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# SUM OF DEFICIENCIES AND THE ORDER OF A MEROMORPHIC FUNCTION

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**Introduction**. Sum of deficiencies of a meromorphic function is treated by Edrei, Fuchs, Ozawa, Pfluger and others. Throughout this paper, a meromorphic function means a function meromorphic in the complex plane $|z|<+\infty$.

Let $f(z)$ be a meromorphic function. It is assumed that the reader is familiar with the following symbols of frequent use in Nevanlinna's theory:

$$m(r,f),\ n(r,f),\ N(r,f),\ T(r,f),\ \delta(a,f),\ \text{etc..}$$

As a relation between the sum of deficiencies and the order of an entire function, the following theorem is well known.

PFLUGER'S THEOREM ([3]). *If $f(z)$ is a transcendental entire function of finite order $\mu$ with $\sum_a \delta(a,f)=2$, then $\mu$ is a positive integer and, for every deficient value $a$ of $f(z)$, $\delta(a,f)$ is an integral multiple of $1/\mu$. In particular, there cannot be more than $\mu$ finite deficient values.*

Now we generalize the concept of the deficiency of $f(z)$.

Let $f(z)$ be a transcendental meromorphic function and $\psi(z)$ a meromorphic function which may be constant. Then we define as follows:

$$\delta(\psi,f)\equiv 1-\limsup_{r\to\infty}\frac{N(r,1/(f-\psi))}{T(r,f)}$$

and

$$\bar{N}_h(r,f)=\int_0^r\frac{\bar{n}_h(t,f)-\bar{n}_h(0,f)}{t}dt+\bar{n}_h(0,f)\log r\,,$$

where $\bar{n}_h(t,f)$ denotes the number of poles of $f(z)$ in $|z|<t$, poles of order $k$ being counted $k$ times if $k\leqq h$ and $h$ times if $k>h$, for a positive integer $h$.

In this paper, we shall prove a similar result to the first part of Pfluger's theorem by replacing $\delta(a,f)$ by $\delta(\psi,f)$.

**1.** We use symbols $\mu_g$ and $\lambda_g$ as the order and lower order of a meromorphic function $g$, respectively.

First we shall state two lemmas which will be used in the later discussion. The following is known.

CHUANG'S THEOREM ([1]). *Suppose that $f(z)$ is a non-constant meromorphic function and $\psi_k(z)(k=1,\cdots,p)$ are $p$ $(2\leqq p\leqq+\infty)$ distinct meromorphic functions such that*

$$T(r,\psi_k)=o(T(r,f)),\ (r\to\infty),\ (k=1,\cdots,p)$$

*and*

$$\psi_k(z)\not\equiv\infty\,.$$

*Then the inequality*

$$[p-1-o(1)]T(r,f)<\sum_{k=1}^{p}\overline{N}_p\left(r,\frac{1}{f-\psi_k}\right)+p\overline{N}_1(r,f)+S(r,f)$$

*holds, where $S(r,f)=O[\log T(r,f)+\log r]$, as $r\to\infty$ through all values if $f(z)$ is of finite order and as $r\to\infty$ outside a set of finite linear measure otherwise.*

Using this theorem we can prove the following lemma. The proof is obtained in the standard way, so it may be omitted here.

LEMMA 1. *Suppose that $f(z)$ is a non-constant meromorphic function satisfying $\delta(\infty,f)=1$ and that $\psi(z)$ is a meromorphic function satisfying the condition of the above Chuang's theorem. Then the set of functions $\psi(z)$ for which $\delta(\psi,f)>0$ is countable and by summing over all such functions $\psi$, it holds that*

$$\sum_{\psi\not\equiv\infty}\delta(\psi,f)\leqq1\,.$$

The following is due to Edrei-Fuchs [2].

LEMMA 2. *Let $f(z)$ be a meromorphic function of finite order $\mu$ and lower order $\lambda$. Let $s$ be the integer defined by*

$$s-\frac{1}{2}\leqq\lambda<s+\frac{1}{2}.$$

*If*

$$K(f)<\frac{\beta}{5e(s+1)}\quad\left(0<\beta\leqq\frac{1}{2}\right),$$

*then* $s\geqq 1$,

$$|\mu-s|<\frac{\beta}{10}$$

*and*

$$s-\beta\leqq\lambda\leqq\mu<s+\frac{\beta}{10}.$$

**2.** Now we can prove the following.

THEOREM 1. *Suppose that $f(z)$ is a transcendental meromorphic function and that $\psi_i(z)$ $(i=1,\ 2)$ are distinct meromorphic functions such that*

$$T(r,\psi_i)=o(T(r,f))\,,\quad(r\to\infty)$$

*and*

$$\delta(\psi_i,f)>0\,,\quad(i=1,2)\,.$$

*Then the lower order of $f(z)$ is positive.*

PROOF. We consider the following function

$$g(z)=\frac{f(z)-\psi_2(z)}{f(z)-\psi_1(z)}\,.$$

Then we have

$$\delta(0,g)=\delta\left(0,\frac{f-\psi_2}{f-\psi_1}\right)=1-\limsup_{r\to\infty}\frac{N(r,(f-\psi_1)/(f-\psi_2))}{T(r,g)}$$

and

$$N\left(r, \frac{f-\psi_1}{f-\psi_2}\right) = N\left(r, 1 + \frac{\psi_2-\psi_1}{f-\psi_2}\right) \leqq N\left(r, \frac{1}{f-\psi_2}\right) + o(T(r,f)) .$$

Since for $\psi \neq \psi_2$,

$$|T(r, g) - T(r,f)| = o(T(r,f)) ,$$

we obtain

$$(1) \qquad [1 - o(1)]T(r,f) \leqq T(r, g) \leqq [1 + o(1)]T(r,f) .$$

Thus we have

$$\delta(0, g) \geqq 1 - \limsup_{r\to\infty} \frac{N(r, 1/(f-\psi_2)) + o(T(r,f))}{T(r,f) + o(T(r,f))} = \delta(\psi_2, f) > 0 .$$

Similarly we see

$$\delta(\infty, g) \geqq \delta(\psi_1, f) > 0 ,$$

so

$$\max(1 - \delta(0, g), \quad 1 - \delta(\infty, g)) < 1 .$$

Thus by a result of Edrei and Fuchs (see [2, Theorem 4]), the lower order $\lambda_g$ is positive. By (1) we have also $\lambda_f = \lambda_g$. Hence $f(z)$ has a positive lower order. This proves Theorem 1.

Next we prove the following lemma which will be used in the proof of Theorem 2.

LEMMA 3. *Suppose that $f(z)$ is a transcendental meromorphic function of finite order $\mu_f$ satisfying $\delta(\infty, f) = 1$ and that $\psi_k(z)$ $(k = 1, \cdots, p)$ are $p$ $(2 \leqq p \leqq +\infty)$ distinct meromorphic functions of finite order $\mu_{\psi_k}$ satisfying*

$$T(r, \psi_k) = o(T(r,f)) , (r \to \infty) .$$

*If $p = +\infty$, then, for any $\varepsilon > 0$, there exists a positive integer $q$ such that*

$$K(L_q(f)) < 1 - \sum_{k=1}^{\infty} \delta(\psi_k, f) + \varepsilon .$$

*If $p < +\infty$, then*

$$K(L_q(f)) \leqq 1 - \sum_{k=1}^{p} \delta(\psi_k, f) .$$

*Here*

$$L_q(f) = \frac{(-1)^n \Delta(f, \psi_{ki_1}, \cdots, \psi_{ki_n})}{\Delta(\psi_{ki_1}, \cdots, \psi_{ki_n})} \equiv f^{(n)} + \frac{A_1}{A_0} f^{(n-1)} + \cdots + \frac{A_n}{A_0} f,$$

*where* $\psi_{ki_1}, \cdots, \psi_{ki_n}$ *are linearly independent meromorphic functions in* $\{\psi_{k_l}\}_{l=1}^{q} \subset \{\psi_k\}_{k=1}^{p}$ $(q<+\infty, 2 \leqq p \leqq +\infty)$, $\Delta(f, \psi_{ki_1}, \cdots, \psi_{ki_n})$ *is the Wronskian determinant of* $f$, $\psi_{ki_l}$ $(l=1, \cdots, n)$ *and* $A_0 \equiv \Delta(\psi_{ki_1}, \cdots, \psi_{ki_n})$ *that of* $\psi_{ki_l}$ $(l=1, \cdots, n)$.

PROOF. First we consider the case $p=+\infty$. Let

$$F(z) = \sum_{j=1}^{q} \frac{1}{f(z) - \psi_j(z)} .$$

Then Chuang ([1]) proved

$$m(r, F) \geqq \sum_{j=1}^{q} m\left(r, \frac{1}{f-\psi_j}\right) - o(T(r, f)) , \tag{2}$$

for any positive integer $q<+\infty$. We now reform $\{\psi_{ki_l}\}_{l=1}^{n}$ in the following way: $\{\psi_l\}_{l=1}^{n}$. Then

$$N(r, A_i) \leqq \sum_{l=1}^{n} N(r, \psi_l^{(n)}) \leqq (n+1) \sum_{l=1}^{n} N(r, \psi_l) = o(T(r, f)) \tag{3}$$

and

$$\begin{aligned} m(r, A_i) &= m\left(r, \prod_{l=1}^{n} \psi_l D_i\right) = \sum_{l=1}^{n} m(r, \psi_l) + m(r, D_i) + O(1) \\ &= o(T(r, f)) , \end{aligned} \tag{4}$$

$(i=0, 1, \cdots, n)$, where

$$D_{n-k} = \begin{vmatrix} 1\ , & 1\ , \cdots , & 1 \\ \dfrac{\psi_1'}{\psi_1}, & \dfrac{\psi_2'}{\psi_2}, \cdots , & \dfrac{\psi_n'}{\psi_n} \\ \vdots & \vdots & \vdots \\ \dfrac{\psi^{(k-1)}}{\psi_1}, & \dfrac{\psi_2{}^{(k-1)}}{\psi_2}, \cdots , & \dfrac{\psi_n{}^{(k-1)}}{\psi_n} \\ \dfrac{\psi^{(k+1)}}{\psi_1}, & \dfrac{\psi_2{}^{(k+1)}}{\psi_2}, \cdots , & \dfrac{\psi_n{}^{(k+1)}}{\psi_n} \\ \vdots & \vdots & \vdots \\ \dfrac{\psi^{(n)}}{\psi_1}, & \dfrac{\psi_2{}^{(n)}}{\psi_2}, \cdots , & \dfrac{\psi_n{}^{(n)}}{\psi_n} \end{vmatrix}$$

We have

$$\sum_{i=1}^{q} \frac{1}{f-\psi_{k_i}} = \frac{1}{L_q(f)} \sum_{i=1}^{q} \frac{L_q(f)}{f-\psi_{k_i}} = \frac{1}{L_q(f)} \sum_{i=1}^{q} \frac{L_q(f-\psi_{k_i})}{f-\psi_{k_i}}$$

and

$$m\left(r, \sum_{i=1}^{q} \frac{L_q(f-\psi_{k_i})}{f-\psi_{k_i}}\right) \leqq \sum_{i=1}^{q} \sum_{j=1}^{n} m\left(r, \frac{(f-\psi_{k_i})^{(j)}}{f-\psi_{k_i}}\right)$$
$$+ \sum_{i=0}^{n} m(r, A_i) + m\left(r, \frac{1}{A_0}\right) + O(1) = o(T(r,f))\,.$$

Hence we have

$$m\left(r, \sum_{i=1}^{q} \frac{1}{f-\psi_{k_i}}\right) \leqq m\left(r, \frac{1}{L_q(f)}\right) + o(T(r,f))\,,$$

so by ( 2 )

$$\sum_{i=1}^{q} m\left(r, \frac{1}{f-\psi_{k_i}}\right) \leqq m\left(r, \frac{1}{L_q(f)}\right) + o(T(r,f))\,.$$

Thus we obtain

$$\sum_{i=1}^{q} \delta(\psi_{k_i}, f) \leqq \liminf_{r\to\infty} \frac{\sum_{i=1}^{q} m(r, 1/f-\psi_{k_i})}{T(r,f)}$$

$$\leqq \liminf_{r\to\infty} \frac{m(r,1/L_q(f))}{T(r,f)}.$$

On the other hand, we have

$$1-K(L_q(f)) = 1-\limsup_{r\to\infty} \frac{N(r,L_q(f))+N(r,1/L_q(f))}{T(r,L_q(f))}$$

$$= \liminf_{r\to\infty} \frac{m(r,1/L_q(f))+O(1)-N(r,L_q(f))}{T(r,L_q(f))}$$

$$\geqq \liminf_{r\to\infty} \frac{m(r,1/L_q(f))-o(T(r,f))}{T(r,L_q(f))} \limsup_{r\to\infty} \frac{T(r,L_q(f))}{T(r,f)}$$

$$\geqq \liminf_{r\to\infty} \frac{m(r,1/L_q(f))-o(T(r,f))}{T(r,f)} = \liminf_{r\to m} \frac{m(r,1/L_q(f))}{T(r,f)},$$

since it holds

$$\limsup_{r\to\infty} \frac{T(r,L_q(f))}{T(r,f)} \leqq 1$$

form (3) and (4). Further, for any $\varepsilon>0$, there exist a positive integer $q_0$ $(0<q_0<+\infty)$ and $\{\psi_{k_i}\}_{i=1}^{q_0} \subset \{\psi_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} \delta(\psi_k,f) < \sum_{i=1}^{q_0} \delta(\psi_{k_i},f)+\varepsilon.$$

Thus we have

$$\sum_{k=1}^{\infty} \delta(\psi_k,f)-\varepsilon < \sum_{i=1}^{q_0} \delta(\psi_{k_i},f) \leqq 1-K(L_q(f)),$$

so

$$K(L_{q_0}(f)) < 1-\sum_{k=1}^{\infty} \delta(\psi_k,f)+\varepsilon$$

for a positive integer $q_0$.

If $p$ is finite, then in the above discussion we may take $q=q_0=p$. This proves Lemma 3.

THEOREM 2. *Suppose that $f(z)$ is a transcendental meromorphic function* *of finite order* $\mu_f$ *satisfying* $\delta(\infty, f)=1$ *and that* $\psi_k(z)(k=1,\cdots,p)$ *are* $p$ $(2 \leqq p \leqq +\infty)$ *distinct meromorphic functions of finite order* $\mu_{\psi_k}$ *such that*

$$T(r, \psi_k) = o(T(r, f)), \quad (r \to \infty),$$

*and*

$$\mu_f > \mu_{\psi_k} \quad (k = 1, \cdots, p)$$

*and further such that*

$$\sum_{\psi_k \neq \infty} \delta(\psi_k, f) = 1 .$$

*Then the order of* $f(z)$ *is a positive integer and* $f(z)$ *is of regular growth.*

PROOF. First we show that $\mu_{L_q} = \mu_f$ for any positive integer $q$, where $L_q = L_q(f)$ is defined as in Lemma 3. Clearly the inequality $\mu_{L_q} \leqq \mu_f$ holds. We reform $\psi_{k i_l}(l=1,\cdots,n)$, which are linearly independent functions in $\psi_{k_i}(i=1,\cdots,q)$, as follows:

$$\psi_1, \cdots, \psi_n \quad (1 \leqq n \leqq q) .$$

We put again

$$L_q(f) = \frac{(-1)^n \Delta(f, \psi_1, \cdots, \psi_n)}{\Delta(\psi_1, \cdots, \psi_n)} = f^{(n)} + \frac{A_1}{A_0} f^{(n-1)} + \cdots + \frac{A_n}{A_0} f .$$

As mentioned already, $L_q(f)$ is a meromorphic function. We put

$$L_q(f)(z) = P(z) .$$

Then this is a normal and linear $n$-th order differential equation of $f(z)$ and $\psi_i(z)$ $(i=1,\cdots,n)$ are clearly linearly independent solutions of $L_q(f)=0$, that is, the fundamental solutions of $L_q(f)=0$. Let $\varphi(z)$ be a solution of $L_q(f)=0$ through a point $(z_0, f(z_0))$ where $z_0$ is not a pole of $f(z)$. Then $\varphi(z)$ is a linear combination of $\psi_i(z)(i=1,\cdots n)$. A solution of $L_q(f)=P$ is given by

$$f(z) = \varphi(z) + \sum_{k=1}^{n} \psi_k(z) \int_{z_0}^{z} \left\{ \frac{\Delta_k(\psi_1, \cdots, \psi_n)(w)}{\Delta(\psi_1, \cdots, \psi_n)(w)} \right\} P(w) dw ,$$

where

$$\Delta_k(\psi_1,\cdots\psi_n)=\begin{array}{c} \begin{array}{ccccc} & & (k) & & \end{array} \\ \left|\begin{array}{ccccc} \psi_1\ , & \psi_2\ , & \cdots,0,\cdots, & \psi_{n-1}, & \psi_n \\ \psi_1'\ , & \psi_2'\ , & \cdots,0,\cdots, & \psi_{n-1}, & \psi_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \psi_1^{(n-2)}, & \psi_2^{(n-2)}, & \cdots,0,\cdots, & \psi_{n-1}^{(n-2)}, & \psi_n^{(n-2)} \\ \psi_1^{(n-1)}, & \psi_2^{(n-1)}, & \cdots,1,\cdots, & \psi_{n-1}^{(n-1)}, & \psi_n^{(n-1)} \end{array}\right| \end{array}$$

If there exists a pole of order one in the integrand of the right-hand side, then

$$\sum_{k=1}^{n} C_{k,i}\psi_k(z)\equiv 0\,,$$

since $f(z)$ and $\varphi(z)$ are single-valued functions, where $C_{k,i}$ are coefficients of poles $z_i$ of order one in the integrand. On the other hand, functions $\psi_i(z)(i=1,\cdots,n)$ are linearly independent. Thus we have

$$C_{k,i}\equiv 0,\quad (k=1,\cdots,n)\,,$$

and so there are no poles of order one there. Hence

$$\int_{z_0}^{z}\left\{\frac{\Delta_k(\psi_1,\cdots,\psi_n)(w)}{\Delta(\psi_1,\cdots,\psi_n)(w)}\right\}P(w)\,dw$$

is a single-valued meromorphic function.

In general it is well known that a meromorphic function $h(z)$ and its derivative $h'(z)$ are of the same order. Thus, for some $k$,

$$\frac{\Delta_k(\psi_1,\cdots,\psi_n)}{\Delta(\psi_1,\cdots,\psi_n)}P(z)$$

is of order $\mu_f$, so $P(z)$ is of order $\mu_f$ since $\mu_{\psi_k}<\mu_f$. Hence $f(z)$ and $L_q(f)(z)$ are of the same order.

We next show that $f(z)$ is of positive integral order. Let $s$ be the integer defined by

$$(5)\qquad s-\frac{1}{2}\leqq\lambda_{Lq}<s+\frac{1}{2}.$$

Then, for any $\varepsilon > 0$, there exists a positive integer $q$ such that

$$K(L_q(f)) < \frac{\varepsilon}{5e(\mu_{L_q} + (1/2) + 1)} \quad \left(0 < \varepsilon \leqq \frac{1}{2}\right)$$

by Lemma 3 and by our hypothesis. Here $\mu_{L_q}$ is independent of $q (0 < q < +\infty)$. Therefore, by Lemma 2, we have $s \geqq 1$ and

$$|\mu_{L_q} - s| < \frac{\varepsilon}{10},$$

since

$$K(L_q(f)) < \frac{\varepsilon}{5e(\mu_{J_q} + (1/2) + 1)} < \frac{\varepsilon}{5e(s+1)}.$$

On the other hand

$$\mu_{L_q} = \mu_f .$$

Thus, for any $\varepsilon > 0$, we have

$$|\mu_f - s| < \frac{\varepsilon}{10},$$

so $\mu_f$ is a positive integer.

Finally we prove that $f(z)$ is of regular growth. As mentioned above, we have

$$\mu_{L_q} = \mu_f \geqq \lambda_f$$

for any positive integer $q$. We see also

$$\begin{aligned} T(r, L_q(f)) &= m(r, L_q(f)) + N(r, L_q(f)) \\ &\leqq m\left(r, \frac{L_q(f)}{f}\right) + m(r, f) + o(T(r, f)) \\ &\leqq T(r, f) + o(T(r, f)), \end{aligned}$$

so

$$\lambda_{L_q} = \liminf_{r \to \infty} \frac{\log T(r, L_q(f))}{\log r}$$

$$\leqq \liminf_{r\to\infty} \frac{\log[1+o(1)]T(r,f)}{\log r} = \lambda_f .$$

Let $s$ be the integer defined by (5). Then, for any $\varepsilon > 0$, there exists a positive integer $q$ such that

$$K(L_q(f)) < \frac{\varepsilon}{5e(\mu_{Lq}+(1/2)+1)} .$$

Thus, by Lemma 2, we have

$$s - \varepsilon \leqq \lambda_{L_l} \leqq \lambda_f \leqq \mu_f = \mu_{Lq} < s + \frac{\varepsilon}{10} ,$$

so $f(z)$ is of regular growth. This proves Theorem 2.

We note that there exist meromorphic functions $f(z)$ of integral order and $\psi(z)$ of order $\mu_\psi(<\mu_f)$ satisfying $T(r,\psi)=o(T(r,f))$ as $r\to\infty$ and further these functions satisfy relations

$$\sum_{\psi \neq \infty} \delta(\psi, f) = 1$$

and

$$\sum_{a\neq\infty} \delta(a,f) = 0$$

for any finite constant $a$.

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