_i, v>]| \\
&\quad = |\sum_i [<u^{(n)}, f_i><b_i, T^*v^{(n)}> - <u, f_i><b_i, T^*v>]| \\
&\quad \leqq |\sum_i <u^{(n)}, f_i><b_i, T^*(v^{(n)}-v)>| + |\sum_i <(u^{(n)}-u), f_i><b_i, T^*v>|.
\end{aligned}$$

Because $T\in C(E,F)$, the set $\{T^*v : v\in \mathfrak{A}_k^0\}$ is compact in the norm topology of E'. We know that $\{T^*v^{(n)}\}_{n=1}^{\infty}$ converges to T^*v in the weak topology $T_s(E', E)$. Hence $\{T^*v^{(n)}\}$ converges to T^*v in the norm topology of E'. Therefore, for each $\varepsilon>0$ we can choose an integer N_1 such that if $n>N_1$ we have

$$\begin{aligned}
|\sum_i <u^{(n)}, f_i><b_i, T^*(v^{(n)}-v)>| &= |<u^{(n)}, T^*(v^{(n)}-v)>| \\
&\leqq \|T^*(v^{(n)}-v)\| \leqq \varepsilon/2.
\end{aligned}$$

On the other hand, the convergence of $\{u^{(n)}\}$ to u in the topology $T_s(E'', E')$ means that $\left\{\{<u^{(n)}, f_i>\}_{i=1}^{\infty}\right\}_{n=1}^{\infty}$ converges to $\{<u, f_i>\}_{i=1}^{\infty}$ in the sequence space $\lambda^{\times\times}$ equipped with the weak topology $T_s(\lambda^{\times\times}, \lambda^{\times})$. Since $\{<b_i, T^*v>\}_{i=1}^{\infty}\in\lambda^{\times}$, we can find an integer N_2 such that if $n>N_2$ we have

$$|\sum_i <(u^{(n)}-u), f_i><b_i, T^*v>| \leqq \varepsilon/2.$$

Then we have, whenever $n>N=\max(N_1, N_2)$,

$$|\sum_i <Tb_i, (a^{(n)}f_i - af_i)>| \leqq \varepsilon.$$

Thus the sequence $\{\mathcal{A}^{(n)}\}$ converges to $\mathcal{A}$ in $\mathcal{L}^1(E', F')$ equipped with the weak topology $T_s(\mathcal{L}^1, C)$. The proof is complete.

NOTES. (1) We assume the normality of the BK-space λ in order that the theory of normal spaces of vector-valued sequences developed in [12] could be applied to the spaces C and $\mathcal{L}^1$.

(2) If we require that the Banach space E be reflexive but drop the condition that the BK-space λ be normal, then we still have $\mathcal{L}^1=C^{\times}$. The proof is based on a numbr of deep results on topological tensor products of locally convex spaces. Since, by Proposition 1, $C(E,F)$ can be identified with $E'\hat{\hat{\otimes}}F$, it amounts to showing that the dual of $E'\hat{\hat{\otimes}}F$ is $\mathcal{L}^1(E', F')$. It is an

immediate consequence of the definition of integral mappings ([2], Chapter 1, page 126, Definition 7) that the dual of $E'\hat{\otimes}F$ is the space of all integral mappings from E' into F', F' equipped with the weak topology $T_s(F', F)$. The dual E of E' is separable in the norm topology. F' is quasicomplete in the weak topology $T_s(F', F)$ because every closed bounded subset of F' is equicontinuous and hence weakly compact. Therefore ([2], Chapter 1, page 134, Corollary 3 to Theorem 10) an integral mapping from E' into F' is nuclear. Since a $T_s(F', F)$-bounded subset of F' is also bounded in the strong $T_b(F', F)$, the nuclear mappings from E' into F' equipped with $T_s(F', F)$ or $T_b(F', F)$ are the same.

COROLLARY. *Suppose that the BK-space λ isometric to the Banach space E is normal. Suppose also that the Fréchet space F is reflexive. Then $\mathcal{L}^1 = \mathcal{L}^\times$, i.e., $\mathcal{L}^1$ is the space of all F-valued sequences $\{y'_i\}$ such that for each $T \in \mathcal{L}(E, F)$ the series $\sum <Tb_i, y'_i>$ converges (absolutely).*

PROOF. From the theory of sequence spaces we know that

$$\mathcal{L}^\times = (C^{\times\times})^\times = C^\times = \mathcal{L}^1 .$$

3. We shall now use Proposition 8 to obtain a representation of nuclear mapping from a reflexive Fréchet space F into a reflexive, normal BK-space λ.

Let $e_n = \{\delta_{ni}\}_{i=1}^\infty$, $n=1, 2, \cdots$ where δ_{ni} is the Kronecker delta. For each $T \in \mathcal{L}(\lambda, F)$, the sequence $\{Te_n\}$ belongs to $\lambda^\times$ scalarly, i.e., $\{<Te_n, y'>\}_{n=1}^\infty \in \lambda^\times$ for each $y' \in F'$. We denote by $\lambda^\times(F)$ the space of all F-valued sequences belonging scalarly to $\lambda^\times$. $\mathcal{L}(\lambda, F)$ can be identified with $\lambda^\times(F)$. In fact, it is sufficient to verify that for each $\{y_i\} \in \lambda^\times(F)$ and each $\{\alpha_i\} \in \lambda$

$$\limsup_{N\to\infty}\left\{\left|\sum_{i=1}^{N} \alpha_i <y_i, y'>\right| : y' \in \mathfrak{A}_k^0\right\} = 0$$

for each $k=1, 2, \cdots$. Since for each $y' \in F'$ the series $\sum \alpha_i <y_i, y'>$ converges and the Fréchet space F is reflexive, hence weakly sequentially complete, there exists $y \in F$ such that

$$\sum \alpha_i <y_i, y'> = <y, y'> \qquad (y' \in F')$$

This shows that the mapping $y' \to \{<y_i, y'>\}$ from F' into $\lambda^\times$ is continuous in the weak topologies $T_s(F', F)$ and $T_s(\lambda^\times, \lambda)$. Thus the set $\{\{<y_i, y'>\} : y' \in \mathfrak{A}_k^0\}$ is bounded in $\lambda^\times$. Hence

$$\limsup_{N\to\infty}\left\{\left|\sum_{i=1}^{N} \alpha_i <y_i, y'>\right| : y' \in \mathfrak{A}_k^0\right\} = 0 .$$

PROPOSITION 9. *Let λ be a normal reflexive BK-space and let F be a reflexive Fréchet space. Let $\lambda^{\times}(F)$ denote the space of all F-valued sequences belonging scalarly to $\lambda^{\times}$. A mapping $\mathcal{A}$ from F into λ is nuclear if and only if it is of the form $y \to \{<y, y_i'>\}$ where $\{y_i'\} \in [\lambda^{\times}(F)]^{\times}$.*

PROOF. $\mathcal{A}$ is nuclear if and only if its transpose $\mathcal{A}^*$ is nuclear ([11], page 483, Propsition 47.4). By the corollary of Proposition 8, $\mathcal{A}^*$ is nuclear if and only if $\{\mathcal{A}^* e_i\} \in \mathcal{L}^{\times} = [\lambda^{\times}(F)]^{\times}$. On the other hand, $\mathcal{A}$ is the mapping $y \to \{<y, \mathcal{A}^* e_i>\}$. In fact, for each $\{\beta_i\} \in \lambda^{\times}$ we have

$$<y, \mathcal{A}^*(\{\beta_i\})> = \sum <y, \mathcal{A}^*(\beta_i e_i)> = \sum \beta_i <y, \mathcal{A}^* e_i> = <\{\beta_i\}, \{<y, \mathcal{A}^* e_i>\}>.$$

4. Let $\{u_i\}$ be a sequence of elements of E such that $\sum \|u_i\| < \infty$. For each seminorm p_k on F, the function $T \to \sum_i p_k(Tu_i)$ is a seminorm on $\mathcal{L}(E, F)$. We call the topology on $\mathcal{L}(E, F)$ determined by these seminorms when p_k runs through the sequence of seminorms determining the topology of F, and $\{u_i\}$ runs through the set of all admissible sequences (i.e., $\sum \|u_i\| < \infty$) the von Neumann topology (von Neumann [6] called it the strongest topology, Dixmier [1] translated that term into "topologie ultra forte"). Since $\sum_i p_k(Tu_i) \leqq q_k(T) \sum_i \|u_i\|$, we see that the von Neumann topology of $\mathcal{L}(E, F)$ is weaker than its metrizable topology. We transfer to $\mathcal{L}$ the von Neumann topology of $\mathcal{L}(E, F)$.

PROPOSITION 10. *Suppose that E is a reflexive Banach space such that the BK-space λ isometric to it is normal. Suppose that the Fréchet space F is reflexive. Then the dual of $\mathcal{L}(E, F)$ equipped with the von Neumann topology is $\mathcal{L}^1(E', F')$. The canonical bilinear form is*

$$(T, a) \to \sum <Tb_i, \mathcal{A} f_i>, \quad T \in \mathcal{L}(E, F), \quad \mathcal{A} \in \mathcal{L}^1(E', F').$$

PROOF. Let $\mathcal{A} = \sum u_i \otimes v_i$, where $\{u_i\}$ is a sequence in E with $\sum \|u_i\| < \infty$, and $\{v_i\}$ is contained in an equicontinuous subset of F', be an element of $\mathcal{L}^1(E', F')$. Since, for each $T \in \mathcal{L}(E, F)$, we have by Proposition 7

$$|\sum <Tb_i, \mathcal{A} f_i>| = |\sum <Tu_i, vi>| \leqq M \sum_i p_k(Tu_i)$$

for a suitable norm p_k on F and a certain constant M, the von Neumann topology on $\mathcal{L}$ is finer than the weak topology $T_s(\mathcal{L}, \mathcal{L}^1)$. Hence the proposition is proved if we show that the von Neumann topology is coarser than the Mackey topology $T_k(\mathcal{L}, \mathcal{L}^1)$. We recall that for each $k = 1, 2, \cdots$ we put $\mathfrak{A}_k = \{y : y \in F, p_k(y) \leqq 1\}$ and denote by $\mathfrak{A}_k^0$ the polar set of $\mathfrak{A}_k$ in F'. Since

$$\sum <Tb_i, \mathcal{A} f_i> = \sum <Tu_i, v_i>$$

for each $\mathcal{A} = \sum u_i \otimes v_i \in \mathcal{L}^1(E', F')$ it is sufficient to show that for each fixed sequence $\{u_i\} \subset E$ with $\sum \|u_i\| < \infty$, the set Γ_k $(k=1, 2, \cdots)$ of all sequences $\{\mathcal{A} f_i\}$ where $\mathcal{A} = \sum u_i \otimes v_i$ with v_i running through $\mathfrak{A}_k^0$ for each $i = 1, 2, \cdots$ is an absolutely convex and relatively $T_S(\mathcal{L}^1, \mathcal{L})$-compact subset in $\mathcal{L}^1$. That it is absolutely convex is clear. Just as has been pointed out in the proof of Proposition 8, it is sufficient to show that Γ_k is relatively, sequentially $T_s(\mathcal{L}^1, \mathcal{L})$-compact. Let $\mathcal{A}^{(n)} = \sum_j u_j \otimes v_j^{(n)}$, $n = 1, 2, \cdots$, be a sequence of Γ_k. Using the weak compactness of $\mathfrak{A}_k^0$ and the diagonal process we can extract from $\{\mathcal{A}^{(n)}\}$ a subsequence, still denoted by $\{\mathcal{A}_n\}$ such that for each $j = 1, 2, \cdots$, $\{v_j^{(n)}\}_{n=1}^{\infty}$ converges to an element $v_i \in \mathfrak{A}_k^0$ in the weak topology $T_s(F', F)$. Let $\mathcal{A} = \sum u_j \otimes v_j$. For each $T \in \mathcal{L}(E, F)$,

$$<Tu_j, (v_j^{(n)} - v_j)> \to 0 \quad \text{as} \quad n \to \infty \ (j = 1, 2, \cdots),$$

$$| <Tu_j, (v_j^{(n)} - v_j)> | \leqq 2p_k(Tu_j).$$

Hence by Lebesgue's dominated convergence theorem

$$\sum_i <Tb_i, (\mathcal{A}^{(n)} - \mathcal{A}) f_i> = \sum_j <Tu_j, (v^{(n)} - v_j)>$$

converges to 0 as $n \to \infty$. Thus $\mathcal{A}$ is the limit of the sequence $\{\mathcal{A}^{(n)}\}$ in the weak topology $T_S(\mathcal{L}^1, \mathcal{L})$.

COROLLARY. *Suppose that E is a reflexive Banach space such that the BK-space λ isometric to it is normal. Suppose that the Fréchet space F is reflexive. Then $\mathcal{L}(E, F)$ is quasicomplete in its von Neumann topology.*

PROOF. The von Neumann topology on $\mathcal{L}$ is finer than the normal topology $T_n(\mathcal{L}, \mathcal{L}^1)$ and coarser than the Mackey topology $T_k(\mathcal{L}, \mathcal{L}^1)$ as we just saw in the proof of the proposition. On the other hand $\mathcal{L}$ is quasicomplete in the normal topology $T_n(\mathcal{L}, \mathcal{L}^1)$ ([12], Proposition 5), hence the statement.

PROPOSITION 11. *Suppose that both E and F are reflexive Banach spaces, E has a shrinking Schauder basis. Then the bidual of $C(E, F)$ can be identified with $\mathcal{L}(E, F)$.*

PROOF. Since E is reflexive ,$\{f_i\}$ is a shrinking Schauder basis of E'. Hence, by Proposition 1, a compact linear mapping from E' into an arbitrary Banach space G is the limit in the norm of continuous mappings of finite rank.

Thus $(E')'=E$ has the approximation property (cf., e.g. [10], Exposé 14). Therefore the natural imbedding of $E\hat{\otimes}F'$ in $\mathcal{L}(E',F')$ is one-to-one. Since E has the approximation property, on $E\hat{\otimes}F'$ the trace norm is equal to the norm induced by the strong dual of $\mathcal{C}(E,F)=E'\hat{\otimes}F$ [3]. Thus $E\hat{\otimes}_{\Pi}F'$ is the normed dual of $\mathcal{C}(E,F)$. The dual of $E\hat{\otimes}_{\Pi}F'$ is the space $B(E,F')$ of all continuous bilinear forms on $E\times F'$. $B(E,F')$ can be identified with $\mathcal{L}(E,F)$ because F is reflexive.

NOTES. (I) For $\mathcal{A}=\sum u_j\otimes v_j\in E\hat{\otimes}F'$, $T\in\mathcal{L}(E,F)$ the canonical bilinear form for the duality between $E\hat{\otimes}F'$ and $\mathcal{L}(E,F)$ is $(\mathcal{A},T)\to\sum <Tu_j,v_j>$. We wish to point out that only under the additional assumption that the BK-space isometric to E be normal, have we managed to prove that $\sum_j <Tu_j,v_j>=\sum_i <Tb_i,\mathcal{A}f_i>$.

(II) If in addition to the hypotheses of Proposition 11, the normality of the BK-space isometric to E is also assumed then the fact that $\mathcal{L}(E,F)$ is the bidual of $\mathcal{C}(E,F)$ can be proved directly without using the deep properties of the Banach spaces with approximation property. As should be expected, the direct proof is rather long compared with the proof given above. It amounts to showing that if $\mathcal{A}\in\mathcal{L}^1(E',F')$ then in $\mathcal{L}^1$(=dual $\mathcal{C}^\times$ of $\mathcal{C}$, by Proposition 8), the sequence (of finite sequences) $\{\{\mathcal{A}f_1,\mathcal{A}f_2,\cdots,\mathcal{A}f_n,0,0,\cdots\}\}_{n=1}^{\infty}$ converges to $\{\mathcal{A}f_n\}_{n=1}^{\infty}$ in the norm topology of $\mathcal{L}^1$, considered as dual of $\mathcal{C}$, for then the dual of the Banach space $\mathcal{L}^1$ is its associate $(\mathcal{L}^1)^\times=\mathcal{L}$. For this purpose one shows that the set

$$\mathcal{O}=\{\{Tb_i\}_{i=1}^{\infty}:\ T\in\mathcal{L}(E,F),\ \|T\|\leqq 1\}$$

is a relatively compact subset of $\mathcal{L}$ equipped with the normal topology $T_n(\mathcal{L},\mathcal{L}^1)$. One then makes use of the fact that in the space l^1 of convergent scalar sequences a subset S is relatively compact if and only if

$$\lim_{n\to\infty}\sup\left\{\sum_{i=n}^{\infty}|x_i|:\ \{x_i\}\in S\right\}=0\,.$$

As we have pointed out in the proof of Proposition 8, to show that $\mathcal{O}$ is relatively $T_n(\mathcal{L},\mathcal{L}^1)$-compact, one only has to show that $\mathcal{O}$ is relatively sequentially compact in the weak topology $T_s(\mathcal{L},\mathcal{L}^1)$.

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