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REMARK ON BEHAVIOR OF SOLUTIONS OF SOME PARABOLIC EQUATIONS

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1. Consider a parabolic equation

$$Lu = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu - \frac{\partial u}{\partial t} = 0$$

in $\Omega = R^n \times (0, \infty)$, where $x = (x_1, \cdots, x_n)$ is a point of the n-dimensional Euclidean space R^n, $t \in (0, \infty)$ the time-variable and $a_{ij} = a_{ji}, b_i$ and c are functions defined in Ω. In this paper, we have some interests in treating behavior of the continuous solution u of the Cauchy problem

$$(1) \qquad \begin{cases} Lu = 0 & \text{in } \Omega, \\ u(x, 0) = f(x) & \text{in } R^n. \end{cases}$$

In the case where $c \leqq 0$ in Ω, some results were obtained by many authors. For instance, we can prove the following.

Suppose that coefficients of the operator L satisfy the following condition in Ω:

$$(2) \qquad \begin{cases} 0 < \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq K_1(|x|^2+1)^{1-\lambda}|\xi|^2 \\ \qquad\qquad \text{for any real vector } \xi = (\xi_1, \cdots, \xi_n) \neq 0, \\ |b_i| \leqq K_2(|x|^2+1)^{1/2}, \quad (i = 1, \cdots, n), \\ c \leqq 0 \end{cases}$$

for some positive K_1, K_2 and $\lambda \in [0, \infty)$. Further, suppose that there exists a positive function $H(x)$ in R^n such that $LH \leqq -\delta$ in R^n for a positive constant δ and such that $H(x)$ tends to infinity as $|x|$ tends to infinity. If a continuous function $u = u(x, t)$ in $\overline{\Omega} = R^n \times [0, \infty)$, satisfying

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$$|u(x,t)| \leqq M_0 \times \begin{cases} (|x|^2+1)^{\mu_0}, & \lambda=0 \\ \exp[\mu_0(|x|^2+1)^{\lambda}] & \lambda>0 \end{cases}$$

in Ω for some positive M_0 and μ_0, is a solution of the Cauchy problem (1) and if $|f(x)|<M$ in R^n for a constant M, then $u(x, t)$ converges to zero uniformly on every compact set in R^n as t tends to infinity.

The special case $\lambda=1$ in the above was proved by Il'in-Kalashnikov-Oleinik [3] and the proof of the above fact is also obtained by using their arguments.

On the other hand, even though c is not non-positive in Ω, we can get the decay of u, similar to the above, under some additional conditions.

The results in this direction were obtained by the writer [2] and by Kuroda [4]. However, in these two works, it was assumed that λ is positive in (2).

In this paper, we shall discuss the asymptotic behavior of solutions of the Cauchy problem (1) under a suitable condition which corresponds to the case $\lambda=0$ in (2) but is different from (2) in the view point that c is not necessarily non-positive.

2. Now suppose that for coefficients of L in (1) there exist positive constants k_1, K_1, K_2, K_3 and K_4 such that

$$(3)\qquad \begin{cases} k_1(|x|^2+1)|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq K_1(|x|^2+1)|\xi|^2 \\ \qquad\qquad \text{for any real vector } \xi=(\xi_1,\cdots,\xi_n), \\ |b_i| \leqq K_2(|x|^2+1)^{1/2}, \qquad (i=1,\cdots,n), \\ c \leqq -K_3(\log(|x|^2+1)+1)^2+K_4. \end{cases}$$

The above condition for c is suggested by Kusano [5]. Throughout this paper, we shall say that $u(x, t)$ is a solution of the Cauchy problem (1) when $u(x, t)$ is continuous in $\overline{\Omega}$, twice continuously differentiable in Ω and satisfies (1).

The purpose of this paper is to prove the following theorem.

THEOREM. *Let $u(x, t)$ be a solution of the Cauchy problem*

$$\begin{cases} Lu=0 & \textit{in } \Omega, \\ u(x,0)=f(x) & \textit{in } R^n \end{cases}$$

such that $|u(x,t)| \leqq \mu \exp(\nu\log(|x|^2+1)+1)^2$ for some positive constants μ and ν. Assume that the coefficients of L in (1) satisfy (3). If the Cauchy data $f(x)$ is bounded in R^n and if

$$(4)\qquad \frac{k_1 n}{2K_1}[(2K_1 + K_2 n) - \sqrt{(2K_1 + K_2 n)^2 + 4K_1K_3}] + K_4 < 0,$$

then $u(x, t)$ *converges to zero uniformly in* $x \in R^n$ *as* t *tends to infinity.*

3. To prove our theorem, we need the following sharpend version of the maximum principle for parabolic equations with unbounded coefficients obtained by Bodanko [1].

LEMMA 1 (Kusano [5]). *Let the differential operator* L *in* (1) *satisfy the condition* (3) *in* Ω. *If a continuous function* $u(x, t)$ *in* $\overline{\Omega}$ *is a solution of* $Lu = 0$ *in* Ω *in the usual sense such that*

$$|u(x, t)| \leqq \mu \exp(\nu \log(|x|^2 + 1) + 1)^2$$

for some positive constants μ *and* ν *in* Ω *and if* $u(x, 0) \geqq 0$ *for* $x \in R^n$, *then* $u(x, t) \geqq 0$ *throughout* Ω.

LEMMA 2. *Let* α *be a positive root of the quadratic equation* $AX^2 + BX + C = 0$ $(A \neq 0)$, *where* $B \geqq 0$ *and* $C < 0$. *Then the function*

$$\varphi(t) = \alpha \tanh A\alpha t$$

satisfies the inequality

$$\varphi'(t) + A\varphi^2(t) + B\varphi(t) + C \leqq 0 .$$

PROOF. Evidently

$$\varphi(t) = 4A\alpha^2 e^{-2A\alpha t}(1 + e^{-2A\alpha t})^{-2},$$

so we get

$$\varphi(t) + A\varphi^2(t) + B\varphi(t) + C$$

$$= [4A\alpha^2 e^{-2A\alpha t} + A\alpha^2(1 - e^{-2A\alpha t})^2 + B\alpha(1 - e^{-4A\alpha t}) + C(1 + e^{-2A\alpha t})^2](1 + e^{-2A\alpha t})^{-2}$$

$$= [A\alpha^2 + B\alpha + C + e^{-2A\alpha t}(4A\alpha^2 - 2A\alpha^2 + 2C) + e^{-4A\alpha t}(A\alpha^2 - B\alpha + C)](1 + e^{-2A\alpha t})^{-2}$$

$$= (e^{-2A\alpha t} + e^{-4A\alpha t}) \frac{-2B\alpha}{(1+e^{-2A\alpha t})^2} \leqq 0 .$$

4. Now we can state the proof of Theorem.

Let $\varphi(t)$ and $\psi(t)$ be functions twice continuously differentiable in $[0, \infty)$. We consider the function

$$H(x, t) = \exp[-\varphi(t)(\log(|x|^2+1)+1)^2 + \psi(t)] . \tag{5}$$

It is easily verified that

$$\begin{aligned} \frac{LH}{H} = {} & 16\varphi^2(t)(\log(|x|^2+1)+1)^2(|x|^2+1)^{-2} \sum_{i,j=1}^{n} a_{ij}x_i x_j \\ & + 8\varphi(t)(\log(|x|^2+1)+1)(|x|^2+1)^{-2} \sum_{i,j=1}^{n} a_{ij}x_i x_j \\ & - 8\varphi(t)(|x|^2+1)^{-2} \sum_{i,j=1}^{n} a_{ij}x_i x_j \\ & - 4\varphi(t)(\log(|x|^2+1)+1)(|x|^2+1)^{-1} \sum_{i=1}^{n} (a_{ii} + b_i x_i) \\ & + c + \varphi'(t)(\log(|x|^2+1)+1)^2 + \psi(t) . \end{aligned}$$

It follows from (3) that

$$\begin{aligned} \frac{LH}{H} \leqq {} & (\log(|x|^2+1)+1)^2[\varphi'(t) + 16K_1\varphi^2(t) + (8K_1 + 4K_2 n)\varphi(t) - K_3] \\ & + (-4k_1 n\varphi(t) + K_4 - \psi'(t)) . \end{aligned}$$

Thus, if we take

$$\varphi(t) = \alpha \tanh 16K_1\alpha t , \tag{6}$$

where α is the positive root of the quadratic equation in X

$$16K_1X^2 + (8K_1 + 4K_2 n)X - K_3 = 0,$$

then we see from Lemma 2 that

$$\varphi'(t) + 16K_1\varphi^2(t) + (8K_1 + 4K_2 n)\varphi(t) - K_3 \leqq 0 .$$

Further, it is easy to see that

(7) $$\psi(t) = -\frac{k_1 n}{4K_1}\log(\cosh 16K_1\alpha t) + K_4 t$$

satisfies

$$-4k_1 n\varphi(t) + K_4 - \psi'(t) = 0$$

for $\varphi(t)$ given by (6). Thus $H(x, t)$ given by (5) for $\varphi(t)$ in (6) and $\psi(t)$ in (7) satisfies

$$LH \leqq 0$$

in Ω.

It is evident that $H(x, 0) = 1$. Further, we can see

(8) $$H(x, t) \leqq 2^{k_1 n/4K_1} \exp[(-4k_1 n\alpha + K_4)t]$$

in Ω. The condition (4) implies boundedness of $H(x, t)$ in Ω. As the Cauchy data $f(x)$ is bounded, we may assume $|f(x)| < M$ in R^n.

If we put

$$W_\pm(x, t) = MH(x, t) \pm u(x, t),$$

then $LW_\pm = MLH \pm Lu \leqq 0$ in Ω and $W_\pm(x, 0) = M \pm u(x, 0) \geqq 0$.

Moreover, we have clearly

$$|W_\pm(x, t)| \leqq \mu^* \exp(\nu^* \log(|x|^2 + 1) + 1)^2$$

in Ω for some constants μ^* and ν^*. Hence we see by Lemma 1 that $W_\pm(x, t) \geqq 0$ in Ω, so from (8) we have

$$\begin{aligned} |u(x, t)| &\leqq MH(x, t) \\ &< M_1 \exp[(-4k_1 n\alpha + K_4)t], \qquad (M = 2^{k_1 n/4K_1} M) \end{aligned}$$

for α in (6) throughout Ω. From the assumption (4), it is obvious that $u(x, t)$ converges to zero uniformly in $x \in R^n$ as t tends to infinity.

5. Finally, in the following we state an example which shows that there is an operator L in (1) satisfying (3) and (4) and having a coefficient c not necessarily non-positive in R^n.

EXAMPLE. Consider a differential equation of the particular form

$$\begin{cases} (|x|^2+1)\sum_{i=1}^{n}\dfrac{\partial^2 u}{\partial x_i{}^2}+cu-\dfrac{\partial u}{\partial t}=0\,, \\ c=\ -K_3(\log(|x|^2+1)+1)^2+K_4. \end{cases}$$

in Ω. Take positive numbers K_3 and K_4 as such as

$$\frac{K_4^2+2nK_4}{n^2}<K_3<K_4\,.$$

This is possible only in the case $0<K_4<n(n-2)$. Then we have

$$K_4^2+2nK_4-n^2K_3<0\,,$$

which is the condition (4) for our equation. Moreover, we see $c(0,t)=-K_3+K_4>0$.

REFERENCES

[1] W. BODANKO, Sur le problème de Cauchy et les problèmes de Fourier pour les équations paraboliques dans un domaine non borné, Ann. Polon. Math., 18(1966), 79-94.

[2] LU-SAN CHEN, Note on the behavior of solutions of parabolic equations with unbounded coefficients, Nagoya Math. J., 37(1970), 1-4.

[3] A. M. IL'IN-A. S. KALASHNIKOV-O. A. OLEINIK, Second order linear equations of parabolic type, Russian Math. Surveys, 17, no. 3 (1962), 1-143.

[4] T. KURODA, Asymptotic behavior of solutions of parabolic equations with unbounded coefficients, Nagoya Math. J., 37(1970), 5-12.

[5] T. KUSANO, Remarks on the behavior of solutions of second order parabolic equations with unbounded coefficients, Funkcial. Ekvac., 11(1968), 197-205.

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