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GENERATORS OF W^*-ALGEBRAS

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Recently Wogen [1] has shown that any properly infinite W^*-algebra on a separable Hilbert space is singly generated. Further results on generators of W^*-algebras have been obtained by Saito [2] and Pearcy [3]. In this note we shall extend these results and present new proofs for them. We shall show for example that any properly infinite W^*-algebra $\mathfrak{A}$ is generated by a single subnormal operator T. We will also see that $\mathfrak{A}$ is generated by two unitary operators U and V with $U^2=1$ and $V^3=1$.These results have some interesting consequences. Throughout H will be a separable Hilbert space and $\mathfrak{A}$ will be a properly infinite W^*-algebra on H. Though some of our results are also valid for certain singly generated W^*-algebras, we formulate them only for the properly infinite ones.

LEMMA 1. *$\mathfrak{A}$ is singly generated.*

PROOF. Since $\mathfrak{A}$ is properly infinite it can be written as $\mathfrak{A}=\mathfrak{B}\otimes B(K')$, where $\mathfrak{B}$ is a properly infinite W^*-algebra on a separable Hilbert space K and where $B(K')$ denotes the algebra of all bounded operators on the infinite dimensional Hilbert space K'. In this notation $H=K\otimes K'$. We shall write K' as the usual sequence space $l^2(N)$, N the positive integers. Then the elements A of $\mathfrak{A}$ can be considered as matrices with entries from $\mathfrak{B}$, $A=(a_{i,j})$ and $a_{i,j}\in\mathfrak{B}$ $\mathfrak{B}$ is clearly generated by a countable number of operators b_i, $i=1,2,\cdots$. Without loss of generality we shall assume that all b_i are positive invertible contractions. Now let $A=(a_{i,j})$ and $B=(b_{i,j})$ with $a_{i,j}=\delta_{i,j}\ 1/i\cdot 1$ and $b_{i+1,i}=b_{i,i+1}=b_i$ and $b_{i,j}=0$ otherwise. Then A and B are selfadjoint. Let $\mathfrak{R}$ be the W^*-algebra generated by A and B and let $C^*=C\in\mathfrak{R}'$. Since $CA=AC$ we see immediately that C is diagonal, $C=(c_{i,j})$ with $c_{i,j}=\delta_{i,j}c_i$. We shall denote this by $C=\text{diag}\,(c_1,c_2,\cdots)$. Then $BC=CB$ shows $b_ic_{i+1}=c_ib_i$ $i=1,2\cdots$. The adjoint of this equation is $c_{i+1}b_i=b_ic_i$ because $c_i=c_i^*$. Thus $c_ib_i^2=b_ic_{i+1}b_i=b_i^2c_i$. However b_i is positive, therefore $c_ib_i=b_ic_i$ and $(c_{i+1}-c_i)b_i=0$. Since b_i is invertible we see $c_{i+1}=c_i$. Thus $c_1=c_2=\cdots=c$ and $cb_i=b_ic$. $\mathfrak{B}$ is generated by the b_i. Therefore $c\in\mathfrak{B}'$ and $\mathfrak{R}'=\mathfrak{B}'\otimes \boldsymbol{C}$ or $\mathfrak{R}=\mathfrak{A}$. Thus $\mathfrak{A}$ is generated by the two hermitean operators A and B.

The above construction actually gives us a continuous family $\{A_k, B_k\}_{k\in(0,1)}$ of pairwise unitarily inequivalent generators of $\mathfrak{A}$. Simply choose for B_k $b_1=k$. These remarks will also apply to all following results.

THEOREM 1. *$\mathfrak{A}$ is generated by a single subnormal operator T.*

PROOF. Let a and b be positive operators on the separable Hilbert space K with $0<a^2<b^2$. Following [4] we define on $(K\oplus K\oplus\cdots)\oplus(K\oplus K)\oplus(K\oplus K\oplus\cdots)$ $=H\oplus H'\oplus H''$ the operator

$$N=\left|\begin{array}{cccc|cc|ccc}
0 & & & & a & 0 & & & \\
a & 0 & & & 0 & c & & & \\
 & b & 0 & & 0 & 0 & & & \\
 & & b & \ddots & \vdots & \vdots & & & \\
\hline
 & & & & 0 & 0 & b & & \\
 & & & & c & 0 & 0 & & \\
\hline
 & & & & & & 0 & b & \\
 & & & & & & & 0 & b \\
 & & & & & & & & \ddots
\end{array}\right|$$

with $c=(b^2-a^2)^{1/2}$. A simple computation shows that N is normal. The subspace H is clearly invariant under N and thus $T=N|H$ is subnormal.

As before write $\mathfrak{A}=\mathfrak{B}\otimes B(K')$, with $\mathfrak{B}$ a properly infinite W^*-algebra. By lemma 1 there exist two positive operators $a, b\in\mathfrak{B}$, which generate $\mathfrak{B}$. We choose them such that $0<a<1$ and $2<b<3$ and form the operator N and $T=N|H$ as above. Again denote by $\mathfrak{R}$ the W^*-algebra generated by T. One sees easily that

$$T^nT^{*n}=\mathrm{diag}(0,\cdots,0,b^{n-1}a^2b^{n-1},b^{2n},b^{2n},\cdots)$$

$$T^{*n}T^n=\mathrm{diag}(ab^{2n-2}a,b^{2n},b^{2n},\cdots)\,.$$

Thus $\mathfrak{R}$ contains any diagonal operator

$$D_n=\mathrm{diag}(0,\cdots,0,1,0,\cdots)\,.$$

Let $C=C^*\in\mathfrak{R}$ then $CD_n=D_nC$ for all n shows $C=\mathrm{diag}(c_1,c_2,\cdots)$. $CT=TC$ implies $c_2a=ac_1$, $c_{i+1}b=bc_i$ $i>1$. With the same trick as in lemma 1 we can now show $c=c_2=\cdots=c$ and $ac=ca, bc=cb$ or $c\in\mathfrak{B}'$. Thus $C=c\otimes\mathbf{1}\in\mathfrak{B}'\otimes\boldsymbol{C}$ and $\mathfrak{R}=\mathfrak{A}$.

Wogen (private communication) has shown that any properly infinite W^*-algebra is generated by a hyponormal operator. However this construction is much simpler than his. Since a quasinormal operator is of type I [5] the subnormality condition in theorem 1 cannot be strengthend. Here we give an independent much shorter proof for the result in [5] using C^*-algebra techniques.

DEF. An operator T on a Hilbert space H is called postliminal if the C^*-algebra $\mathfrak{C}$ generated by T is is postliminal [6, 7].

Postliminal operators are clearly of type I, i. e. they generate a W^*-algebra of type I. However the converse is not true.

THEOREM 2. *A quasinormal operator T is postliminal.*

PROOF. Since T is quasinormal T and T^* commute with (T^*T).

Let π be an irreducible representation of $\mathfrak{C}$. Then $\pi(T^*T)$ commutes with $\pi(T)$ and $\pi(T^*)$ and thus with every element in $\pi(\mathfrak{C})$. By irreducibility $\pi(T^*T) = c1$ with $c>0$. Then $\pi(T)\ 1/\sqrt{c}$ is an irreducible isometry. Thus $\pi(T)\ 1/\sqrt{c}$ is unitary or a simple unilateral shift. In the first case $\dim \pi = 1$, whereas in the second case $\pi(\mathfrak{C})$ contains all compact operators. If one applies now the method of the direct integral decomposition to the identity representation of $\mathfrak{C}$, one obtains the result of [5].

Now let $\mathfrak{A}$ be a factor of type III and T a hyponormal generator of $\mathfrak{A}$. By $\mathfrak{C}$ denote again the C^*-algebra generated by T. Then $\mathfrak{C}$ is antiliminal [6]. Let π be a representation of $\mathfrak{C}$ such that $\pi(\mathfrak{C})$ generates a W^*-algebra of finite type. Then $\pi(\mathfrak{C})''$ has a complete family of normal traces. Let φ be a trace, then $0 \leqq \varphi(\pi(T^*T - TT^*)) = \varphi(\pi(T)^*\pi(T)) - \varphi(\pi(T)\pi(T)^*) = 0$. Hence $\pi(T)$ is normal. This gives us many examples of C^*-algebras with representations of type III, but none of type II_1. This also shows that no finite nonabelian W^*-algebra is generated by a hyponormal operator.

The following corollaries are corollaries of lemma 1 or theorem 1.

COROLLARY 1. *$\mathfrak{A}$ is generated by an operator T with $p(T) = 0$, where $p(x)$ is a polynom of degree three or higher.*

PROOF. Let $p(x) = \prod_{i=1}^{n} (x - a_i)$ and let $\mathfrak{A} = \mathfrak{B} \otimes M_n$.

Then consider the operator matrix $T = (t_{i,j})$ $i, j = 1, \cdots, n$ with $t_{i,i} = a_i$, $t_{1,2} = a$, $t_{n-1,n} = b$, $t_{i,i+1} = 1$ $i = 2, \cdots, n-2$ and $t_{i,j} = 0$ otherwise Here a and b are positive invertible generators of $\mathfrak{B}$. If a and b were commuting the theorem of Hamilton and Cayley would show $p(T)=0$. It is easy to see however that the only matrix elements of T^k $k\leqq n$, where a and b appear together are $(T^n)_{,n}$ and $(T^{n-1})_{1,n}$. Thus the theorem of Hamilton and Cayley is also applicable in this case and $p(T)=0$. Simple matrix computation shows then as in theorem 1 that T generates $\mathfrak{A}$.

We should remark that an operator T satisfying a polynomial identity of degree 2 is binormal and thus generates a W^*-algebra of type $\mathrm{I}_{\leqq 2}$. In particular $\mathfrak{A}$ is generated by an operator T with $T^n=1, n=3, 4, \cdots$. Using Weyl's trick on the bounded representation $k\to T^k$ of the cyclic group of order n this shows that T is similar to a unitary operator U with $U^n=1$. Thus $\mathfrak{A}$ is generated by an operator T which is similar to a unitary operator.

COROLLARY 2. *$\mathfrak{A}$ is generated by a partial isometry.*

PROOF. We write $\mathfrak{A}=\mathfrak{B}\otimes M_2$. By lemma 1 $\mathfrak{B}$ is generated by a positive operator $1/2\geqq a\geqq 1/4$ and a unitary operator u. Let $b=(1-a^2)^{1/2}u$ then $T=\begin{pmatrix} b & a\\ 0 & 0\end{pmatrix}$ is a partial isometry with $TT^*=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}$. Thus any operator $C=C^*$ commuting with T must be diagonal, because it commutes with $TT^*, C=\operatorname{diag}(c_1, c_2)$. Then $TC=CT$ shows $bc_1=c_1b$ and $ac_2=c_1a$. As before this gives $c_1=c_2=c$ because a is positive and invertible. Thus $ac=ca$ and $cu=uc$ or $c\in\mathfrak{B}'$.

COROLLARY 3. *$\mathfrak{A}$ is generated by an (infinite) projection P and a positive operator S.*

PROOF. Let $\mathfrak{A}=\mathfrak{B}\otimes M_2$ and let $P=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}$ and $S=\begin{pmatrix} a & b\\ b & 1\end{pmatrix}$ where a and b are generators of $\mathfrak{B}$ with $2\geqq a\geqq 1$ and $1/2\geqq b>0$. The remainder is shown as in the previous corollary.

THEOREM 3. *$\mathfrak{A}$ is generated by three projections P_1, P_2 and P_3, two of which may be chosen to be orthogonal, $P_1\cdot P_2=0$.*

PROOF. Again we write $\mathfrak{A}=\mathfrak{B}\otimes M_2$. Then by corollary 3 $\mathfrak{B}$ is generated by a projection p and a positive invertible operator a with $1/2\geqq a>0$. Then let $P_1=\begin{pmatrix} p & 0\\ 0 & 0\end{pmatrix}$, $P_2=\begin{pmatrix} 1-p & 0\\ 0 & 0\end{pmatrix}$, $P_3=\begin{pmatrix} a & t\\ t & 1-a\end{pmatrix}$ with $t=(a(1-a))^{1/2}$. These are obviously projections and $P_1\cdot P_2=0$. Let $\mathfrak{R}$ be the W^*-algebra generated by P_1, P_2 and P_3 and let $C=C^*\in\mathfrak{R}$. Then $C(P_1+P_2)=(P_1+P_2)C$ shows as before that C is diagonal, $C=\operatorname{diag}(c_1, c_2)$. Then $CP_1=P_1C$ and $CP_3=P_3C$ show $c_1p=pc_1$ and $c_1a=ac_1$. By assumption $c_1\in\mathfrak{B}'$. $CP_3=P_3C$ gives further $tc_1=c_2t$. We have already shown $c_1 \in \mathfrak{B}$, thus $c_1t = c_2t$ or $c = c_2$ since t is invertible.

We should remark that the projections P_i $i = 1, 2, 3$ are infinite with infinite complement.

COROLLARY. *$\mathfrak{A}$ is generated by two unitary operators U and V, which can be chosen such that $U^2 = 1$ and $V^3 = 1$.*

PROOF. Let P_1, P_2 and P_3 be the projection generators of $\mathfrak{A}$ as we have determined them above. Let $U = 1-2P_3$ and $V = P_1 + P_2e^{2\pi i/3} + (1-P_1-P_2)e^{4\pi i/3}$. Since the P_i $i = 1, 2, 3$ generate $\mathfrak{A}$ also U and V generate $\mathfrak{A}$.

Theorem 3 improves a result by Saito [2], who did not show that two of the generating projections may be chosen orthogonal.

This corollary has an interesting consequence. Let $\mathfrak{A}$ be a properly infinite W^*-algebra on the separable Hilbert space H and let U and V be unitary generators of $\mathfrak{A}$ with $U^2 = 1 = V^3$. Let $G = Z_2 * Z_3$ be the free product of the cyclic group Z_2 of order 2 and Z_3, the cyclic group of order 3, with generators α and β, $\alpha^2 = e = \beta^3$. Let $\pi(\alpha) = U$ and $\pi(\beta) = V$, then π determines a unitary representation of G such that $\pi(G)$ generates $\mathfrak{A}$. This shows that any properly infinite W^*-algebra $\mathfrak{A}$ arises from a representation π of G. Because of our earlier remarks there exists even a continuous family π_k $k \in (0, 1)$ of representations of G such that $\pi_k(G)$ generates $\mathfrak{A}$ and that the π_k are pairwise unitarily inequivalent. Conversely any W^*-algebra which comes from a representation of G is generated by two unitaries U, V with $U^2 = 1 = V^3$.

In the above theorems the separability of H cannot be dropped in general, because there exist properly infinite factors, which are not even countably generated. For example let G be the group of all finite permutations of an uncountable set and let $\mathfrak{B}$ be the left ring of G Then $\mathfrak{A} = \mathfrak{B} \otimes B(K)$, with K a separable infinite dimensional Hilbert space, is properly infinite. But $\mathfrak{A}$ is not countably generated. To see this assume indirectly that $\mathfrak{A}$ is generated by the operators $\{A_i\}_{i=1}^{\infty}$ with $A_i = (a_{j,k}^{(i)})$. Then $\mathfrak{B}$ is generated by the countable set $\{a_{j,k}^{(i)}\}, i, j, k = 1, 2 \cdots$. Every $a_{j,k}^{(i)}$ can be written as $a_{j,k}^{(i)} = \sum_{g \in G} \alpha_g^{(i,j,k)} U_g$ with $\sum_{g \in G} |\alpha_g^{(i,j,k)}|^2 < \infty$, where U_g is the translation by g on $l^2(G)$. Thus $\mathfrak{B}$ is generated by the countable set $S = \{U_g \mid \alpha_g^{(i,j,k)} \neq 0$ for some $(i, j, k)\}$. However the set of all $\{g \in G \mid U_g \in S\}$ generates a countable and thus proper subgroup H of G and thus the $U_g \in S$ do not generate $\mathfrak{B}$. This contradiction shows that $\mathfrak{A}$ is not countably generated. Using some cardinal, arithmetic this result can be improved slightly.

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