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A GENERALIZATION OF THE CLASSICAL THEORY OF PRIMARY GROUPS

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In this paper we develop a theory that generalizes those familiar results about primary abelian groups that depend on the notions of purity, basic subgroups, pure-projectivity and pure-injectivity. All groups considered are assumed to be additively written p-primary abelian groups for some fixed prime p. Throughout, λ denotes a fixed but arbitrary countable limit ordinal. We shall mainly be concerned with that class C_λ consisting of all p-primary groups G such that $G/p^\alpha G$ is a direct sum of countable groups for all $\alpha<\lambda$. Groups in the class C_λ will be referred to as C_λ-groups. C_ω is, of course, the class of all primary groups. Moreover, as we shall see, the classical theory of primary abelian groups apparently has its roots in the gratuitous fact that, for any group G, G/p^nG is a direct sum of cyclic groups for all $n<\omega$.

By a subsocle of G we shall mean a subgroup of $G[p]$. A subsocle S of G will be said to be *summable* if there exists a direct decomposition. $S=\bigoplus_{\alpha<\mu} S_\alpha$ where $S_\alpha-0\subseteq p^\alpha G-p^{\alpha+1}G$ for each ordinal α. A group G itself is said to be *summable* if its socle $G[p]$ is summable. The notion of summability is crucial in the following treatment and we refer the reader to [4] and [6] for the pertinent results related to the concept. In particular, we mention that a summable group has length at most Ω (where, as usual, Ω denotes the first uncountable ordinal) and that a direct sum of countable reduced primary groups is necessarily summable. In [4] there is given the barest outline of a proof of the following generalization of the Kulikov criterion.

THEOREM 1. *A C_λ-group of length λ is a direct sum of countable groups if and only if it is summable.*

That this theorem fails for Ω is shown by Hill in [3]. A more potent formulation of this generalized Kulikov criterion requires the notion of a

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p^{α}-high subgroup. H is said to be a *p^{α}-high subgroup* of G if H is maximal among the subgroups of G that intersect $p^{\alpha}G$ trivially. In [6] there is given a detailed proof of both Theorem 1 and the following

THEOREM 2. *A summable p-primary group G of length λ is a direct sum of countable groups provided, for each $\alpha<\lambda$, G contains a p^{α}-high subgroup which is a direct sum of countable groups.*

The proofs of these theorems are combinatorial in nature and, as given in [6], eschew all homological notions.

Call a subgroup H of G *isotype* if $H\cap p^{\alpha}G=p^{\alpha}H$ for all ordinals α. Hill [2] has shown that an isotype subgroup of a direct sum of countable reduced primary groups is itself a direct sum of countable groups provided also that the subgroup has countable length. Simpler proofs of Hill's theorem appear in [4] and [6]. That this theorem fails for isotype subgroups of length Ω prevents a complete extension of the results of this paper to uncountable ordinals. Compare, however, remarks at the end of this paper about possible further generalizations. Recall that a subgroup H of G is said to be a *p^{α}-pure subgroup* if $H\rightarrowtail G\twoheadrightarrow G/H$ represents an element of $p^{\alpha}\mathrm{Ext}(G/H,H)$. This notion is due to Nunke and shall assume the same role in our theory as that played by ordinary purity ($=p^{\omega}$-purity for p-primary groups) in the classical theory. Indeed this paper may be viewed as a vindication of Nunke's definition of p^{α}-purity. If H is a p^{α}-pure subgroup of G, then $H\cap p^{\beta}G=p^{\beta}H$ for all $\beta\leqq\alpha$ (see [1] or [8]). If H is a p^{α}-high subgroup of G, then H is $p^{\alpha+1}$-pure subgroup of G (see [2]).

An observation very much in the spirit of the generalizations of this paper, but which we shall require only in obtaining a subsidiary result, is the following

PROPOSITION 1. *Every infinite subgroup of a C_{λ}-group is contained in a p^{λ}-pure subgroup of the same cardinality.*

Proposition 1 is actually quoted in [4] where there is given a correct though sketchy indication of a proof. The idea is as follows: Given an infinite subgroup H of the C_{λ}-group G, one constructs a group K of G such that $H\subseteqq K$, $|K|=|H|$, $K\cap p^{\alpha}G=p^{\alpha}K$ for all $\alpha<\lambda$ and $K+p^{\alpha}G/p^{\alpha}G$ is a direct summand of $G/p^{\alpha}G$ for all $\alpha<\lambda$. That such a K is necessarily p^{λ}-pure in G follows by Proposition 4 in [2]. The details of the combinatorial construction of such a K should be clear from a perusal of [6].

PROPOSITION 2. *If H is a p^{λ}-pure subgroup of a C_{λ}-group G, then H is itself a C_{λ}-group.*

PROOF. We actually only need that $H \cap p^\beta G = p^\beta H$ for all $\beta < \lambda$. For then it is a simple calculation to show that $H + p^\alpha G/p^\alpha G$ is isotype in $G/p^\alpha G$ for each $\alpha < \lambda$. And therefore, by Hill's theorem, $H + p^\alpha G/p^\alpha G \cong H/p^\alpha H$ is a direct sum of countable groups for all $\alpha < \lambda$.

A much deeper result to be proved below is that G/H is a C_λ-group provided H is a p^λ-pure subgroup of the C_λ-group G.

To generalize the familiar concept of a basic subgroup, we introduce the notion of a λ-basic subgroup. B is said to be a *λ-basic subgroup* of G if

(1) B is a direct sum of countable groups of length at most λ,

(2) B is a p^λ-pure subgroup of G, and

(3) G/B is divisible.

We shall also require for technical convenience the notion of a *λ-high tower*, by which we shall mean a well-ordered ascending sequence $\{G_\alpha\}_{\alpha<\lambda}$ of subgroups of G such that, for each α, G_α is a p^α-high subgroup of G. In order to establish the existence of λ-basic subgroups we require the following lemma.

LEMMA 1. *If $\{G_\alpha\}_{\alpha<\lambda}$ is a λ-high tower of G and if each G_α is summable, then $H = \bigcup_{\alpha<\lambda} G_\alpha$ is summable.*

PROOF. As λ is a countable limit ordinal, we may choose a strictly increasing sequence $\alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots$ of ordinals having λ as its limit. Then $H = \bigcup_{n<\omega} G_{\alpha_n}$. Set $T_0 = G_{\alpha_1}[p]$ and, for $n > 1$, let T_n be such that $(p^{\alpha_n}G)[p] = T_n \oplus (p^{\alpha_{n+1}}G)[p]$ with $T_n \subseteqq G_{\alpha_{n+1}}$. Then we have a direct decomposition $H[p] = \bigoplus_{n<\omega} T_n$ which is normal in the sense that $h_G(t_1 + \cdots + t_n) = \min[h_G(t_1), \cdots, h_G(t_n)]$ provided $t_i \in T_i$ for $i = 1, \cdots, n$. Now each G_α is isotype, summable and of countable length. Therefore, by remarks in section 1 of [6] each subsocle of G_α is a summable subsocle of G. In particular, each T_n is a summable subsocle of G. Since the decomposition $H[p] = \bigoplus_{n<\omega} T_n$ is normal, it follows once again from section 1 of [6] that $H[p]$ is a summable subsocle of G. Since each G_α is isotype, H itself is an isotype subgroup of G and consequently H is summable.

COROLLARY. *If $\{G_\alpha\}_{\alpha<\lambda}$ is a λ-high tower of G where each G_α is a direct sum of countable groups, then $H = \bigcup_{\alpha<\lambda} G_\alpha$ is a direct sum of countable groups of length at most λ.*

PROOF. As noted above, H is an isotype subgroup of G and clearly H has length at most λ. Thus G_α is also a p^α-high subgroup of H for each $\alpha < \lambda$.

Since H is summable by our lemma, Theorem 2 implies that H is in fact a direct sum of countable groups.

THEOREM 3. *A primary group G contains a λ-basic subgroup if and only if G is a C_λ-group.*

PROOF. If B is a p^λ-pure subgroup of G and if G/B is divisible, then by Theorem 16 in [1] it follows that $G/p^\alpha G \cong B/p^\alpha B$ for all $\alpha < \lambda$. Consequently, only C_λ-groups can have λ-basic subgroups. Suppose now that G is a C_λ-group and select a λ-tower $\{G_\alpha\}_{\alpha<\lambda}$. Now $G_\alpha \cong G_\alpha + p^\alpha G/p^\alpha G$ and, since G_α is isotype in G, $G_\alpha + p^\alpha G/p^\alpha G$ is isotype in $G/p^\alpha G$. By the preceding corollary, $B = \bigcup_{\alpha<\lambda} G_\alpha$ is a direct sum of countable groups. It is easily seen that $G[p] \subseteqq B[p] + p^\alpha G$ for each $\alpha < \lambda$ and therefore by Proposition 1 in [2], B is p^λ-pure in G. Moreover, $B \cap pG = pB$ and $G[p] \subseteqq B[p] + p^\alpha G$ for $\alpha < \omega$ imply that G/B is divisible. Thus, B is the desired λ-basic subgroup of G.

LEMMA 2. *Suppose H is an isotype subgroup of G and that $\{H_\alpha\}_{\alpha<\lambda}$ is a λ-high tower of H. Then there exists a λ-high tower $\{G_\alpha\}_{\alpha<\lambda}$ of G such that, for each α, $H_\alpha \subseteqq G_\alpha$ and $H_\alpha = H \cap G_\alpha$.*

PROOF. Let us first note that $H \cap G_\alpha = H_\alpha$ is a consequence of $H_\alpha \subseteqq G_\alpha$. Indeed $H_\alpha \subseteqq G_\alpha$ implies $H_\alpha \subseteqq H \cap G_\alpha$ and $(H \cap G_\alpha) \cap p^\alpha H = (H \cap G_\alpha) \cap p^\alpha G = 0$. The maximality of a p^α-high subgroup then yields the equality. Assume now that $\alpha < \lambda$ and that for each $\beta < \alpha$ we have a p^β-high subgroup G_β of G such that $H_\beta \subseteqq G_\beta$ and $G_\gamma \subseteqq G_\beta$ for all $\gamma < \beta$. In order to be able to choose the desired G_α, it suffices to show that $(H_\alpha + \cup_{\beta<\alpha} G_\beta) \cap (p^\alpha G)[p] = 0$. Suppose $x + g \in (p^\alpha G)[p]$ where $x \in H_\alpha$ and $g \in G_\beta$ for some $\beta < \alpha$. Then $px = -pg \in pG \cap H \cap G_\beta = pG \cap H_\beta = pH_\beta$ and hence there is an $h_1 \in H_\beta$ such that $x - h_1 \in H[p] = H_\beta[p] \oplus (p^\beta H)[p]$. Thus we can write $x = h_1 + h_2 + z$ where $h_2 \in H_\beta[p]$ and $z \in (p^\beta H)[p]$. Then $h_1 + h_2 + g = x + g - z \in p^\beta G \cap G_\beta = 0$ and $x + g = z \in H$. Therefore $g \in H \cap G_\beta = H_\beta \subseteqq H_\alpha$ and, consequently, $x + g \in H_\alpha \cap p^\alpha G = H_\alpha \cap p^\alpha H = 0$ as desired.

LEMMA 3. *Suppose G is a direct sum of countable groups and that $G = \bigcup_{\alpha<\lambda} G_\alpha$ where $\{G_\alpha\}_{\alpha<\lambda}$ is a λ-high tower. If H is a p^λ-pure subgroup of G such that for each α, $H \cap G_\alpha$ is a p^α-high subgroup of H, then H is a direct summand of G.*

PROOF. We need only show that G/H is a direct sum of countable groups having length at most λ. Since $H \cap G_\alpha$ is $p^{\alpha+1}$-pure in H and H is p^λ-pure in

G, $H\cap G_\alpha$ is $p^{\alpha+1}$-pure in G and, a fortiori, $p^{\alpha+1}$-pure in G_α. Since G_α is a direct sum of countable groups (by Hill's theorem), G_α is p^α-projective. Therefore, by Proposition 3.1 of [8], there is a direct decomposition $G_\alpha=(H\cap G_\alpha)\oplus K_\alpha$ for each $\alpha<\lambda$. Now $G/H=\bigcup_{\alpha<\lambda} G_\alpha+H/H$ and $G_\alpha+H/H\cong G_\alpha/G_\alpha\cap H\cong K_\alpha$ is a direct sum of countable groups for each α. By the corollary to Lemma 1, it is enough to show that $G_\alpha+H/H$ is a p^α-high subgroup of G/H whenever $\omega\leqq\alpha<\lambda$. Since H is p^λ-pure in G, we have $p^\alpha(G/H)[p]=p^\alpha G[p]+H/H$ for $\alpha<\lambda$ and it then easily follows that $(G/H)[p]=(G_\alpha+H/H)[p]\oplus p^\alpha(G/H)[p]$. Because of this direct decomposition, it is enough to show that $G_\alpha+H/H$ is a pure subgroup of G/H for $\alpha\geqq\omega$. Now $(G_\alpha+H)[p]=(K_\alpha\oplus H)[p]=K_\alpha[p]\oplus H[p]$ $=K_\alpha[p]\oplus(H\cap G_\alpha)[p]\oplus(p^\alpha H)[p]=G_\alpha[p]\oplus(p^\alpha H)[p]$. If $\alpha\geqq\omega$ and if $x\in(G_\alpha+H)[p]$, then we can write $x=y+z$ where $y\in G_\alpha[p]$ and $z\in(p^\alpha H)[p]\subseteqq p^\omega H$. If x has finite height in G, then this height is just the height of y in $G(=$ height of y in $G_\alpha)$ and thus just the height of $x=y+z$ in $G_\alpha+H$. On the other hand, if x has infinite height in G, then y has infinite height in G_α and $x=y+z$ has infinite height in $G_\alpha+H$. By a well-known theorem, it follows that $G_\alpha+H$ is a pure subgroup of G. Thus $G_\alpha+H/H$ is pure in G/H.

PROPOSITION 3. *Let A be a direct sum of countable groups of length at most λ and suppose A is a p^λ-pure subgroup of the C_λ-group G. Then there exists a subgroup C of G such that $A\oplus C$ is a λ-basic subgroup of G.*

PROOF. Since A is a direct sum of countable groups of length $\leqq\lambda$, A is the union of a λ-high tower $\{A_\alpha\}_{\alpha<\lambda}$ of itself. By Lemma 2, there exists a λ-high tower $\{G_\alpha\}_{\alpha<\lambda}$ of G such that $A_\alpha=A\cap G_\alpha$ for each α. Let $B=\bigcup_{\alpha<\lambda} G_\alpha$. By the proof of Theorem 3, B is a λ-basic subgroup of G. But $\{G_\alpha\}_{\alpha<\lambda}$ is also a λ-high tower of B and, by Lemma 3, we have the desired direct decomposition $B=A\oplus C$.

It is now a simple matter to prove

THEOREM 4. *If H is a p^λ-pure subgroup of the C_λ-group G, then G/H is a C_λ-group.*

PROOF. Let A be a λ-basic subgroup of H and choose C such that $A\oplus C$ is a λ-basic subgroup of G. Now if $x\in(H\cap C)[p]$, we can write, for each $\alpha<\lambda$, $x=a_\alpha+z_\alpha$ where $a_\alpha\in A[p]$ and $z_\alpha\in p^\alpha H$. Thus $-a_\alpha+x\in p^\alpha(A\oplus C)=p^\alpha A\oplus p^\alpha C$ and $x\in\bigcap_{\alpha<\lambda} p^\alpha C=p^\lambda C=0$. We then have a direct decomposition $H\oplus C$. If pg

$\in H \oplus C$, then $pg = a + ph + c$ where $a \in A$, $h \in H$ and $c \in C$. Since $pG \cap (A \oplus C) = p(A \oplus C)$, we conclude that $pG \cap (H \oplus C) = p(H \oplus C)$. Now $G[p] \subseteq (A \oplus C)[p] + p^\alpha G \subseteq (H \oplus C)[p] + p^\alpha G$ for all $\alpha < \lambda$ and therefore, by Proposition 1 of [2], $H \oplus C$ is a p^λ-pure subgroup of G. Consequently, $H \oplus C/H$ is p^λ-pure in G/H. Also $H \oplus C/H \cong C$ and $(G/H)/(H \oplus C/H) \cong (G/A \oplus C)/(H \oplus C/A \oplus C)$ is divisible. We have constructed a λ-basic subgroup of G/H and we conclude that G/H is indeed a C_λ-group.

As an easy consequence of the foregoing theorem, we have the following striking analog of a familiar property of pure subgroups.

COROLLARY. *A subgroup H of a C_λ-group G is a p^λ-pure subgroup if and only if $H + p^\alpha G/p^\alpha G$ is a direct summand of $G/p^\alpha G$ for all $\alpha < \lambda$.*

PROOF. $H + p^\alpha G/p^\alpha G$ being a direct summand of $G/p^\alpha G$ implies that $H + p^\alpha G/p^\alpha G$ is p^α-pure in $G/p^\alpha G$ which is equivalent to H being p^α-pure in G. Since λ is a limit, H is p^λ-pure in G if and only if H is p^α-pure in G for all $\alpha < \lambda$. Conversely, let us assume that H is p^λ-pure in G. Then G/H is a C_λ-group and therefore, for $\alpha < \lambda$, $(G/H)/p^\alpha(G/H) = (G/H)/(p^\alpha G + H/H) \cong (G/p^\alpha G)/(H + p^\alpha G/p^\alpha G)$ is a direct sum of countable groups of length at most α. Since $H + p^\alpha G/p^\alpha G$ is p^α-pure in $G/p^\alpha G$, $H + p^\alpha G/p^\alpha G$ is a direct summand of $G/p^\alpha G$.

PROPOSITION 4. *If H is a p^λ-pure subgroup of the C_λ-group G and if $p^\alpha H$ is a direct summand of $p^\alpha G$ for some $\alpha < \lambda$, then H is a direct summand of G.*

PROOF. Assuming the conditions of the theorem, we have for some $\alpha < \lambda$:
(1) $(G/H)/p^\alpha(G/H)$ is a direct sum of countable groups;
(2) $H \cap p^\alpha G = p^\alpha H$;
(3) $H + p^\alpha G/p^\alpha G$ is a direct summand of $G/p^\alpha G$; and
(4) $p^\alpha G = p^\alpha H \oplus C$.
It follows from Theorem 2.18 of [4] that $G = H \oplus L$ where $L \supseteq C$.

As an immediate corollary, we have the following remarkable generalization of the well-known fact that bounded pure subgroups are direct summands.

COROLLARY. *If H is a p^λ-pure subgroup of the C_λ-group G and if $p^\alpha H = 0$ for some $\alpha < \lambda$, then H is a direct summand of G.*

COROLLARY. *If G is a C_λ-group of length λ, then every finite subset of G is contained in a countable direct summand.*

PROOF. Let S be a finite subset of G. By Proposition 1, $S \subseteq A$ for some countable, p^{λ}-pure subgroup A of G. We may assume that A has length λ. Then A is a direct sum of groups of length less than λ. Consequently, S is contained in a direct summand C of A having length less than λ. By the preceding corollary, C is a direct summand of G.

The latter corollary tells us that C_{λ}-groups of length λ are both transitive and fully transitive in the sense of Kaplansky. This, of course, is merely a reflection of the fact that groups of length $\leqq \lambda$ behave in the C_{λ} context exactly as groups without elements of infinite height in the classical situations.

We shall call a C_{λ}-group G a C_{λ}-projective if $p^{\lambda}\mathrm{Ext}(G, K) = 0$ for all C_{λ}-groups K and a C_{λ}-injective if $p^{\lambda}\mathrm{Ext}(K, G) = 0$ for all C_{λ}-groups K. Each definition corresponds to a splitting condition on short exact sequences $A \overset{\sigma}{\rightarrowtail} B \twoheadrightarrow C$ with $\mathrm{Im}\,\sigma$ p^{λ}-pure in B. In particular, a C_{λ}-group is a C_{λ}-injective if and only if it is a direct summand of every C_{λ}-group in which it occurs as a p^{λ}-pure subgroup. Note, however, that the proof of this equivalence requires Theorem 4. Theorem 2.9 and Proposition 2.11 in [8] give us immediately

PROPOSITION 5. *Every C_{λ}-group is the homomorphic image of a direct sum of countable groups of length λ under a map with p^{λ}-pure kernel.*

Thus there are "enough projectives" and we obviously have the following characterization of C_{λ}-projectives.

THEOREM 5. *A C_{λ}-group is a C_{λ}-projective if and only if it is a direct sum of countable groups of length at most λ.*

To characterize the C_{λ}-injectives we must generalize the notion of a closed p-group. On an arbitrary abelian group G we define the λ-topology by taking as neighborhoods of zero the members of the family $\{p^{\alpha}G\}_{\alpha<\lambda}$. We call a group a λ-closed group if it is (under the canonical imbedding) the maximal torsion subgroup of its completion in the λ-topology. This, of course, is equivalent to requiring every Cauchy net with elements uniformly bounded in order to converge. Observe that λ-closed groups have length at most λ and that groups of length less than λ are necessarily λ-closed as the λ-topology is then discrete.

PROPOSITION 6. *A λ-closed C_{λ}-group is a C_{λ}-injective.*

PROOF. Let G be a λ-closed C_{λ}-group. We first show that $p^{\lambda}\mathrm{Ext}(C(p^{\infty}),G) = 0$. Assume then that G is a p^{λ}-pure subgroup of K with $K/G \cong C(p^{\infty})$. Since

λ is a limit ordinal, it follows that $K=p^\alpha K+G$ for all $\alpha<\lambda$. Therefore, if $k\in K$, we can find for each $\alpha<\lambda$ a $g_\alpha\in G$ such that $k-g_\alpha\in p^\alpha K$. Moreover, we can assume that the order of g_α does not exceed that of k. Indeed if k has order p^n, then $p^n g_\alpha\in p^{\alpha+n}K\cap G=p^{\alpha+n}G$ and $p^n g_\alpha=p^n z_\alpha$ for some $z_\alpha\in p^\alpha G$. Then $\bar{g}_\alpha=g_\alpha-z_\alpha$ has order at most n and $k-\bar{g}_\alpha\in p^\alpha K$. But $\{g_\alpha:\alpha<\lambda\}$ is a Cauchy net in G with elements uniformly bounded in order and, therefore, converges to some $g\in G$. Hence $k-g\in\bigcap_{\alpha<\lambda}p^\alpha K=p^\lambda K$. We conclude that $K=G\oplus p^\lambda K$.

Now let K be an arbitrary C_λ-group and let B be a λ-basic subgroup of K. We then have an exact sequence (see [9])

$$p^\lambda \mathrm{Ext}(K/B,G)\longrightarrow p^\lambda \mathrm{Ext}(K,G)\longrightarrow p^\lambda \mathrm{Ext}(B,G)\,.$$

The left hand term of the above sequence vanishes since K/B is isomorphic to a direct sum of copies of $C(p^\infty)$ and the right hand term vanishes since B is a C_λ-projective. Thus, $p^\lambda\mathrm{Ext}(K,G)=0$ and we conclude that G is a C_λ-injective.

We can now show that there are "enough C_λ-injectives" and that a C_λ-injective is the sum of a λ-closed group and a divisible group.

THEOREM 6. *Every C_λ-group is a p^λ-pure subgroup of a C_λ-injective and a C_λ-group is a C_λ-injective if and only if it is the direct sum of a divisible group and a λ-closed C_λ-group.*

PROOF. It is evident from Proposition 6 that the direct sum of a divisible p-group and a λ-closed C_λ-group is necessarily a C_λ-injective. Next we need the observation (see [7]) that every C_λ-group G of length at most λ can be imbedded as a p^λ-pure subgroup of a λ-closed group $T_\lambda(G)$ with $T_\lambda(G)/G$ divisible. Indeed $T_\lambda(G)$ may be taken as the maximal torsion subgroup of $\varprojlim_{\alpha<\lambda} G/p^\alpha G$, the completion of G in the λ-topology. It follows, by the same reasoning as in the proof of Theorem 3, that $T_\lambda(G)/p^\alpha T_\lambda(G)\cong G/p^\alpha G$ for all $\alpha<\lambda$ and therefore that $T_\lambda(G)$ is a C_λ-group.

Now let G be an arbitrary C_λ-group. Let D be minimal divisible containing $p^\lambda G$. Then take M to be the amalgamated sum of G and D over $p^\lambda G$, that is, construct the push-out diagram

$$\begin{array}{ccc} P^\lambda G & \rightarrowtail & G \\ \downarrow & & \downarrow \\ D & \rightarrowtail & M \end{array}$$

Then $M=K\oplus D$ where $K\cong G/p^\lambda G$ and $K\cap G$ is a p^λ-high subgroup of G.. the details are similar to those in the proof of Lemma 1 in [5]. Also, M/G is divisible and $M[p]\subseteqq G[p]+p^\alpha M$ for all $\alpha<\lambda$. It follows that G is a p^λ-pure subgroup of M. By the transitivity of p^λ-purity, G is p^λ-pure in the C_λ-injective $T_\lambda(K)\oplus D$. Finally, assume that G is itself a C_λ-injective and that we have it imbedded, as above, as a p^λ-pure subgroup of $\bar{M}=T_\lambda(K)\oplus D$. Since G is a C_λ-injective, $\bar{M}=G\oplus E$ where $E\cong\bar{M}/G$ is obviously divisible since both M/G and $\bar{M}$/M are divisible. But then $E\subseteqq D$ and since $D[p]\subseteqq p^\lambda G$, we conclude that $E=0$ and $G=T_\lambda(K)\oplus D$.

Our final result was first obtained by Waller [10] for the case when $G/p^\alpha G$ is countable for all $\alpha<\lambda$.

THEOREM 7. *If G and K are λ-closed C_λ-groups with the same Ulm invariants, then $G\cong K$.*

PROOF. Our proof is the obvious generalization of the standard proof of the corresponding result for closed p-groups. Take B and C to be λ-basic subgroups of G and K respectively. It is easily seen that B and C have the same Ulm invariants as G and K. Therefore, by Kolettis' theorem, there is an isomorphism ϕ of B onto C. Since B is a p^λ-pure subgroup of G, we have an exact sequence

$$\mathrm{Hom}(G, K)\longrightarrow \mathrm{Hom}(B, K)\longrightarrow p^\lambda\mathrm{Ext}(G/B, K)=0$$

Thus, there is a homomorphism $\bar{\phi}: G\rightarrow K$ that extends ϕ. Let $x\in\mathrm{Ker}\,\bar{\phi}$ and assume that $x\neq 0$. Then x has some height $\alpha<\lambda$ and we can write $x=b+z$ where $b\in B$ and $z\in p^{\alpha+1}G$. But then b has height α and $\phi(b)=\bar{\phi}(b)=-\bar{\phi}(z)$ has height at least $\alpha+1$. This, however, is a contradiction since ϕ is an isomorphism of B onto C and C is an isotype subgroup of K. We conclude that $\mathrm{Ker}\,\bar{\phi}=0$. Then $\bar{\phi}(G)/C=\bar{\phi}(G)/\bar{\phi}(B)\cong G/B$ is divisible. Hence $\bar{\phi}(G)/C$ is a direct summand of K/C and, since C is a p^λ-pure subgroup of K, it follows that $\bar{\phi}(G)$ is a p^λ-pure subgroup of K. Since $\bar{\phi}(G)\cong G$ is a C_λ-injective, we have a direct decomposition $K=\bar{\phi}(G)\oplus E$ where $E\cong K/\bar{\phi}(G)$ is divisible. But K is reduced and therefore $E=0$ and $\bar{\phi}(G)=K$, that is, $\bar{\phi}$ is an isomorphism of G onto K.

We have now developed the C_λ-theories to roughly the same level as the classical C_ω-theory. The reader should have no difficulty in establishing the appropriate analog of his favorite C_ω-theorem. But likewise, familiar pathologies surely translate from ω to λ; for example, there evidently exist C_λ-groups of length λ that are isomorphic to none of their proper subgroups and C_λ-groups of length $\lambda+1$ that are neither transitive nor fully transitive.

We close with a few remarks about possible generalizations of results in this paper. It is obvious what we wish the class C_Ω to be. Moreover, as direct sums of countable reduced p-groups are just the totally-projective groups of length at most Ω (see [8]), it is clear that we should define C_λ, for an arbitrary limit ordinal λ, to be the class of all p-primary abelian groups G such that, for each $\alpha < \lambda$, $G/p^\alpha G$ is a totally projective. There are, however, already serious difficulties at $\lambda = \Omega$ which show that the theory we have developed cannot be generalized intact for uncountable λ. First, Theorems 1 and 2, which are indispensable tools in our treatment, fail for $\lambda = \Omega$. But independent of this fact, it is already known that a C_Ω-group can contain no Ω-basic subgroup distinct from itself. Indeed, by Theorem 1.9 of [4], if B is a proper p^Ω-pure subgroup of a reduced p-group G with G/B divisible, then B is not a direct sum of countable groups, On the other hand, Theorem 4 and its several striking consequences generalize trivially to $\lambda = \Omega$ simply because Ω is the supremum of all countable limit ordinals. It would not be suprising then if Theorem 4 could be established for arbitrary limit ordinals λ. However, it seems likely that combinatorial techniques such as those used in this paper will not be adequate for the task and that an approach more homological in spirit will be required.

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