Tôhoku Math. Journ.
22(1970), 635-658.

ON A MODIFIED DEFICIENCY OF MEROMORPHIC FUNCTIONS

NOBUSHIGE TODA*)

(Received July 13, 1970)

1. Introduction. Let $f(z)$ be a meromorphic function in the finite plane $|z|<\infty$. If the order of $f(z)$ is infinite, then the so-called "exc eptional set" appears in the second fundamental theorem of Nevanlinna. Therefore, in contrast to the case of finite order, we have many difficulties and troubles in the investigation of value distribution of meromorphic functions of infinite order. To avoid some of them, we introduce some new notions, modified characteristic function and deficiencies etc., to the Nevanlinna theory. Applying them to the classical cases, we obtain, for example, that $\sum_{a\neq\infty}\delta(a,f)=1$ and $\delta(\infty,f)=1$ imply that $f(z)$ is of regular growth even if the order of $f(z)$ is infinite. This was proved by Edrei and Fuchs [2] when the order of $f(z)$ is finite.

We use freely the symbols

$$T(r,f),\ m(r,a),\ N(r,a),\ N(r,f),\ \delta(a,f),\ S(r,f)\ \text{etc.}$$

of the Nevanlinna theory of meromorphic functions [5].

2. Definitions. Let $f(z)$ be a meromorphic function in $|z|<\infty$ of order ρ, $0\leqq\rho\leqq\infty$, lower order μ. Denote by α any non-negative number smaller than ρ if ρ is not zero, and zero if $\rho=0$. Take any positive number r_0.

DEFINITION 1. We put

$$T_\alpha(r,r_0;f)=\int_{r_0}^{r}\frac{T(t,f)}{t^{1+\alpha}}dt,\ N_\alpha(r,r_0;a)=\int_{r_0}^{r}\frac{N(t,a)}{t^{1+\alpha}}dt,$$

$$m_\alpha(r.r_0;\ a)=\int_{r_0}^{r}\frac{m(t,a)}{t^{1+\alpha}}dt,\ \overline{N}_\alpha(r,r_0;a)=\int_{r_0}^{r}\frac{\overline{N}(t,a)}{t^{1+\alpha}}dt,$$

and

*) Supported in part by the Sakkokai Foundation

$$S_\alpha(r, r_0; f) = \int_{r_0}^{r} \frac{S(t, f)}{t^{1+\alpha}} dt .$$

The function $T_\alpha(r, r_0; f)$ is called the modified α-characteristic function of $f(z)$.

DEFINITION 2. For any complex number a, finite or not, we define as follows:

$$\delta_\alpha(a, f) = \liminf_{r\to\infty} \frac{m_\alpha(r, r_0; a)}{T_\alpha(r, r_0; f)},$$

$$\Delta_\alpha(a, f) = \limsup_{r\to\infty} \frac{m_\alpha(r, r_0; a)}{T_\alpha(r, r_0; f)}$$

and

$$\Theta_\alpha(a, f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_\alpha(r, r_0; a)}{T_\alpha(r, r_0; f)} .$$

We call $\delta_\alpha(a, f)$ the modified α-deficiency of $f(z)$ at a and the value a satisfying $\delta_\alpha(a, f) > 0$ an α-deficient value of $f(z)$.

In this paper, using these notations given above, we reform some parts of the Nevanlinna theory.

3. Basic properties. In this section, we give some basic properties and fundamental results using the notations in Definitions 1 and 2, which will be needed later.

PROPOSITION 1. 1) *For $1 < k < \infty$, it holds that*

$$T(r, f) \leqq c(k, \alpha) T_\alpha(kr, r_0; f) r^\alpha ,$$

where

$$c(k, \alpha) = \begin{cases} \dfrac{k^\alpha \alpha}{k^\alpha - 1} & for \quad \alpha > 0 \\ \log k & for \quad \alpha = 0 . \end{cases}$$

2) *If $\rho > 0$, then $T_\alpha(r, r_0; f)$ is increasing in r and satisfies the following*:

$$\limsup_{r\to\infty} \frac{\log T_\alpha(r, r_0; f)}{\log r} = \rho - \alpha$$

and

$$\liminf_{r\to\infty} \frac{\log T_\alpha(r, r_0; f)}{\log r} \geqq \max(\mu - \alpha, 0).$$

Further

$$\lim_{r\to\infty} \frac{T_0(r, r_0; f)}{(\log r)^2} = \infty$$

if and only if $f(z)$ is transcendental.

3)

$$N(r, a) \leqq c(k, \alpha)\ N_\alpha(kr, r_0;\ a)\ r^\alpha$$

and

$$\bar{N}(r, a) \leqq c(k, \alpha)\ \bar{N}_\alpha(kr, r_0; a)\ r^\alpha.$$

PROOF. 1) By the definition, for $1 < k < \infty$, we have

$$T_\alpha(kr, r_0;\ f) \geqq \int_r^{kr} \frac{T(t, f)}{t^{1+\alpha}}\, dt \geqq T(r, f) \int_r^{kr} \frac{1}{t^{1+\alpha}}\, dt\ ,$$

because $T(r, f)$ is increasing. Here

$$\int_r^{kr} \frac{1}{t^{1+\alpha}}\, dt = \begin{cases} \log k, & \text{for } \alpha = 0 \\ \dfrac{1}{\alpha r^\alpha}\left(1 - \dfrac{1}{k^\alpha}\right), & \text{for } \alpha \neq 0\,. \end{cases}$$

Therefore, we have 1).

2) It is trivial that $T_\alpha(r, r_0;\ f)$ is increasing in r. We can prove easily the remainder using 1) and definitions.

The proof of 3) is as same as that of 1).

REMARK. 2) implies that $T_\alpha(r, r_0; f)$ is unbounded if $f(z)$ is transcendental.

Many properties concerning the quantities defined in Definition 1 are independent of "r_0" as Proposition 1 shows, so we omit "r_0" in the sequel except when it is necessary.

PROPOSITION 2. *For any finite number a, the first fundamental theorem holds*:

$$T_\alpha(r,f) = m_\alpha(r,f) + N_\alpha(r,f)$$
$$= m_\alpha(r, a) + N_\alpha(r, a) + \varepsilon(r)$$

where

$$\varepsilon(r) = \begin{cases} O(\log r) & \text{for} \quad \alpha = 0 \,, \\ O(1) & \text{for} \quad \alpha \neq 0 \,. \end{cases}$$

PROOF. We obtain this proposition by dividing the both sides of the first fundamental theorem of Nevanlinna by $t^{1+\alpha}$ and integrating from r_0 to r.

COROLLARY. *If $f(z)$ is transcendental, then*

$$\delta_\alpha(a) = 1 - \limsup_{r\to\infty} \frac{N_\alpha(r, a)}{T_\alpha(r,f)}$$

and

$$\Delta_\alpha(a) = 1 - \liminf_{r\to\infty} \frac{N_\alpha(r, a)}{T_\alpha(r,f)} \,.$$

It is trivial by Proposition 1-2) that the right-hand sides of these equalities or of Definition 2 are independent of "r_0".

PROPOSITION 3. *For q complex numbers $a_1, \cdots, a_q$ (finite or not), the second fundamental theorem holds*:

$$(q-2)T_\alpha(r,f) \leqq \sum_{i=1}^{q} N_\alpha(r, a_i) + S_\alpha(r,f) \,,$$

where, for any positive number $r \geqq r_0$,

$$S_\alpha(r,f) = \begin{cases} O\left(\displaystyle\int_{r_0}^{r} \frac{\log^+ T(t, f)}{t^{1+\alpha}}\right) dt & \text{for} \quad \alpha > 0 \,, \\ O((\log r)^2) & \text{for} \quad \alpha = 0 \quad \text{and} \quad \rho < \infty. \end{cases}$$

PROOF. We have only to prove the fact concerning the error term $S_\alpha(r,f)$. Nevanlinna [5] proved

$$\int_{r_0}^{r} \frac{S(t,f)}{t^{1+\alpha}}\,dt = O\left(\int_{r_0}^{r} \frac{\log^+ T(t,f)}{t^{1+\alpha}}\,dt\right)$$

for $\alpha>0$ and $r_0>0$. We have only to use this relation for the case $\alpha\neq 0$. For $\alpha=0$ and $\rho<\infty$, we see

$$S(r,f) = O(\log r),$$

so we have

$$S_0(r,f) = O((\log r)^2).$$

LEMMA 1. *Let $f(z)$ be transcendental. Then*

$$\lim_{r\to\infty} \frac{S_\alpha(r,f)}{T_\alpha(r,f)} = 0$$

for $\alpha>0$ or for $\alpha=0$ and $\rho<\infty$.

PROOF. $T(r,f)$ being increasing and unbounded, for any positive number ε there is a positive number r_1 such that

$$\log T(r,f) \leqq \varepsilon\, T(r,f)$$

for $r\geqq r_1$. Therefore we get

$$S_\alpha(r,r_0;f) = S_\alpha(r_1,r_0;f) + S_\alpha(r,r_1;f)$$

$$= O(1) + O\left(\int_{r_1}^{r} \frac{\log^+ T(t,f)}{t^{1+\alpha}}\right)dt \leqq O(1) + O(\varepsilon T_\alpha(r,r_1;f)).$$

As $T_\alpha(r,f)$ is increasing and unbounded, we see

$$\lim_{r\to\infty} \frac{T_\alpha(r,r_1;f)}{T_\alpha(r,r_0;f)} = 1.$$

From the above inequality, we have the required for $\alpha>0$. In the case when $\alpha=0$ and $\rho<\infty$, the proof is very easy by virtue of Proposition 1-2).

As is easily seen, Lemma 1 gives the following defect relation.

PROPOSITION 4. *If $f(z)$ is transcendental, then the set*

$$N_\alpha = \{a\,;\ \delta_\alpha(a) > 0\}$$

is countable and the inequality

$$\sum_{a \in N_\alpha} \delta_\alpha(a) \leqq 2$$

holds for any admissible α. (Hereafter, we use "admissible α" for $\alpha > 0$ or $\alpha = 0$ and $\rho < \infty$.)

In fact, we can prove this proposition by Lemma 1 as usual.

PROPOSITION 5. *If $f(z)$ is transcendental and if $0 \leqq \alpha < \beta < \rho$, then*

$$\delta(a) \leqq \delta_\alpha(a) \leqq \delta_\beta(a) \leqq \Delta_\beta(a) \leqq \Delta_\alpha(a) \leqq \Delta(a)$$

for any complex number a finite or not.

PROOF. First we shall prove $\delta(a) \leqq \delta_\alpha(a)$. By the definition of $\delta(a)$, for any positive number ε, there is a positive number r_0 such that

$$(\delta(a) - \varepsilon)T(r, f) \leqq m(r, a)$$

for any $r \geqq r_0$. Therefore, we have

$$(\delta(a) - \varepsilon)T_\alpha(r, r_0\,;\ f) \leqq m_\alpha(r, r_0\,;\ a)$$

so that

$$\delta(a) \leqq \delta_\alpha(a)\,.$$

Next, we shall show $\delta_\alpha(a) \leqq \delta_\beta(a)$. Let $\beta - \alpha = \gamma$, which is positive. Integration by parts yields

$$T_\beta(r) = \gamma \int_{r_0}^{r} \frac{T_\alpha(t)}{t^{1+\gamma}}\, dt + \frac{T_\alpha(r)}{r^\gamma}$$

and

$$N_\beta(r, a) = \gamma \int_{r_0}^{r} \frac{N_\alpha(r, a)}{t^{1+\gamma}} dt + \frac{N_\alpha(r, a)}{r^\gamma}$$

where $T_\alpha(r) = T_\alpha(r, f)$.

On the other hand, by the definition of $\delta_\alpha(a)$, for any $\varepsilon > 0$, there is an r_1 such that

$$N_\alpha(r, a) \leqq (1 - \delta_\alpha(a) + \varepsilon) T_\alpha(r, f)$$

for any $r \geqq r_1$. Using these three relations, we have easily

$$\delta_\alpha(a) \leqq \delta_\beta(a) .$$

Similarly, we can prove the remainder.

4. The sum of $\delta_\alpha(a)$. We can prove the following

THEOREM 1. *Let $f(z)$ be a transcendental meromorphic function in $|z| < \infty$. Then,*

$$\sum_{a \neq \infty} \delta_\alpha(a) \leqq \liminf_{r \to \infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqq \limsup_{r \to \infty} \frac{T_\alpha(r, f')}{T_\alpha(r, f)} \leqq 2 - \Theta_\alpha(\infty)$$

for any admissible α to $f(z)$.

PROOF. We note first that $T_\alpha(r, f')$ etc. can be defined well because of the identity of the order of $f(z)$ and that of $f'(z)$. We know that the following inequalities hold for any positive r:

$$(1) \qquad N(r, 1/f') + \sum_{i=1}^{q} m(r, a_i) - S(r, f) \leqq T(r, f')$$

and

$$(2) \qquad T(r, f') \leqq T(r, f) + \overline{N}(r, f) + S(r, f) ,$$

where $a_i (i = 1, \cdots, q)$ are q distinct finite complex numbers. (See [7].)

From (1), we have

$$N_\alpha(r, 1/f') + \sum_{i=1}^{q} m_\alpha(r, a_i) - S_\alpha(r, f) \leqq T_\alpha(r, f) ,$$

so Lemma 1 implies

$$\sum_{a\neq\infty}\delta_\alpha(a) \leqq \liminf_{r\to\infty}\frac{T_\alpha(r,f')}{T_\alpha(r,f)}\,.$$

On the other hand, from (2), we have

$$T_\alpha(r,f') \leqq T_\alpha(r,f) + \bar{N}_\alpha(r,f) + S_\alpha(r,f)$$

and by using Lemma 1

$$\limsup_{r\to\infty}\frac{T_\alpha(r,f')}{T_\alpha(r,f)} \leqq 2-\Theta_\alpha(\infty)\,.$$

THEOREM 2. *Let $f(z)$ be meromorphic and transcendental in $|z|<\infty$. Then for any admissible α,*

$$\frac{1}{2-\Theta_\alpha(\infty)}\sum_{a\neq\infty}\delta_\alpha(a) \leqq \delta_\alpha'(0)\,,$$

where $\delta_\alpha'(a)=\delta_\alpha(a,f')$.

PROOF. From the inequality (1), we have

$$\sum_{i=1}^{q} m_\alpha(r,a_i) - S_\alpha(r,f) \leqq m_\alpha(r,1/f')\,.$$

Hence

$$\sum_{i=1}^{q}\liminf_{r\to\infty}\frac{m_\alpha(r,a_i)}{T_\alpha(r,f)} \leqq \liminf_{r\to\infty}\frac{m_\alpha(r,1/f')}{T_\alpha(r,f')}\limsup_{r\to\infty}\frac{T_\alpha(r,f')}{T_\alpha(r,f)}\,.$$

Using Theorem 1, we obtain this theorem.

5. The order of meromorphic functions with several α-deficient values. We investigate relations between the order and the α-deficient value of $f(z)$.

DEFINITION 3. Let $f(z)$ be transcendental meromorphic in $|z|<\infty$. We define, for admissible α,

$$K_\alpha(f)=\limsup_{r\to\infty}\frac{N_\alpha(r,1/f)+N_\alpha(r,f)}{T_\alpha(r,f)}\,.$$

We can show easily that the right-hand side of the above is independent of r_0.

PROPOSITION 6. *For admissible α and $\beta(>\alpha)$,*

$$K_\beta(f) \leqq K_\alpha(f) \leqq K(f)$$

where

$$K(f) = \limsup_{r\to\infty} \frac{N(r, 1/f) + N(r, f)}{T(r, f)}.$$

We can prove this proposition as in the proof of Proposition 5.

THEOREM 3. *Let $f(z)$ be a meromorphic function of non-integral order ρ, $0<\rho<\infty$. Then, for admissible α, the inequality*

$$K_\alpha(f) \geqq K_\alpha(\rho)$$

holds, where

$$K_\alpha(\rho) \begin{cases} \geqq 1-\rho & \text{for } 0<\rho<1; \\ \geqq \dfrac{(q+1-\rho)(\rho-q)}{\rho c_1(q)} & \text{for } \rho>1 \text{ and } [\rho]=q; \end{cases}$$

and

$$c_1(q) \begin{cases} = 2(p+1)(2+\log(q+1)) & \text{if } q>0, \\ = 1 & \text{if } q=0. \end{cases}$$

PROOF. Let $[\rho]=q$. Then we have the inequality

$$T(r,f) \leqq c_1(q)\left\{qr^q \int_0^r \frac{N(t)}{t^{q+1}}dt + (q+1)r^{q+1}\int_r^\infty \frac{N(l)}{t^{q+2}}dt\right\} + O(r^q)$$

where $N(t) = N(t,0)+N(t,f)$ (See [4] p. 102).

Dividing both sides by $r^{1+\alpha}$ and integrating by parts from r_0 to r, we see easily

$$(3)\quad T_\alpha(r,f) \leqq c_1(q)\left\{qr^{q-\alpha}\int_0^r \frac{N_\alpha(t)}{t^{q+1-\alpha}}\,dt + (q+1)r^{q+1-\alpha}\int_r^\infty \frac{N_\alpha(t)}{t^{q+2-\alpha}}dt\right\} + O(r^{q-\alpha}).$$

We proceed as in the proof of Theorem 4.5 [4]. Given a sufficiently small positive ε and applying Lemma 4.7 [4] for functions

$$\phi(t) = \frac{N_\alpha(t)t^\alpha}{t^{\rho-\varepsilon}}$$

and

$$\psi(t) = t^{2\varepsilon}$$

which satisfy the conditions of the lemma, we can see that there is a sequence $\{r_n\}_{n=1}^\infty$ increasing to infinity such that, for any n,

$$N_\alpha(t)t^\alpha \leqq \left(\frac{t}{r_n}\right)^{\rho-\varepsilon} N_\alpha(r_n)r_n^\alpha \quad (t_0 \leqq t \leqq r_n)$$

and

$$N_\alpha(t)t^\alpha \leqq \left(\frac{t}{r_n}\right)^{\rho+\varepsilon} N_\alpha(r_n)r_n^\alpha \quad (r_n \leqq t < \infty)\,.$$

1) If $\rho<1$, then $q=0$. Hence $c_1(q)=1$ and (3) yields

$$\begin{aligned} T_\alpha(r_n, f) &\leqq r_n^{1-\rho} N_\alpha(r_n) r_n^\alpha \int_{r_n}^\infty \left(\frac{t}{r_n}\right)^{\rho+\varepsilon-2} dt + O(1) \\ &= \frac{N_\alpha(r_n)}{1-\rho-\varepsilon} + O(1)\,. \end{aligned}$$

Since ε may be chosen as small as we please, this implies

$$K_\alpha(f) \geqq \limsup_{n\to\infty} \frac{N_\alpha(r_n)}{T_\alpha(r_n, f)} \geqq 1-\rho\,.$$

2) If $q>0$, then $q<\rho<q+1$. Then we obtain from (3)

$$T_\alpha(r_n, f) - O(r_n^{q-\alpha}) \leqq c_1(q) N_\alpha(r_n) \left\{\frac{q}{\rho-q-\varepsilon} + \frac{q+1}{q+1-\rho-\varepsilon}\right\}.$$

Since there is a constant c such that, for any n,

$$\frac{N_\alpha(r_n)r_n^\alpha}{r_n^{\rho-\varepsilon}} \geqq c > 0\,,$$

we have

$$r_n^{q-\alpha} = o(N_\alpha(r_n)) .$$

(We may consider as $O(r^{q-\alpha}) = O(1)$ if $q \leqq \alpha$.)

Therefore, we obtain

$$K_\alpha(f) \geqq \frac{(q+1-\rho)(\rho-q)}{\rho c_1(q)} .$$

COROLLARY. *If $K_\alpha(f) = 0$ for some admissible α, then the order of $f(z)$ is integer or infinite.*

THEOREM 4. *Let $f(z)$ be meromorphic of order ρ and lower order μ in $|z| < \infty$. If*

$$K_\alpha(f) < 1$$

for some admissible α, then

$$\rho \geqq 1 \quad \textit{and} \quad \mu \geqq 1-\alpha \quad \textit{for} \quad K_\alpha(f) = 0$$

and

$$\rho \geqq \frac{\log \dfrac{1}{K_\alpha(f)(2-K_\alpha(f))}}{\log\left(1 + \dfrac{4}{K_\alpha(f)(1-K_\alpha(f))}\right)} \quad \textit{for} \quad K_\alpha(f) > 0 .$$

PROOF. We use the method of Edrei and Fuchs ([2], Th. 3a). They proved the following inequality

$$T(r,f) \leqq \frac{4}{\sigma-1} T(\sigma r, f) + \max(N(\sigma r, 0), N(\sigma r, f)) + O(\log r)$$

for $\sigma > 1$ and $r > 2$.

From this, we can deduce easily

$$T_\alpha(r,f) \leqq \frac{4\sigma^\alpha}{\sigma-1} T_\alpha(\sigma r, f) + \sigma^\alpha N_\alpha(\sigma r) + S_{1,\alpha}(r)$$

where

$$S_{1,\alpha}(r) = O(1) \quad \text{for} \quad \alpha > 0, = O((\log r)^2) \quad \text{for} \quad \alpha = 0\,.$$

Let c and c be two positive number such that

$$K_\alpha(f) < c' < c < 1\,.$$

Then, by the definition of $K_\alpha(f)$, it holds that

$$N_\alpha(r) < c\, T_\alpha(r, f)$$

for all sufficiently large r.

Since $f(z)$ is clearly not a rational function, we see

$$\lim_{r\to\infty} \frac{S_{1,\alpha}(r)}{T_\alpha(r)} = 0$$

by Proposition 1, and hence

$$c' + \frac{S_{1,\alpha}(r)}{T_\alpha(r)} < c$$

for all sufficiently large r. Let

$$\sigma = 1 + \frac{4}{c(1-c)}\,.$$

Then we have for all sufficiently large r

$$T_\alpha(r, f) \leqq T_\alpha(\sigma r, f)\sigma^\alpha c(2-c)\,.$$

Therefore, as in the proof of Theorem 4 [2], we obtain

$$\liminf_{r\to\infty} \frac{\log T_\alpha(r,f)}{\log r} \geqq -\alpha + \frac{\log \dfrac{1}{c(2-c)}}{\log \sigma}\,.$$

By Proposition 1, we have our theorem by letting $c \to K_\alpha(f)$.

THEOREM 5. *Let $f(z)$ be a transcendental meromorphic function in $|z| < \infty$ of order $\rho(\leqq \infty)$ and lower order μ, $0 < \mu < \rho$ and let τ and $\alpha(>0)$ be any number such that*

1) *τ is not an integer such that $\mu<\tau$;*
2) *$\alpha<\mu$ and $\tau+\alpha<\rho$*

and

3) $[\tau]=[\tau+\alpha]$.

Then

$$K_\alpha(f) \geqq \frac{2|\sin \pi(\tau+\alpha)|}{K(\alpha,\tau)+|\sin \pi(\tau+\alpha)|}$$

where $K(\alpha,\tau)$ is a positive constant depending only on α and τ.

PROOF. We use some inequalities proved by Edrei [3]. Let $[\tau]=q$. Edrei proved that

$$(4) \qquad 2T(r,f)-N(r) \leqq r^q \int_0^\infty \frac{n_R{}^*(t)}{t^{q+1}} \phi\left(\frac{t}{r}\right) dt + K(r^q+\log r) + 14\left(\frac{r}{R}\right)^{q+1} T(2R) \quad \text{for} \quad r_0 \leqq r=|z| \leqq \frac{1}{2}R,$$

where

$$n_R^*(t) = \sum_{0<|a_\mu|\leqq \min(t,R)} 1 + \sum_{0<|b_\nu|\leqq \min(t,R)} 1 ,$$

a_μ and b_ν being the zeros and the poles of $f(z)$,

$$\phi(t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{(t^2-2t\cos\theta+1)^{1/2}}$$

and K is an absolute constant.

Let

$$\mu<\sigma<\tau .$$

Since, the order of $T_\alpha(r,f)$ is equal to $\rho-\alpha$ and lower order of $T_\alpha(r,f)$ is at most μ, we can apply Lemma 1 [3] for $T_\alpha(r,f)$ and σ, τ chosen as above. We have a sequence $\{r_l\}_{l=1}^\infty$ increasing to infinity such that

$$(5) \qquad \frac{T_\alpha(t,f)}{t^\tau} \leqq \frac{T_\alpha(r_l,f)}{r_l^\tau} \quad (1 \leqq t \leqq r_l^{\tau/\sigma}).$$

and

$$(6) \qquad \lim_{l\to\infty} \frac{T_\alpha(r_l, f)}{r_l^\tau} = \infty .$$

We estimate the right-hand side of the inequality (4). First let

$$I(r) = r^q \int_0^\infty \frac{n_R^*(t)}{t^{q+1}} \phi\left(\frac{t}{r}\right) dt .$$

Then, using the inequality

$$\phi(t) \leqq 2/t \quad (2 \leqq t),$$

we obtain

$$I(r) \leqq r^q \int_\beta^R \frac{n(t)}{t^{q+1}} \phi\left(\frac{t}{r}\right) dt + KT(2R)\left(\frac{r}{R}\right)^{q+1} \quad (r_0 \leqq r \leqq R/2),$$

where $n(t) = n(t, 0) + n(t, \infty) - n(0, 0) - n(0, \infty)$ and $n(\beta) = n(0)$, $\beta > 0$.

Since

$$\frac{n(t)}{\tau+1} \leqq \int_t^{t(1+1/\tau)} \frac{n(x)}{x} dx \leqq N(t(1+1/\tau)),$$

we have

$$I(r) \leqq r^q (1+1/\tau)^q (\tau+1) \int_{\beta(1+1/\tau)}^{R(1+1/\tau)} \frac{N(t)}{t^{q+1}} \phi\left(\frac{t}{r(1+1/\tau)}\right) dt$$
$$+ KT(2R)\left(\frac{r}{R}\right)^{q+1} \quad (r_0 \leqq r \leqq R/2).$$

Using, here, Proposition 1-3) for $k=2$, that is, the inequality

$$N(t) \leqq \frac{2^\alpha \alpha}{2^\alpha - 1} t^\alpha N_\alpha(2t),$$

we obtain

$$\int_{\beta(1+1/\tau)}^{R(1+1/\tau)} \frac{N(t)}{t^{q+1}} \phi\left(\frac{t}{r(1+1/\tau)}\right) dt \leqq \frac{2^\alpha \alpha}{2^\alpha - 1} \int_{\beta(1+1/\tau)}^{R(1+1/\tau)} \frac{N_\alpha(2t)}{t^{q+1-\alpha}} \phi\left(\frac{t}{r(1+1/\tau)}\right) dt$$
$$= \frac{2^{q+1}\alpha}{2^\alpha - 1} \int_{2\beta(1+1/\tau)}^{2R(1+1/\tau)} \frac{N_\alpha(t)}{t^{q+1-\alpha}} \phi\left(\frac{t}{2r(1+1/\tau)}\right) dt = I_1 .$$

For any positive number ε, there is a t_0 such that

$$N_\alpha(t) \leqq (K_\alpha(f)+\varepsilon)T_\alpha(t,f) \quad (t \geqq 2t_0).$$

So

$$(7) \qquad I_1 \leqq \frac{2^{q+1}\alpha}{2^\alpha-1}\left\{(K_\alpha(f)+\varepsilon)\int_{2t_0}^{2R(1+1/\tau)} \frac{T_\alpha(t)}{t^{q+1-\alpha}}\phi\left(\frac{t}{2r(1+1/\tau)}\right)dt + \int_{2\beta(1+1/\tau)}^{2t_0} \frac{N_\alpha(2t_0)}{t^{q+1-\alpha}}\phi\left(\frac{t}{2r(1+1/\tau)}\right)dt\right\}.$$

Now, we estimate the first term of the right-hand side of the above inequality by using (5) for the sequence $\{r_l\}_{l=1}^\infty$.

We put

$$4R_l(1+1/\tau) = r_l^{\tau/\sigma} \quad (l=0,1,\cdots).$$

Then, for $r_0 \leqq r \leqq R_l/2$, it holds that

$$I_2 = \int_{2t_0}^{2R_l(1+1/\tau)} \frac{T_\alpha(t)}{t^{q+1-\alpha}}\phi\left(\frac{t}{2r(1+1/\tau)}\right)dt \leqq \frac{T_\alpha(r_l)}{r_l^\tau}\int_{2t_0}^{2R_l(1+1/\tau)} \frac{1}{t^{q+1-\alpha-\tau}}\phi\left(\frac{t}{2r(1+1/\tau)}\right)dt.$$

Changing $t/(2r(1+1/\tau))$ to u, we see that the right-hand side of the above is equal to

$$\frac{T_\alpha(r_l)}{r_l^\tau}(2(1+1/\tau))^{\tau+\alpha-q}r^{\tau+\alpha-q}\int_{2t_0/r(1+1/\tau)}^{R_l/r} u^{\tau+\alpha-q-1}\phi(u)du.$$

Since

$$\int_0^\infty t^{\beta-1}\phi(t)dt < \frac{4.4}{\sin\pi\beta} \quad (0<\beta<1),$$

for $\beta=\tau+\alpha-q$, we see

$$I_2 \leqq \frac{T_\alpha(r_l)}{r_l^\tau}(2(1+1/\tau))^{\tau+\alpha-q}r^{\tau+\alpha-q}\frac{4.4}{|\sin\pi(\tau+\alpha)|}.$$

Next, we estimate the second term of the right-hand side of (7):

$$(8)\qquad I_3 = N_\alpha(2t_0)\int_{2\beta(1+1/\tau)}^{2t_0} t^{-q-1+\alpha}\phi\left(\frac{t}{2r(1+1/\tau)}\right)dt$$

$$\leqq (2t_0)^\alpha(2(1+1/\tau)r)^{-q}N_\alpha(2t_0)\int_{\beta/r}^{t_0/r(1+1/\tau)} u^{-q-1}\phi(u)du$$

$$= (2t_0)^\alpha(2(1+1/\tau)r)^{-q}N_\alpha(2t_0)\int_{\beta/r}^{t_0/r(1+1/\tau)} u^{-q-2}\phi\left(\frac{1}{u}\right)du$$

because ϕ satisfies the relation

$$u\phi(u) = \phi\left(\frac{1}{u}\right).$$

Changing $1/u$ to t, we see that the last term in (8) equals

$$(2t_0)^\alpha(2(1+1/\tau)r)^{-q}N_\alpha(2t_0)\int_{r(1+1/\tau)/t_0}^{r/\beta} t^q\phi(t)dt\,.$$

Since

$$\phi(t) < 2/t$$

for $t \geqq 2$, we see for $r(\geqq 2t_0)$ that

$$(9)\qquad I_3 \leqq (2t_0)^\alpha 2(2(1+1/\tau)r)^{-q}N_\alpha(2t_0)\int_2^{r/\beta} t^{q-1}dt$$

$$\leqq \begin{cases} \dfrac{2(2t_0)^\alpha N_\alpha(2t_0)}{q(2\beta(1+1/\tau))^q} & \text{for } q>0 \\ (2t_0)^\alpha 2N_\alpha(2t_0)\log r/2\beta & \text{for } q=0\,. \end{cases}$$

Therefore, we have

$$(10)\quad I(r) \leqq r^q(1+1/\tau)(1+\tau)\frac{2^{q+1}\alpha}{2^\alpha-1}(K_\alpha(f)+\varepsilon)\frac{T_\alpha(r_l)}{r_l^\tau}(2(1+1/\tau)r)^{\tau+\alpha-q}\frac{4.4}{|\sin\pi(\tau+\alpha)|}$$

$$+ I_3 + KT(2R_l)\left(\frac{r}{R_l}\right)^{q+1}$$

for $\gamma = \max(r_0, 2t_0) \leqq r \leqq R_l/2$.

Combining (4), (9) and (10), dividing by $r^{1+\alpha}$ and integrating with respect to r from γ to $r(\leqq R_l/2)$, we have

$$2T_\alpha(r,\gamma;f)-N_\alpha(r,\gamma)$$

$$\leqq (1+1/\tau)^{\tau+\alpha+1}(\tau+1)\frac{2^{\tau+\alpha+1}\alpha}{2^\alpha-1}(K_\alpha(f)+\varepsilon)\frac{T_\alpha(r_l)}{r_l^\tau}\frac{4.4}{|\sin\pi(\tau+\alpha)|}\int_\gamma^r r^{\tau-1}dr$$

$$+K_1\int_\gamma^r\left(r^{q-1-\alpha}+\frac{\log r}{r^{1+\alpha}}\right)dr+K_2\frac{T(2R_l)}{R_l^{q+1}}\int_\gamma^r r^{q-\alpha}dr\,,$$

where K_1 and K_2 are positive constants.

This reduces to

(11) $$2T_\alpha(r,\gamma;f)-N_\alpha(r,\gamma)$$

$$\leqq\frac{2^{\tau+\alpha+1}\alpha}{2^\alpha-1}(1+1/\tau)^{\tau+\alpha+2}(K_\alpha(f)+\varepsilon)\frac{4.4}{|\sin\pi(\tau+\alpha)|}\frac{T_\alpha(r_l,r_0;f)}{r_l^\tau}r^\tau$$

$$+K_1\int_\gamma^r\left(r^{q-1-\alpha}+\frac{\log r}{r^{1+\alpha}}\right)dr+\frac{K_2}{q+1-\alpha}\frac{T(2R_l)}{R^{q+1}}r^{q+1-\alpha}.$$

By the definition, we have for any positive ε

(12) $$(2-K_\alpha(f)-\varepsilon)T_\alpha(r,\gamma;f)\leqq 2T_\alpha(r,\gamma;f)-N_\alpha(r,\gamma)$$

for all sufficiently large r.

As

$$\frac{R_l}{r_l}=\frac{r_l^{\tau/\sigma-1}}{4(1+1/\tau)}\nearrow\infty\quad(l\to\infty)\,,$$

we can take

$$r=r_l$$

in the above discussion for all sufficiently large l. We can prove easily

$$\lim_{r\to\infty}\frac{T_\alpha(r,\gamma,f)}{T_\alpha(r,r_0;f)}=1$$

and by (6) we get

$$K_1\int_\gamma^{r_l}\left(r^{q-1-\alpha}+\frac{\log r}{r^{1+\alpha}}\right)dt=o(T_\alpha(r_l,f))\quad(l\to\infty)\,.$$

Further

$$\frac{T(2R_l)}{T_\alpha(r_l, r_0; f)} \cdot \frac{r_l^{q+1-\alpha}}{R_l^{q+1}} \leqq K_3 \frac{T_\alpha(4R_l, r_0; f)}{T_\alpha(r_l, r_0; f)} \left(\frac{r_l}{R_l}\right)^{q+1-\alpha}$$

$$\leqq K_3 \frac{(4R_l)^\tau}{r_l^\tau} \left(\frac{r_l}{R_l}\right)^{q+1-\alpha} = K_4 \left(\frac{r_l}{R_l}\right)^{q+1-\tau-\alpha}$$

by Proposition 1-1), (5) and the choice of R_l. Since $R_l = r_l^{\tau/\sigma}/4(1+1/\tau)$, the last term of the above inequalities equals

$$K_5 r_l^{\left(1-\frac{\tau}{\sigma}\right)(q+1-\tau-\alpha)}$$

which tends to zero as $l \to 0$ by virtue of $\left(1 - \frac{\tau}{\sigma}\right)(q+1-\tau-\alpha) < 0$, where K_3, K_4 and K_5 are positive constants.

Putting $r = r_l$ in (11), combining (11) and (12) and using above properties, we have, ε being as small as we please,

$$2 - K_\alpha(f) \leqq \frac{K(\alpha, \tau)\, K_\alpha(f)}{|\sin\pi(\tau+\alpha)|}$$

where

$$K(\alpha, \tau) = 2.\, 2(2(1+1/\tau))^{\tau+\alpha+2} \cdot \alpha/(2^\alpha - 1)\,.$$

This implies

$$K_\alpha(f) \geqq \frac{2|\sin\pi(\tau+\alpha)|}{K(\alpha, \tau) + |\sin\pi(\tau+\alpha)|}\,,$$

which is the desired.

COROLLARY 1. *Let $f(z)$ be a meromorphic function in $|z| < \infty$ of order $\rho(\leqq \infty)$ and lower order μ. If, for some admissible α, $0 < \alpha < 1/2$,*

$$K_\alpha(f) = 0\,,$$

1) *when $\rho < \infty$, then*

$$\rho - \mu \leqq \alpha\,;$$

2) *when $\rho=\infty$, then $f(z)$ is of regular growth.*

PROOF. 1) The case $\rho<\infty$. By Corollary of Theorem 3, ρ is an integer. Further, by Theorem 4, we have

$$\mu \geqq 1-\alpha,$$

so

$$\mu-\alpha>0.$$

If $\rho \neq \mu$, then clearly $\rho>[\mu]$. First, we shall prove that

$$\rho-[\mu]=1.$$

Suppose that

$$\rho-[\mu] \geqq 2.$$

Let

$$\tau=[\mu]+1+\varepsilon, \quad 0<\varepsilon<1/2.$$

Then

$$[\tau]=[\tau+\alpha]=[\mu]+1, \tau+\alpha<\rho.$$

Therefore, Theorem 5 implies that

$$K_\alpha(f)>0,$$

which is a contradiction. This shows

$$\rho-[\mu]=1.$$

We have also

$$\rho \geqq 1.$$

Now, we prove that

$$\rho-\mu \leqq \alpha.$$

If

$$\rho - \mu > \alpha ,$$

then, there is a non integral number τ such that

$$\mu < \tau, \quad \tau + \alpha < \rho$$

and so

$$[\tau] = [\tau + \alpha] = [\mu] .$$

As these numbers satisfy the conditions of Theorem 5, we obtain by Theorem 5

$$K_\alpha(f) > 0 ,$$

which is a contradiction. This implies $\rho - \mu \leqq \alpha$.

2) The case $\rho = \infty$. Suppose that $\mu < \infty$. By Theorem 4,

$$\mu \geqq 1 - \alpha .$$

Hence

$$\mu - \alpha > 0 .$$

Let

$$\tau = [\mu] + 1 + \varepsilon , \quad 0 < \varepsilon < 1/2 .$$

Then

$$[\tau] = [\tau + \alpha] = [\mu] + 1, \tau + \alpha < \infty.$$

As these numbers satisfy the conditions of Theorem 5, we have by Theorem 5

$$K_\alpha(f) > 0 ,$$

which is a contradiction. Thus we have

$$\mu = \infty .$$

COROLLARY 2. *If, for any* $\alpha > 0$,

$$K_\alpha(f) = 0 ,$$

then, $f(z)$ *is of regular growth; and if* $\rho < \infty$, *then* ρ *is a positive integer.*

LEMMA 2. *Let $f(z)$ be a transcendental meromorphic function in $|z|<\infty$. Then,*

$$\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log T(r,f')}{\log r}$$

$$\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log T(r,f')}{\log r}$$

(See [1]).

THEOREM 6. *Let $f(z)$ be a transcendental meromorphic function in $|z|<\infty$ of order ρ and lower order μ. If, for some admissible α, $0<\alpha<1/2$,*

$$\sum_{a\neq\infty}\delta_\alpha(a)=1 \quad and \quad \delta_\alpha(\infty)=1\,,$$

then, 1) *if $\rho<\infty$, then*

$$\rho-\mu\leqq\alpha\,;$$

2) *if $\rho=\infty$, then $f(z)$ is of regular growth, that is, $\rho=\mu$.*

PROOF. By the definition of $\Theta_\alpha(\infty)$, we see

$$1=\delta_\alpha(\infty)\leqq\Theta_\alpha(\infty)\leqq 1\,.$$

By Theorem 2, we have

$$\delta_\alpha'(0)=1\,. \tag{13}$$

On the other hand, it holds that

$$N(r,f')\leqq 2N(r,f)\,,$$

so we obtain

$$N_\alpha(r,f')\leqq 2N_\alpha(r,f)\,.$$

In our case, by Theorem 1,

$$\lim_{r\to\infty}\frac{T_\alpha(r,f')}{T_\alpha(r,f)}=1\,,$$

so, by the hypothesis,

$$0 \leqq \limsup_{r\to\infty} \frac{N_\alpha(r, f')}{T_\alpha(r, f')} \leqq 2 \limsup_{r\to\infty} \frac{N_\alpha(r, f)}{T_\alpha(r, f)} = 0 .$$

This shows that

$$\delta_\alpha'(\infty) = 1 . \tag{14}$$

By (13) and (14)

$$0 \leqq K_\alpha(f') \leqq 2 - \delta_\alpha'(0) - \delta_\alpha'(\infty) = 0 .$$

Therefore, by Corollary of Theorem 5 and Lemma 2, we have this theorem.

COROLLARY. *Let $f(z)$ be transcendental meromorphic in $|z|<\infty$. If*

$$\sum_{a \neq \infty} \delta(a) = 1 \quad \textit{and} \quad \delta(\infty) = 1 ,$$

then, for any admissible α,

$$\sum_{a \neq \infty} \delta_\alpha(a) = 1 \quad \textit{and} \quad \delta_\alpha(\infty) = 1 .$$

Therefore, in this case, $f(z)$ is of regular growth even if the order of $f(z)$ is infinite.

PROOF. We have by Proposition 4

$$\delta_\alpha(\infty) = 1 ,$$

and using Corollary of Proposition 3 we see

$$1 = \sum_{a \neq \infty} \delta(a) \leqq \sum_{a \neq \infty} \delta_\alpha(a) \leqq 1 .$$

PROPOSITION 7. *Let $f(z)$ be a meromorphic function in $|z|<\infty$. If, for any admissible α,*

$$\sum_{a} \delta_\alpha(a) = 2 ,$$

then

$$\delta_\alpha(a) = \Delta_\alpha(a)$$

for any complex number a finite or not.

We can prove this easily by using Theorems 1 and 2.

6. Supplement to [6]. In our former paper [6], there are some gaps in the proofs of Theorems 1, 2, 4 and 5, where we used Lemmas 2, 4 and 7 which are valid only for meromorphic functions of finite order. That is, we concluded that, if a meromorphic function in $|z|<\infty$ is of lower order finite and has the deficient values such that

$$\sum_{a \neq \infty} \delta(a) = 1 \quad \text{and} \quad \delta(\infty) = 1\,,$$

then the function is of order finite and of regular growth. This can not be proved by Lemmas 2, 4 and 7 in [6]. But, we can prove this by Corollary of Theorem 6 and can cover these gaps in the proofs of Theorems 1, 2, 4 and 5 in [6].

Addendum. We can give an example of meromorphic functions in $|z|<\infty$ such that

$$\delta_\alpha(a) \neq \delta(a)$$

for some admissible α and some value a.

In fact, let $f_1(z)$ be an entire function of order ρ and lower order μ such that $\rho \neq \mu$, $f_2(z)$ an entire function of regular growth and of order λ, $\mu<\lambda<\rho$. There is a finite value w_0 not exceptional in the sense of Valiron such that $f_1(z)$ and $f_3(z)=f_2(z)-w_0$ have no common zero. Then the meromorphic function

$$f(z) = f_1(z)/f_3(z)$$

is of order ρ and lower order not less than λ. Further,

$$N(r,f) = N(r,0,f_3)$$

and

$$T(r,f) \leqq T(r,f_1) + T(r,f_3) + O(1)\,.$$

By the properties of $f_1(z)$ and $f_3(z)$, we can verify easily that

$$\delta(\infty,f) \leqq 1/2$$

and that if $\lambda < \alpha < \rho$, then

$$\delta_{\alpha}(\infty, f) = 1 .$$

S. Mori told me that by another method he could also construct such an example of meromorphic functions as mentioned above.

REFERENCES

[1] C.-T. CHUANG, Sur la comparaison de la croissance d'une fonction méromorphe et de celle de sa dérivée, Bull. Sci. Math., 75(1951), 171–190.
[2] A. EDREI AND W. H. J. FUCHS, On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc., 93(1959), 292–328.
[3] A. EDREI, The deficiencies of meromorphic functions of finite lower order, Duke Math. J., 31(1964), 1–22.
[4] W. K. HAYMAN, Meromorphic functions, Oxford Math. Mono. 1964.
[5] R. NEVANLINNA, Le théoréme de Picard-Borel et la théorie des fonctions méromorphes, Gauthier-Villars, Paris, 1929.
[6] N. TODA, Sur quelques fonctions entières ou méromorphes à l'égard de la conjecture de Paley, Tôhoku Math. J., 21(1969), 654–667.
[7] E. ULLRICH, Über die Ableitung einer meromorphen Funktionen, Sitzung des phys.-math. Klasse 7(1929), 592–608.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, JAPAN