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ON A RIEMANNIAN SPACE ADMITTING MORE THAN ONE SASAKIAN STRUCTURES

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Introduction. Let M^n be a connected n-dimensional Riemannian space. A unit Killing vector field ξ^h is called a Sasakian structure if it satisfies

$$\nabla_j \nabla_i \xi^h = \xi_i \delta_j^h - \xi^h g_{ji},$$

where g_{ji} is the Riemannian metric and ∇_j means the Levi-Civita covariant differentiation[1]. Recently Y. Y. Kuo proved that if M^n admits two Sasakian structures orthogonal to each other it admits one more Sasakian structure orthogonal to them[2].

Our interest and purpose of this paper are to study M^n admitting (i) $r(>3)$ Sasakian structures orthogonal to one another and (ii) 2 Sasakian structures not orthogonal to each other.

1. Preliminaries. Consider a Riemannian space M^n with a Sasakian structure ξ^h. If we put $\varphi_i{}^h = \nabla_i \xi^h$, the following relations hold good:

$$(1)\qquad \begin{aligned} &\xi^r \xi_r = 1, \qquad \varphi_i{}^r \xi_r = 0, \qquad \xi^r \varphi_r{}^h = 0, \\ &\varphi_{ji} \equiv \varphi_j{}^r g_{ri} = -\varphi_{ij}, \\ &\varphi_i{}^r \varphi_r{}^h = -\delta_i^h + \xi_i \xi^h, \\ &\nabla_j \varphi_i{}^h = \xi_i g_{jh} - \xi^h g_{ji}. \end{aligned}$$

Let $M^{n+1} = M^n \times R$ be the product manifold of M^n with a line, and define a tensor Φ of type (1, 1) by

$$\Phi = \begin{pmatrix} \varphi_i{}^h & -\xi^h \\ \xi_i & 0 \end{pmatrix},$$

1) We follow the notations in [5] and [6].
2) Y. Y. Kuo, [1].

then Φ is an almost complex structure on M^{n+1}, i. e., $\Phi^2=-I$ holds good, where I means the unit tensor[3].

Suppose a Riemannian space M^n admits 2 Sasakian structures ξ^h and η^h which are orthogonal to each other (at every point). Putting $\varphi_i{}^h=\nabla_i\xi^h$ and $\psi_i{}^h=\nabla_i\eta^h$ we know that ζ^h defined by $\zeta^h=\xi^r\psi_r{}^h$ is Sasakian too and constitutes an orthonormal field together with ξ^h and η^h. We shall represent this process by $\{\xi,\eta\}=\zeta$. They satisfy the following equations:

$$\xi^h=\eta^r\theta_r{}^h=-\zeta^r\psi_r{}^h, \quad \text{where} \quad \theta_r{}^h=\nabla_r\zeta^h,$$

$$\eta^h=\zeta^r\varphi_r{}^h=-\xi^r\theta_r{}^h,$$

$$\zeta^h=\xi^r\psi_r{}^h=-\eta^r\varphi_r{}^h,$$

$$\varphi_i{}^h=\psi_i{}^r\theta_r{}^h-\eta_i\zeta^h=-\theta_i{}^r\psi_r{}^h+\zeta_i\eta^h,$$

$$\psi_i{}^h=\theta_i{}^r\varphi_r{}^h-\zeta_i\xi^h=-\varphi_i{}^r\theta_r{}^h+\xi_i\zeta^h,$$

$$\theta_i{}^h=\varphi_i{}^r\psi_r{}^h-\xi_i\eta^h=-\psi_i{}^r\varphi_r{}^h+\eta_i\xi^h.$$

Here ξ,η and ζ appear symmetrically. Hence if $\{\xi,\eta\}=\zeta$, then $\{\eta,\zeta\}=\xi$ and $\{\zeta,\xi\}=\eta$. We call the collection $\{\xi,\eta,\zeta\}$ of such properties a Sasakian 3-structure. This case the dimension n must be of the form $n=4p+3$ for a non-negative integer p[4].

We need the following lemma later.

LEMMA. *Let M be a differentiable manifold with an almost quaternion structure $\Phi_{(\lambda)}$, $(\lambda=1,2,3)$, i.e., three almost complex structures satisfying $\Phi_{(1)}\Phi_{(2)}=-\Phi_{(2)}\Phi_{(1)}=\Phi_{(3)}$. Then there does not exist an almost complex structure $\Phi_{(4)}$ such that $\Phi_{(\lambda)}\Phi_{(4)}=-\Phi_{(4)}\Phi_{(\lambda)}$, $\lambda=1,2,3$.*

In fact, we have

$$\Phi_{(3)}\Phi_{(4)}=\Phi_{(1)}\Phi_{(2)}\Phi_{(4)}=-\Phi_{(1)}\Phi_{(4)}\Phi_{(2)}=\Phi_{(4)}\Phi_{(1)}\Phi_{(2)}=\Phi_{(4)}\Phi_{(3)}.$$

2. More than one Sasakian structures.

THEOREM 1. *There does not exist a Sasakian structure $\xi_{(4)}$ which is orthogonal to every $\xi_{(\lambda)}$ of a Sasakian 3-structure $\{\xi_{(1)},\xi_{(2)},\xi_{(3)}\}$.*

PROOF. Assume the existence of $\xi_{(4)}$ and let $\Phi_{(\lambda)}$ be the almost complex

3) S. Sasaki and Y. Hatakeyama, [4].
4) Y. Y. Kuo, [1].

structures on $M^{n+1} = M^n \times R$ corresponding to $\xi_{(\lambda)}, \lambda = 1, 2, 3, 4$. Then $\Phi_{(\lambda)}$ for $\lambda = 1, 2, 3$ gives an almost quaternion structure. As $\xi_{(\ )}$ and $\xi_{(4)}$ produce another Sasakian 3-structure, we have $\Phi_{(1)}\Phi_{(4)} = -\Phi_{(4)}\Phi_{(1)}$ and similarly $\Phi_{(\lambda)}\Phi_{(4)} = -\Phi_{(4)}\Phi_{(\lambda)}$ hold good for $\lambda = 1, 2, 3$. Thus we have a contradiction by virtue of Lemma.

REMARK. The corresponding theorem holds good for the case of almost contact 3-structures.

Now suppose that M^n admits r Sasakian structures $\xi_{(\lambda)}, (\lambda = 1, \cdots, r \geqq 3)$, which are orthogonal to one another and that there exist no more Sasakian structures orthogonal to every $\xi_{(\lambda)}$. Consider $\eta = \{\xi_{(1)}, \xi_{(2)}\}$ and put $f_{(\lambda)} = <\eta, \xi_{(\lambda)}>$ for $\lambda = 3, \cdots, r$, where $< \, , \, >$ denotes the inner product. We assume that all $f_{(\lambda)}$ are constant.

If η is of the form

$$\eta = \sum_{\lambda=3}^{r} f_{(\lambda)} \xi_{(\lambda)}, \tag{2}$$

we can find a Sasakian structure ζ orthogonal to each member of the 3-structure $\{\xi_{(1)}, \xi_{(2)}, \eta\}$ for $r > 3$ which lead us to a contradiction, and for the case $r = 3$ we have $\eta = \pm \xi_{(3)}$.

If η is not of the form (2), we define ζ by

$$\zeta = \left(\eta - \sum_{\lambda=3}^{r} f_{(\lambda)} \xi_{(\lambda)}\right) \Big/ \left|\eta - \sum_{\lambda=3}^{r} f_{(\lambda)} \xi_{(\lambda)}\right|,$$

where $|A|$ means the length of A. ζ is Sasakian and orthogonal to $\xi_{(\lambda)}, \lambda = 1, \cdots, r$, which is a contradiction too.

Thus our structures reduce to a Sasakian 3-structure except the case when at least one of $f_{(\lambda)}$ is not constant.

Hence it comes into a problem to study a Riemannian space admitting 2 Sasakian structures whose inner product is not constant.

3. Condition to be a sphere.

Consider a Riemannian space M^n which admits 2 Sasakian structures ξ and η. Assuming that $\rho = <\xi, \eta>$ is not constant, we shall get the differential equation for ρ to satisfy. We have

$$\rho_j \equiv \nabla_j \rho = \eta^i \nabla_j \xi_i + * ,$$

$$\nabla_k \rho_j = \nabla_k \eta^i \nabla_j \xi_i + \eta^i \nabla_k \nabla_j \xi_i + * ,$$

where * means the sum of terms which are obtained from terms written exactly in the same side by interchanging ξ and η.

$$\nabla_l \nabla_k \rho_j = \nabla_l \nabla_k \eta^i \nabla_j \xi_i + \nabla_k \eta^i \nabla_l \nabla_j \xi_i + \nabla_l \eta^i \nabla_k \nabla_j \xi_i + \eta^i \nabla_l \nabla_k \nabla_j \xi_i + *.$$

As ξ, η satisfies (1) and the equation

$$\nabla_l \nabla_k \eta_i = \eta_k g_{li} - \eta_i g_{lk} ,$$

we can get

$$\nabla_l \nabla_k \rho_j + 2\rho_l g_{kj} + \rho_k g_{lj} + \rho_j g_{lk} = 0 . \tag{3}$$

On the other hand the following theorem is known.

THEOREM (Obata [2]). *Let M be a complete simply connected Riemannian space of dimension n. In order that M admit a non-trivial solution ρ for the system of differential equations*

$$\nabla_l \nabla_k \rho_j + c(2\rho_l g_{kj} + \rho_k g_{lj} + \rho_j g_{lk}) = 0 , \quad c > 0,\ \rho_j = \nabla_j \rho ,$$

it is necessary and sufficient that M is isometric with a sphere S^n of radius $1/\sqrt{c}$ in the Euclidean $(n+1)$-space E^{n+1}.

Thus we get

THEOREM 2. *Let M be a complete simply connected Riemannian space of dimension n. If M admits Sasakian structures ξ and η with non-constant $<\xi, \eta>$, then M is isometric with a sphere of radius 1 in E^{n+1}.*

Taking account of the argument in §2 we have

THEOREM 3. *Let M^n be a complete simply connected Riemannian space. If M^n admits r (>1) Sasakian structures orthogonal to one another, then either $r=3$ and n is equal to $4p+3$ for a non-negative integer p or M^n is isometric with a unit sphere in E^{n+1}.*

REMARK. The equation (3) has appeared in Okumura's paper [3], in which he states that if a Sasakian space admits an infinitesimal projective transformation $X=(\xi^h)$ then its associated function ($=$ div X) satisfies (3).

We shall generalize the above theorem, by taking notice of the fact that the constancy of $|\xi|$ (or $|\eta|$) did not play any role in the proof.

Let $u_{i_1 \cdots i_r}$ be a Killing tensor field in a Riemannian space. It is defined as a skew symmetric tensor whose covariant derivative is skew symmetric[5].

We call a Killing tensor $u_{i_1 \cdots i_r}$ is special if it satisfies

$$\nabla_a \nabla_b u_{i_1 \cdots i_r} = -c \left(g_{ab} u_{i_1 \cdots i_r} + \sum_{h=1}^{r} (-1)^h g_{a i_h} u_{b i_1 \cdots \hat{i}_h \cdots i_r} \right),$$

where c is a constant and $\hat{i}_h$ means that i_h is omitted.

In a space of constant curvature, any Killing tensor is special with $c=R/n(n-1)$, where R is the scalar curvature.[6]

By the similar way as the proof of Theorem 2, we can get the following

THEOREM 4. *Let a complete simply connected Riemannian space M^n admit special Killing tensor $u_{i_1 \cdots i_r}$ and $v_{i_1 \cdots i_r}$ with a positive constant c. If $u_{i_1 \cdots i_r} v^{i_1 \cdots i_r}$ is not constant, M^n is isometric with a sphere of radius $1/\sqrt{c}$ in E^{n+1}.*

COROLLARY. *If a complete simply connected Riemannian space M^n admits a special Killing tensor with $c>0$ of non-constant length, it is isometric with a sphere of radius $1/\sqrt{c}$ in E^{n+1}.*

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5), 6) S. Tachibana and T. Kashiwada, [6].