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REMARKS ON THE RIESZ DECOMPOSITION FOR SUPERMARTINGALES

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In this paper we shall give an another proof of the Riesz decomposition theorem for supermartingales and we shall consider on the Riesz-type decomposition for local supermartingales.

1. Let $(\Omega, \mathfrak{F}, P)$ be the basic P-complete probability space and let $\mathfrak{F}_n$ be a sub σ-field of $\mathfrak{F}$ such that $\mathfrak{F}_m \subset \mathfrak{F}_n$ whenever $m < n$. It is clear that $E[x_n]$ decreases if $(x_n, \mathfrak{F}_n)$ is a supermartingale. We assume here the integrability of x_n for each n.

THEOREM 1. *Let $(x_n, \mathfrak{F}_n)$ be a supermartingale. Then x_n can be written as*

$$x_n = x_n^* + y_n$$

where $(x_n^*, \mathfrak{F}_n)$ is a martingale and $(y_n, \mathfrak{F}_n)$ is a positive supermartingale if and only if

$$\text{(A)} \qquad \inf_n E[x_n] > -\infty$$

(there is no uniqueness)

PROOF. The condition is obviously necessary. Let us prove the sufficiency. Since $(x_n, \mathfrak{F}_n)$ is a supermartingale, we have

$$E[x_{n+k+1} | \mathfrak{F}_n] \leqq E[x_{n+k} | \mathfrak{F}_n] \leqq x_n .$$

Put for each n

$$x_n^* = \lim_{k\to\infty} E[x_{n+k} | \mathfrak{F}_n] .$$

Clearly $x_n - x_n^* \geqq 0$ and x_n^* is $\mathfrak{F}_n$-measurable. If the condition (A) is fulfilled, then from the monotone convergence theorem we have

(1) $$E[x_n - x_n^*] = E[\lim_{n\to\infty}(x_n - E[x_{n+k} \mid \mathfrak{F}_n])]$$
$$= \lim_{k\to\infty} E[x_n - E[x_{n+k} \mid \mathfrak{F}_n]]$$
$$= E[x_n] - \lim_{k\to\infty} E[x_{n+k}]$$
$$= E[x_n] - \inf_m E[x_m] < +\infty .$$

Therefore $x_n - x_n^*$ is integrable and so x_n^* is integrable. Moreover for each pair $m < n$

$$E[x_n^* \mid \mathfrak{F}_m] = \lim_{k\to\infty} E[\{E[x_{n+k} \mid \mathfrak{F}_n]\} \mid \mathfrak{F}_m]$$
$$= \lim_{k\to\infty} E[x_{n+k} \mid \mathfrak{F}_m]$$
$$= \lim_{k\to\infty} E[x_{m+k} \mid \mathfrak{F}_m]$$
$$= x_m^* .$$

This implies that $(x_n^*, \mathfrak{F}_n)$ is a martingale and so it follows from $x_n^* \leqq x_n$ that $(y_n, \mathfrak{F}_n)$, where $y_n = x_n - x_n^*$,is a positive supermartingale. This completes the proof.

COROLLARY. *If the condition* (A) *is fulfilled, then one may assume that* $(y_n, \mathfrak{F}_n)$ *is a potential.* (the Riesz decomposition theorem)

PROOF. In order to prove this corollary, it is sufficient to prove that the process $(y_n, \mathfrak{F}_n)$ constructed in the proof of Theorem 1 is a potential. It follows from (1) that

$$E[y_n] = E[x_n] - \inf_m E[x_m]$$

and so $\lim_{n\to\infty} E[y_n] = 0$. This implies that $(y_n, \mathfrak{F}_n)$ is a potential.

REMARK. If a supermartingale $(x_n, \mathfrak{F}_n)$ is decomposable into a martingale and a potential, then the decomposition is unique. Indeed we suppose that $(x_n, \mathfrak{F}_n)$ has two such decompositions:

$$x_n = x_n^{*(1)} + y_n^{(1)}$$
$$= x_n^{*(2)} + y_n^{(2)} .$$

Then for each k, we have

$$x_n^{*(1)} - x_n^{*(2)} = E[y_{n+k}^{(2)} \mid \mathfrak{F}_n] - E[y_{n+k}^{(1)} \mid \mathfrak{F}_n] .$$

Since each $(y_n^{(i)}, \mathfrak{F}_n)$, $(i = 1, 2)$, is a potential, we have

$$E[\lim_{k\to\infty} E[y_{n+k}^{(i)} \mid \mathfrak{F}_n]] \leqq \lim_{k\to\infty} E[y_{n+k}^{(i)}] = 0 .$$

Thus $\lim_{k\to\infty} E[y_{n+k}^{(i)} \mid \mathfrak{F}_n] = 0$. This implies that $x_n^{*(1)} = x_n^{*(2)}$ *a. s.* and so $y_n^{(1)} = y_n^{(2)}$ *a. s.*

2. We assume here that we are given on the basic probability space $(\Omega, \mathfrak{F}, P)$ a right continuous, increasing family $(\mathfrak{F}_t)_{0 \leqq t < \infty}$ of sub σ-fields of $\mathfrak{F}$. We may, and do, suppose that each $\mathfrak{F}_t$ contains all $\mathfrak{F}$-sets of P-measure zero.

To begin with, we shall consider on the Riesz decomposition for right continuous supermartingales.

DEFINITION 1. Let $X = (x_t, \mathfrak{F}_t)$ and $Y = (y_t, \mathfrak{F}_t)$ be two stochastic processes. We say that Y is a modification of X if for each t $P(x_t = y_t) = 1$.

In the followings we assume the integrability of x_t for each t if $X = (x_t, \mathfrak{F}_t)$ is a supermartingale.

LEMMA. *Let $X = (x_t, \mathfrak{F}_t)$ be a supermartingale. Then there exists a right continuous modification of X if and only if the function $t \to E[x_t]$ is right continuous.*

PROOF. We designate by S a countable set which is dense in $[0, \infty[$. We consider a sequence $(t_n)_{n=1,2,\dots}$ of elements of S such that $t_n > t$ which decreases to t. Then the random variables x_{t_n} are uniformly integrable. Thus it follows from $E[x_{t_n} \mid \mathfrak{F}_t] \leqq x_t$ for each n that we have

$$E[x_{t+} \mid \mathfrak{F}_t] \leqq x_t .$$

From the assumption on the right continuity of the family $(\mathfrak{F}_t)$ we have

$$P(x_{t+} \leqq x_t) = 1$$

for each t. Clearly $P(x_{t+} = x_t) = 1$ if and only if

$$E[x_t] = E[x_{t+}] = \lim_{n\to\infty} E[x_{t_n}] .$$

Therefore if there exists a right continuous modification $Y = (y_t, \mathfrak{F}_t)$ of X, then it follows from $E[x_t]=E[y_t]$ for each t that the function $t \to E[x_t]$ is right continuous. Conversely if the mapping $t \to E[x_t]$ is right continuous, then the stochastic process $\widetilde{X}=(x_{t+}, \mathfrak{F}_t)$ is a desired right continuous modification of X. Hence the lemma is established. (This proof is due to P. A. Meyer [1]).

THEOREM 2. *Let $X=(x_t, \mathfrak{F}_t)$ be a right continuous supermartingale. Then there exist a right continuous martingale $X^*=(x_t^*, \mathfrak{F}_t)$ and a positive right continuous supermartingale $Y=(y_t, \mathfrak{F}_t)$ satisfying*

$$P(x_t = x_t^* + y_t, \quad \forall t \geqq 0) = 1$$

if and only if

$$\text{(B)} \qquad \inf_{0 \leqq t < +\infty} E[x_t] > -\infty .$$

PROOF. The condition (B) is obviously necessary. Let us prove the sufficiency. For each t $E[x_{n \vee t} | \mathfrak{F}_t]$ decreases with respect to n. We define:

$$x_t^* = \lim_{n \to \infty} E[x_n | \mathfrak{F}_t] .$$

Clearly x_t^* is $\mathfrak{F}_t$-measurable and $P(x_t - x_t^* \geqq 0) = 1$ for each t. It follows from the condition (B) that

$$(2) \qquad E[x_t - x_t^*] = E[x_t] - \inf_{0 \leqq t < +\infty} E[x_t] < +\infty .$$

Thus $x_t - x_t^*$ is integrable and so x_t^* is integrable. Moreover for each pair $s < t$ we have

$$\begin{aligned} E[x_t^* | \mathfrak{F}_s] &= E[\lim_{n \to \infty} E[x_n | \mathfrak{F}_t] | \mathfrak{F}_s] \\ &= \lim_{n \to \infty} E[x_n | \mathfrak{F}_s] \\ &= x_s^* \end{aligned}$$

from the monotone convergence theorem. Thus $X^*=(x_t^*, \mathfrak{F}_t)$ is a martingale. From the assumption on the right continuity of the family $(\mathfrak{F}_t)$ there exists a right continuous modification of X^*. Without loss of generality we may assume that X^* is right continuous. Then the stochastic process $Y=(y_t, \mathfrak{F}_t)$, where $y_t = x_t - x_t^*$, is a desired positive right continuous supermartingale. This completes the proof.

COROLLARY. *If the condition* (B) *is fulfilled, then one may assume that* *the positive supermartingale* $Y=(y_t, \mathfrak{F}_t)$ *is a potential.* (the Riesz decomposition theorem).

If a right continuous supermartingale $X=(x_t, \mathfrak{F}_t)$ is decomposable into a right continuous martingale and a potential, then it is easy to show that the decomposition is unique.

We are now going to investigate the Riesz-type decomposition for local supermartingales. Let u be a real number, $0 \leqq u < +\infty$, and let $X=(x_t, \mathfrak{F}_t)$ be a right continuous stochastic process. We shall say that it belongs to the class (D) if all the random variables x_τ are uniformly integrable, τ being any finite-valued stopping times with respect to the family $(\mathfrak{F}_t)$.

DEFINITION 2. A right continuous process $X=(x_t, \mathfrak{F}_t)$ is a local supermartingale if and only if there exists an increasing sequence (τ_n) of stopping times with respect to the family $(\mathfrak{F}_t)$, such that

1) $P(\lim_{n\to\infty}\tau_n = +\infty) = 1$
2) for every n, the process $(x_{t\wedge\tau_n}, \mathfrak{F}_{t\wedge\tau_n})$ is a supermartingale which belongs to the class (D).

To be short, we shall say that a stopping time τ reduces the right continuous process $X=(x_t, \mathfrak{F}_t)$ if $(x_{t\wedge\tau})_{0\leqq t<\infty}$ belongs to the class (D). Note that, in what follows, we shall not use the uniform integrability of the family $(x_{t\wedge\tau_n})_{0\leqq t<\infty}$ for each n.

THEOREM 3. *Let* $X=(x_t, \mathfrak{F}_t)$ *be a local supermartingale. Then there exist a local martingale* $X^*=(x_t^*, \mathfrak{F}_t)$ *and a positive supermartingale* $Y=(y_t, \mathfrak{F}_i)$ *satisfying*

$$P(x_t = x_t^* + y_t, \forall t\geqq 0) = 1$$

if and only if there exists an increasing sequence (τ_n) *of stopping times with respect to the family* $(\mathfrak{F}_t)$, *almost surely finite, reducing* $X=(x_t, \mathfrak{F}_t)$ *such that* $P(\lim_{n\to\infty}\tau_n=\infty)=1$ *and*

$$\text{(C)} \qquad \inf_n E[x_{\tau_n}] > -\infty\,.$$

PROOF. Necessity. Since $X^*=(x_t^*, \mathfrak{F}_t)$ is a local martingale, there exists an increasing sequence (τ_n) of stopping times, almost surely finite, reducing X^* such that $P(\lim_{n\to\infty}\tau_n=\infty)=1$. We may assume, without loss of generality, that for each n $P(\tau_n\leqq n)=1$ and τ_n reduces the process $X=(x_t, \mathfrak{F}_t)$. Then for each k $x_{\tau_k}^*$ is integrable and for each pair $m<n$

$$E[x^*_{n\wedge\tau_n}\,|\,\mathfrak{F}_{(\tau_m\wedge\tau_n)\wedge\tau_n}] = x^*_{(\tau_m\wedge\tau_n)\wedge\tau_n}$$

because $\tau_m\wedge\tau_n$ is a stopping time with respect to the family $(\mathfrak{F}_{t\wedge\tau_n})_{0\leqq t<\infty}$. As $n\wedge\tau_n=\tau_n$ and $\tau_m\wedge\tau_n=\tau_m$, we have

$$E[x^*_{\tau_n}\,|\,\mathfrak{F}_{\tau_m}] = x^*_{\tau_m} \quad \text{a. s.}$$

Thus $(x^*_{\tau_n}, \mathfrak{F}_{\tau_n})$ is a martingale and it follows from $P(x_t\geqq x_t^*,\ \forall\, t\geqq 0)=1$ that

$$P(x_{\tau_n}\geqq x^*_{\tau_n}) = 1$$

for each n. This implies that $-\infty<E[x^*_{\tau_1}]\leqq \inf_n E[x_{\tau_n}]$.

Sufficiency. Without loss of generality, we may assume that for each $\omega\in\Omega$, the trajectory $t\to x_t(\omega)$ is right continuous. We may also assume that $P(\tau_n\leqq n)=1$ for all n. Then it is easy to show that $(x_{\tau_n}, \mathfrak{F}_{\tau_n})$ is a supermartingale. For each t and each k, we have

$$\begin{aligned}(\forall m=1,2,\cdots),\ E[x_{\tau_{m+k+1}}\,|\,\mathfrak{F}_{t\wedge\tau_k}] &= E[\{E[x_{\tau_{m+k+1}}\,|\,\mathfrak{F}_{\tau_{m+k}}]\}\,|\,\mathfrak{F}_{t\wedge\tau_k}]\\ &\leqq E[x_{\tau_{m+k}}\,|\,\mathfrak{F}_{t\wedge\tau}] \quad \text{on } (N_{t,k})^c\end{aligned}$$

where $N_{t,k}$ is a $\mathfrak{F}$-set of P-measure zero which may depend on t and k, and $(N_{t,k})^c$ is the complement of $N_{t,k}$ with respect to Ω. Since $E[x_{\tau_{m+k}}\,|\,\mathfrak{F}_{t\wedge\tau_k}]$ decreases with respect to m on $(N_{t\,k})^c$, we can now give the following definition:

$$x_t^k = \begin{cases}\lim_{m\to\infty} E[x_{\tau_m}\,|\,\mathfrak{F}_{t\wedge\tau_k}] & \text{on} \quad (N_{t,k})^c\\ \quad x_{t\wedge\tau_k} & \text{on} \quad N_{t,k}.\end{cases}$$

Clearly x_t^k is $\mathfrak{F}_{t\wedge\tau_k}$-measurable. It follows from $P(E[x_{\tau_{m+k}}\,|\,\mathfrak{F}_{t\wedge\tau_k}]\leqq x_{t\wedge\tau_k},\ \forall\, m)=1$ that for each t and each k we have

$$P(x_{t\wedge\tau_k}-x_t^k\geqq 0) = 1\,.$$

From the monotone convergence theorem we have for each pair $s<t$ and each k

$$\begin{aligned}E[x_t^k\,|\,\mathfrak{F}_{s\wedge\tau_k}] &= E[\{\lim_{m\to\infty} E[x_{\tau_m}\,|\,\mathfrak{F}_{t\wedge\tau_k}]\}\,|\,\mathfrak{F}_{s\wedge\tau_k}]\\ &= \lim_{m\to\infty} E[x_{\tau_m}\,|\,\mathfrak{F}_{s\wedge\tau_k}]\end{aligned}$$

$$= x_s^k \quad \text{a. s.}$$

Moreover it follows from the condition (C) that

$$(3) \qquad E[x_{t\wedge\tau_k} - x_t^k] = E[x_{t\wedge\tau_k}] - \inf_m E[x_{\tau_m}] < +\infty .$$

This implies that $x_{t\wedge\tau_k} - x_t^k$ is integrable. Thus x_t^k is integrable. Therefore for each k $X^k = (x_t^k, \mathfrak{F}_{t\wedge\tau_k})$ is a martingale. From the assumption on the right continuity of the family $(\mathfrak{F}_t)$ there exists a right continuous modification $\tilde{X}^k = (\tilde{x}_t^k, \mathfrak{F}_{t\wedge\tau_k})$ of X^k. It is clear that for each t and each k we have

$$P(\tilde{x}_t^k = \lim_{m\to\infty} E[x_{\tau_m} | \mathfrak{F}_{t\wedge\tau_k}]) = 1$$

Next we shall investigate on the relation of $\tilde{X}^k$ and $\tilde{X}^{k+p}$ $(p=1, 2, \cdots)$. Since for each $\Lambda \in \mathfrak{F}_{t\wedge\tau_{k+p}}$ $(\Lambda \cap [t \leqq \tau_k]) \cap [t\wedge\tau_k \leqq u] \in \mathfrak{F}_u (\forall u \geqq 0)$, we have

$$\Lambda \cap [t \leqq \tau_k] \in \mathfrak{F}_{t\wedge\tau_k} \subset \mathfrak{F}_{t\wedge\tau_{k+p}} .$$

Thus it follows that for each $\Lambda \in \mathfrak{F}_{t\wedge\tau_{k+p}}$

$$\int_{\Lambda\cap[t\leqq\tau_k]} E[x_{\tau_m} | \mathfrak{F}_{t\wedge\tau_k}] dP = \int_{\Lambda\cap[t\leqq\tau_k]} x_{\tau_m} dP$$

$$= \int_{\Lambda\cap[t\leqq\tau_k]} E[x_{\tau_m} | \mathfrak{F}_{t\wedge\tau_{k+p}}] dP .$$

Since both $E[x_{\tau_m} | \mathfrak{F}_{t\wedge\tau_k}]$ and $E[x_{\tau_m} | \mathfrak{F}_{t\wedge\tau_{k+p}}]$ are $\mathfrak{F}_{t\wedge\tau_{k+p}}$-measureable, we have that for each t

$$P(E[x_{\tau_m} | \mathfrak{F}_{t\wedge\tau_k}] \neq E[x_{\tau_m} | \mathfrak{F}_{t\wedge\tau_{k+p}}], t \leqq \tau_k) = 0 .$$

Thus $P(\tilde{x}_t^k \neq \tilde{x}_t^{k+p}, t \leqq \tau_k) = 0$ for each t. Let Q^+ be the set of all positive rational numbers and we now put:

$$N_{r,k,p} = [\tilde{x}_r^k \neq \tilde{x}_r^{k+p}, r \leqq \tau_k]$$

Then $P(N) = 0$ where $N = \bigcup_{\substack{k,p=1,2,\cdots \\ r\in Q^+}} N_{r,k,p}$, and for each $\omega \notin N$ we have

$$\tilde{x}_r^k(\omega) = \tilde{x}_r^{k+p}(\omega)$$

for all $p=1, 2, \cdots$. From the right continuities of $\tilde{X}^k$ and $\tilde{X}^{k+p}$ it follows that

$$P(\exists\, t \geqq 0, \exists\, k, \exists\, p;\ \tilde{x}_t^k(\omega) \neq \tilde{x}_{t\wedge\tau_k(\omega)}^{k+p}(\omega)) = 0\,.$$

We may assume, without loss of generality, that for each $\omega \in \Omega$ the trajectories $t \to \tilde{x}_t^k(\omega)$ and $t \to \tilde{x}_t^{k+p}(\omega)$ are right continuous. This implies that

$$(\forall t \geqq 0),\ \tilde{x}_t^k = \tilde{x}_{t\wedge\tau_k}^{k+p} \quad \text{on} \quad N^c \quad (k, p = 1, 2, \cdots)\,.$$

Now we can give the following definition:

$$x_t^* = \begin{cases} \lim_{j\to\infty} \tilde{x}_t^j & \text{on } N^c \\ x_t & \text{on } N\,. \end{cases}$$

Then clearly x_t^* is $\mathfrak{F}_t$-measurable and we have

$$P(x_{t\wedge\tau_k}^* = \tilde{x}_t^k,\ \forall\, t \geqq 0) = 1\,.$$

Since $X^k = (\tilde{x}_t^k, \mathfrak{F}_{t\wedge\tau_k})$ is a right continuous martingale which belongs to the class (D), $X^* = (x_t^*, \mathfrak{F}_t)$ is a local martingale. Then $Y = (y_t, \mathfrak{F}_t)$, where $y_t = x_t - x_t^*$, is a positive local supermartingale. It is easy to see that for each pair $s < t$ and each k we have

$$E[y_{t\wedge\tau_k} | \mathfrak{F}_s] \leqq y_{s\wedge\tau_k}\,.$$

From the Fatou's lemma we have

$$E[y_t | \mathfrak{F}_s] \leqq y_s\,.$$

Since $y_0 = x_{0\wedge\tau_k} - x_{0\wedge\tau_k}^*$ is integrable, $Y = (y_t, \mathfrak{F}_t)$ is a positive right continuous supermartingale. This completes the proof.

COROLLARY. *If the condition* (C) *is fulfilled, then one may assume that the positive supermartingale is a potential.* (the Riesz-type decomposition theorem for local supermartingales).

PROOF. In order to prove this corollary it is sufficient to prove that the process $Y = (y_t, \mathfrak{F}_t)$ constructed in the proof of Theorem 3 is a potential. It follows from (3) that

$$\begin{aligned} \lim_{t\to\infty} E[y_t] &= \lim_{t\to\infty} E[\lim_{k\to\infty}(x_{t\wedge\tau_k} - x_{t\wedge\tau_k}^*)] \\ &\leqq \lim_{t\to\infty} \liminf_{k\to\infty} E[x_{t\wedge\tau_k} - x_{t\wedge\tau_k}^*] \\ &\leqq \lim_{t\to\infty} \lim_{k\to\infty} E[x_{t\wedge\tau_k}] - \inf_m E[x_{\tau_m}]\,. \end{aligned}$$

Since $P(\tau_n \leqq n)=1$ and $\lim_{k\to\infty} E[x_{t\wedge\tau_k}] \leqq E[x_{t\wedge\tau_n}]$ for every n, we have

$$\lim_{t\to\infty}\lim_{k\to\infty} E[x_{t\wedge\tau_k}] \leqq E[x_{\tau_n}] .$$

for every n. Therefore for every n

$$\lim_{t\to\infty} E[y_t] \leqq E[x_{\tau_n}] - \inf_m E[x_{\tau_m}] .$$

This inequality implies that $\lim_{t\to\infty} E[y_t]=0$. Hence the corollary is established.

REMARK. If a local supermartingale $X=(x_t, \mathfrak{F}_t)$ is decomposable into a local martingale and a potential which belongs to the class (D), then the decomposition is unique. In fact, we suppose that X has two such decompositions:

$$\begin{aligned} x_t &= x_t^{*(1)} + y_t^{(1)} \\ &= x_t^{*(2)} + y_t^{(2)} . \end{aligned}$$

Then there exists an increasing sequence (τ_n) of stopping times with respect to the family $(\mathfrak{F}_t)$ reducing $X=(x_t, \mathfrak{F}_t)$ and $X^{*(i)}=(x_t^{*(i)}, \mathfrak{F}_t)$, $(i=1,2)$, such that $P(\lim_{n\to\infty} \tau_n=\infty)=1$. Without loss of generality, we may assume that $P(\tau_n \leqq n)=1$ for every n. It is easy to see that for each $u \geqq 0$ we have

$$x^{*(1)}_{t\wedge\tau_n} - x^{*(2)}_{t\wedge\tau_n} = E[y^{(2)}_{(t+u)\wedge\tau_n} \mid \mathfrak{F}_{t\wedge\tau_n}] - E[y^{(1)}_{(t+u)\wedge\tau_n} \mid \mathfrak{F}_{t\wedge\tau_n}] .$$

Since each $Y^{(i)}=(y_t^{(i)}, \mathfrak{F}_t)$, $(i=1,2)$, is a potential which belongs to the class (D), we have

$$\begin{aligned} & E[\lim_{n\to\infty}\lim_{u\to\infty} E[y^{(i)}_{(t+u)\wedge\tau_n} \mid \mathfrak{F}_{t\wedge\tau_n}]] \\ &= E[\lim_{n\to\infty} E[y^{(i)}_{\tau_n} \mid \mathfrak{F}_{t\wedge\tau_n}]] \\ &\leqq \lim_{n\to\infty} E[y^{(i)}_{\tau_n}] = 0 . \end{aligned}$$

Therefore $x_t^{*(1)} = x_t^{*(2)}$ a.s. and so $y_t^{(1)} = y_t^{(2)}$ a.s. for each t.

REFERENCE

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