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# A NOTE ON THE ALGEBRA OF MEASURABLE OPERATORS OF AN $AW^*$-ALGEBRA

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1. **Introduction**. This note is concerned with the relation between an $AW^*$-algebra $A$ and the algebra $M$ of 'measurable operators' affiliated with $A$ [4], the general idea being that properties of $A$ are reflected in analogous properties of the larger algebra $M$. We consider here two specific properties: (1) the property of being a Baer*-ring, and (2) the monotonicity of the square root operation on positive elements.

More precisely, to say that $A$ is an $AW^*$-algebra means that $A$ is both a $C^*$-algebra and a Baer*-ring. K. Saitô has shown that $M$ is also a Baer *-ring ([4], Theorem 6. 4); we offer here a simpler proof, of a nominally more general result. The algebra $A$ has the property that $0\leqq a\leqq b$ implies $a^{1/2}\leqq b^{1/2}$ (this is true in any $C^*$-algebra); assuming the set of self-adjoint elements of $A$ has a certain 'monotone convergence property' (automatically verified when $A$ is a von Neumann algebra), we show that $M$ has the property that $0\leqq x\leqq y$ implies $x^{1/2}\leqq y^{1/2}$. The latter result plays a role in D. Topping's theory of vector lattices of self-adjoint operators ([5], see p. 27, Proposition 10).

2. **The Baer*-ring property**. Let $A$ be an $AW^*$-algebra and let $M$ be the algebra of measurable operators affiliated with $A$ [4]. We extract as axioms the properties of $A$ and $M$ that are relevant for the discussion in this section: (1) $M$ is a *-algebra with unity element 1, (2) $A$ is a *-subalgebra of $M$, (3) if $x, y\in M$ and $x^*x+y^*y=1$, then $x, y\in A$ ([4], Lemma 5. 2), (4) $A$ is a Baer*-ring in the sense of I. Kaplansky [3], and (5) for any $x\in M$, $1+x^*x$ is invertible and $(1+x^*x)^{-1}\in A$ ([4], Lemmas 4. 1, 5. 2).

Taking (1)-(5) as axioms, we show in this section that $M$ is also a Baer*-ring. We write

$$a_x = (1 + x^*x)^{-1} \qquad (x\in M).$$

Note that $x^*x=0$ implies $x=0$. {Proof: $1=0+1=x^*x+1^*1$, therefore $x\in A$ by (3), hence $x=0$ by a property of Baer*-rings ([3], p. 31, Theorem 21).} Also, $M$ has no new partial isometries (i. e., none not already in $A$), hence no new projections.

{Proof: If $w^*w=e$, $e$ a projection, then $1=e+(1-e)=w^*w+(1-e)^*(1-e)$, therefore $w \in A$ by (3).} It follows that the set of projections of $M$ (i.e., of $A$) is a complete lattice ([3], p. 29, Theorem 19). Every element of $A$ has a right projection ([3], p. 28); so does every element of $M$:

LEMMA 2.1. *If $x \in M$ and $e=RP(1-a_x)$ (the right projection of $1-a_x$, calculated in $A$), then $xe=x$ and $e$ is the smallest such projection.*

PROOF. If $g$ is a projection, the following conditions imply one another: $xg=0$, $x^*xg=0$, $(1+x^*x)g=g$, $g=(1+x^*x)^{-1}g=a_xg$, $(1-a_x)g=0$, $eg=0$. The largest such projection is $g=1-e$, thus $1-e$ is the largest projection such that $x(1-e)=0$; that is, $e$ is the smallest projection such that $xe=x$.

With notation as in Lemma 2.1, we write $e=RP(x)$; that is, $RP(x)$ is defined to be $RP(1-a_x)$. We also define $LP(x)=RP(x^*)$; thus, $LP(x)$ is the smallest projection $f$ such that $fx=x$. By definition,

$$LP(x) = RP(x^*) = RP(1-a_{x^*}).$$

LEMMA 2.2. *If $x, y \in M$, $e=RP(x)$ and $g=LP(y)$, then $xy=0$ iff $eg=0$.*

PROOF. If $eg=0$ then $xy=(xe)(gy)=0$. Conversely, if $xy=0$ then $x^*xyy^*=0$, $(1+x^*x)yy^*=yy^*$, $yy^*=(1+x^*x)^{-1}yy^*=a_xyy^*$, $(1-a_x)yy^*=0$, $(1-a_x)(1+yy^*)=1-a_x$, $1-a_x=(1-a_x)(1+yy^*)^{-1}=(1-a_x)a_{y^*}$, $(1-a_x)(1-a_{y^*})=0$; since $A$ is a Baer*-ring this implies $[RP(1-a_x)][RP(1-a_{y^*})]=0$ (for a self-adjoint element, we need not distinguish between $LP$ and $RP$), that is, $RP(x)LP(y)=0$.

Generalizing Saitô's theorem for the case that $A$ is an $AW^*$-algebra ([4], Theorem 6.4), we have:

THEOREM 2.3. *If $A$ and $M$ satisfy axioms* (1)-(5), *then $M$ is a Baer*-ring.*

PROOF. Let $S$ be any subset of $M$ and write $R(S)=\{y \in M: Sy=0\}$ for the right-annihilator of $S$; the problem is to show that the right ideal $R(S)$ is generated by a projection. Define $e=\sup\{RP(x): x \in S\}$. In view of Lemma 2.2, the following conditions imply one another: $y \in R(S)$, $xy=0$ for all $x \in S$, $RP(x)LP(y)=0$ for all $x \in S$, $eLP(y)=0$, $ey=0$, $(1-e)y=y$, $y \in (1-e)M$. Thus $R(S)=(1-e)M$.

3. **Monotonicity of square roots.** Let $A$ be an $AW^*$-algebra, $M$ the

algebra of measurable operators of $A$. In the application [5], $A$ is assumed to be of finite class; we do not make this restriction here, but, for the reader who is interested only in the finite case, we cite alternative references to [1].

LEMMA 3.1. *If $x \in M$, $x \geqq 0$ and $x$ is invertible in $M$, then $x^{-1} \geqq 0$.*

PROOF. Write $x = z^2$, with $z$ self-adjoint ([4], Corollary 5.2; [1], Corollary 6.2). Then $z$ is invertible and $x^{-1} = (z^{-1})^2 = (z^{-1})^*(z^{-1}) \geqq 0$.

LEMMA 3.2. *If $a \in A$, $0 \leqq a \leqq 1$, and $a$ has an inverse in $M$, then $a^{-1} \geqq 1$.*

PROOF. Same as ([2], p. 179, Lemma 2).

LEMMA 3.3. *If $x$ and $y$ are invertible elements of $M$ such that $0 \leqq x \leqq y$, then $x^{-1} \geqq y^{-1} \geqq 0$.*

PROOF. By Lemma 3.1, $y^{-1} \geqq 0$. Write $x = s^2$, $y = t^2$, with $s$ and $t$ self-adjoint, and set $w = st^{-1}$. Then $w^*w = t^{-1}s^2t^{-1} = (t^{-1})^*xt^{-1} \leqq (t^{-1})^*yt^{-1} = t^{-1}yt^{-1} = 1$, thus $w \in A$, $\|w\| \leqq 1$ ([4], Lemma 5.2; [1], Lemma 5.1). Since $w$ is invertible in $M$, so is $w^*w$; citing Lemma 3.2 we have $1 \leqq (w^*w)^{-1} = w^{-1}(w^{-1})^* = ts^{-1}s^{-1}t = tx^{-1}t$, therefore $y^{-1} = t^{-1}1t^{-1} \leqq t^{-1}(tx^{-1}t)t^{-1} = x^{-1}$.

LEMMA 3.4. *If $x, y \in M$, $0 \leqq x \leqq y$, $xy = yx$, and $y \geqq \varepsilon 1$ for some real number $\varepsilon > 0$, then $x^{1/2} \leqq y^{1/2}$.*

PROOF. We can suppose $\varepsilon = 1$. Then $y = 1 + [(y-1)^{1/2}]^2$ shows that $y$ is invertible ([4], Lemma 4.1; [1], Corollary 3.1), and $y^{-1} \leqq 1$ by Lemma 3.3. Write $x = s^2, y = t^2$, with $s \geqq 0$ and $t \geqq 0$, $s \in \{x\}''$, $t \in \{y\}'$ ([4], Corollary 5.2; [1], Corollary 6.2); since $xy = yx$ it follows that $st = ts$, therefore

$$(*) \qquad 0 \leqq y - x = t^2 - s^2 = (t+s)(t-s).$$

Since $(t^{-1})^*t^{-1} = (t^2)^{-1} = y^{-1} \leqq 1$, we have $t^{-1} \in A$ and $0 \leqq t^{-1} \leqq 1$, therefore $t \geqq 1$ by Lemma 3.2; then $t+s \geqq t \geqq 1$ shows that $t+s$ is invertible. Let $z = (t+s)^{-1}$; by Lemma 3.1, $z \geqq 0$. Clearly $z$ commutes with $t$ and $s$, and therefore with $t^2 - s^2 = y - x \geqq 0$; since the product of commuting positives is positive {e.g., in the above notation, $xy = s^2t^2 = ts^2t = (st)^*(st) \geqq 0$} it follows that $z(y-x) \geqq 0$, i.e., in view of $(*)$, $0 \leqq z(t+s)(t-s) = t-s$.

LEMMA 3.5. *If $x, y \in M$ and $0 \leqq x \leqq y$, then $x^{1/2} \leqq (y + \varepsilon 1)^{1/2}$ for every real number $\varepsilon > 0$.*

PROOF. Since $x \leqq x+\varepsilon 1$, where $x$ and $x+\varepsilon 1$ commute and $x+\varepsilon 1 \geqq \varepsilon 1$, Lemma 3. 4 yields

$$\text{(i)} \qquad x^{1/2} \leqq (x+\varepsilon 1)^{1/2}.$$

On the other hand, $\varepsilon 1 \leqq x+\varepsilon 1 \leqq y+\varepsilon 1$, where $x+\varepsilon 1$ and $y+\varepsilon 1$ are invertible (see the proof of Lemma 3. 4), therefore $0 \leqq (y+\varepsilon 1)^{-1} \leqq (x+\varepsilon 1)^{-1} \leqq \varepsilon^{-1} 1$ by Lemma 3. 3; moreover, $(y+\varepsilon 1)^{-1}$, $(x+\varepsilon 1)^{-1} \in A$, therefore $(y+\varepsilon 1)^{-1/2} \leqq (x+\varepsilon 1)^{-1/2}$ by the monotonicity of square roots in a $C^*$-algebra, and Lemma 3. 3 then yields

$$\text{(ii)} \qquad (x+\varepsilon 1)^{1/2} \leqq (y+\varepsilon 1)^{1/2}.$$

Combining (i), (ii), we obtain the desired inequality.

The $AW^*$-algebra $A$ is said to have the *monotone convergence property* if it satisfies the following condition: if $a_n$ is a sequence of self-adjoint elements in $A$ such that $a_1 \leqq a_2 \leqq a_3 \leqq \cdots$ and $a_n \leqq b (n=1,2,3,\cdots)$ for some self-adjoint element $b$ of $A$, then sup $a_n$ exists (with respect to the usual ordering of self-adjoints). The notation $a_n \uparrow a$ means that $a_1 \leqq a_2 \leqq a_3 \leqq \cdots$, sup$a_n$ exists, and $a = \sup a_n$. We remark that $a$ also serves as a supremum for the $a_n$ in $M$. {The point is that if $x$ is a self-adjoint element of $M$ such that $a_n \leqq x$ for all $n$, then $a \leqq x$. To prove this, it suffices to consider the case that $a_1 \geqq 0$. Then $1 \leqq 1+a_n \uparrow 1+a$ implies that $1 \geqq (1+a_n)^{-1} \downarrow (1+a)^{-1}$. Also, $1 \leqq 1+a_n \leqq 1+x$ implies, by Lemma 3. 3, that $1 \geqq (1+a_n)^{-1} \geqq (1+x)^{-1}$. Writing $b=(1+a)^{-1}$, we have $(1+x)^{-1} \leqq \inf(1+a_n)^{-1} = b$, therefore $1+x \geqq b^{-1}$, $x \geqq b^{-1}-1=a$.}

The foregoing definition makes sense also for $M$: we say that $M$ has the *monotone convergence property* iff sup $x_n$ exists for any increasing sequence of self-adjoint elements of $M$ such that $x_n \leqq y (n=1,2,3,\cdots)$ for some self-adjoint $y$ in $M$.

LEMMA 3. 6. *If $A$ has the monotone convergence property, then so does $M$.*

PROOF. Suppose $x_n, y \in M$ are self-adjoint, $x_1 \leqq x_2 \leqq x_3 \leqq \cdots$, and $x_n \leqq y$ for all $n$. Subtracting $x_1$ throughout, we can suppose $x_1 \geqq 0$. Then

$$1 \leqq 1+x_1 \leqq 1+x_2 \leqq \cdots \leqq 1+y,$$

therefore by Lemma 3. 3,

$$1 \geqq (1+x_1)^{-1} \geqq (1+x_2)^{-1} \geqq \cdots \geqq (1+y)^{-1} \geqq 0.$$

Since $(1+x_n)^{-1} \in A$, we may set $z = \inf\,(1+x_n)^{-1}$, and it follows readily from Lemma 3.3 that $x_n \uparrow z^{-1}-1$. {An essential point is that $z$ is invertible. Indeed, $z \geqq (1+y)^{-1} \geqq 0$. Setting $t = (1+y)^{1/2}$, we have $t^2 = 1+y$ invertible, hence $t$ is invertible, and $t^*zt \geqq t^*(1+y)^{-1}t = t(t^2)^{-1}t = 1$; thus $tzt = t^*zt$ is invertible, hence so is $t^{-1}(tzt)t^{-1} = z$}.

We remark that the foregoing considerations can be cast more generally in terms of directed families (of any cardinality); the directed family formulation of Lemma 3.6 generalizes an earlier result for algebras of finite class ([2], p. 179, Corollary).

LEMMA 3.7. *If $A$ has the monotone convergence property, and $y$ is any positive element of $M$, then*

$$y^{1/2} = \inf_{\varepsilon>0}(y + \varepsilon 1)^{1/2}\,.$$

PROOF. By Lemma 3.5 we have $y^{1/2} \leqq (y+\varepsilon 1)^{1/2}$ for all $\varepsilon > 0$.

Consider the sequence $y_n = y + (1/n)1$ $(n = 1, 2, 3, \cdots)$; thus $y_1 \geqq y_2 \geqq y_3 \geqq \cdots \geqq y \geqq 0$. By Lemma 3.4,

$$(y_1)^{1/2} \geqq (y_2)^{1/2} \geqq \cdots \geqq y^{1/2}\,;$$

in view of Lemma 3.6, we may set $z = \inf(y_n)^{1/2}$, and it will clearly suffice to show that $z = y^{1/2}$.

We note first that $z \in \{y\}''$. {Proof: Assuming $s \in \{y\}'$ we are to show that $zs = sz$. Since $\{y\}$ is a *-subalgebra of $M$, we can suppose $s$ is self-adjoint. If $u$ is the Cayley transform of $s$ ([4], Theorem 5.1; [1], Theorem 4.1), one has $\{s\}' = \{u\}'$, thus it will suffice to show that $zu = uz$. Since $\{u\}'' = \{s\}'' \subset \{y\}''' = \{y\}' = \{y_n\}$, we have $(y_n)^{1/2} \in \{y_n\}'' \subset \{u\}''' = \{u\}'$, that is, $u$ commutes with $(y_n)^{1/2}$; then

$$uzu^* = u[\inf(y_n)^{1/2}]u^* = \inf[u(y_n)^{1/2}u^*] = \inf(y_n)^{1/2} = z\,,$$

as was to be shown.}

Since $(y_n)^{1/2} \geqq y^{1/2}$ for all $n$, we have $z \geqq y^{1/2}$; since, moreover, $z$ commutes with $y^{1/2}$ (by the preceding paragraph) it follows that $z^2 \geqq y$. {Indeed, $z^2 - y = (z+y^{1/2})(z-y^{1/2})$ is the product of commuting positives.} On the other hand, $z \leqq (y_n)^{1/2}$ for all $n$, and since $z$ commutes with $(y_n)^{1/2}$ it follows that $z^2 \leqq y_n$; thus

$$0 \leqq z^2 - y \leqq y_n - y = (1/n)1\,,$$

therefore $z^2 - y \in A$ and $\|z^2 - y\| \leqq 1/n$ for all $n$, i.e., $z^2 - y = 0$, $z = y^{1/2}$.

THEOREM 3.8. *If $A$ has the monotone convergence property, and if $x, y \in M$ satisfy $0 \leqq x \leqq y$, then $x^{1/2} \leqq y^{1/2}$.*

PROOF. For any $\varepsilon > 0$, Lemma 3.5 yields $x^{1/2} \leqq (y + \varepsilon 1)^{1/2}$; in view of Lemma 3.7, this implies $x^{1/2} \leqq y^{1/2}$.

For the case that $A$ is of finite class, Theorem 3.8 evolved in correspondence (1963) with Professor Topping, to whom I am indebted for several of the key ideas.

REMARK. In Lemma 3.3, the assumption that $y$ is invertible is redundant; that is, if $y \geqq x \geqq 0$ and $x$ is invertible, then $y$ is also invertible. {Proof: By Lemma 3.1 we may write $x^{-1} = t^2$ with $t^* = t$; then $tyt \geqq txt = 1$, therefore $tyt$ is invertible, hence so is $t^{-1}(tyt)t^{-1} = y$.}

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