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CLASS NUMBERS OF IMAGINARY ABELIAN NUMBER FIELDS, II

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In the previous paper [9], we proved the following theorem.

THEOREM 1. *For any integer N, there exist only a finite number of imaginary abelian number fields whose first factors h_1 of class numbers are not greater than N.*

We give here another proof derived from Landau's estimate for $L(1,\chi)$ and Siegel's theorem. By applying Tatuzawa's estimate for $L(1,\chi)$ with real character χ, we can also prove

THEOREM 1′. *For any integer N, we can compute an upper bound of the conductors of the imaginary abelian number fields for which $h_1 \leqq N$, except following two cases:*

(i) *imaginary quadratic fields.*
(ii) *imaginary biquadratic fields with Galois groups of type* (2, 2).

Let l be an odd prime number and let K_l be the field of the l-th roots of unity. It has been conjectured that the class number of K_l is greater than 1 if $l \geqq 23$. An upper bound for l such that h_1 of K_l is equal to 1 is computable by Theorem 1′. This has been known by [1] and [7]. We now compute an upper bound, i.e., we have

THEOREM 2. *Let K_l be the field of the l-th roots of unity. Then its first factor h_1 of the class number is greater than 1 if $l > 2400$.*

Let K be an imaginary abelian number field. Let $L(s,\chi)$ be an L-function with character χ corresponding to K. We put

$$L_1(s) = \prod_{\chi_1} L(s,\chi_1),$$

where χ_1 runs over characters such that $\chi_1(-1) = -1$. Also we put

$$L_2(s) = \prod_{\chi_2} L(s, \chi_2),$$

where χ_2 runs over non-trivial characters such that $\chi_2(-1)=1$. Let $\zeta(s)$ be Riemann ζ-function. Then

$$\zeta_K(s) = \zeta(s)L_1(s)L_2(s)$$

is Dedekind ζ-function of K. Theorem 2 will be proved by an estimate for $L_1(1)$, i. e.,

PROPOSITION. *Let $l \equiv 1 \ (mod\ 4)$, and K_l be as above. Then it holds*

$$L_1(1)^{-1} < 215(l-2)(\log l)^{(l-1)/2}$$

for $l > 410.$[1]

1. Proofs of Theorems. Let K be an imaginary abelian number field of degree $n = 2n_0$. Let K_0 be its maximal real subfield. Let h_1 and h_0 be first and second factors of the class number of K. Let R and R_0 be regulators of K and K_0 respectively. Then $R = q^{-1} \cdot 2^{n_0 - 1} R_0$ holds for $q = 1$ or 2 [3]. Let d and d_0 be absolute values of discriminants of K and K_0 respectively. Let w be the number of the roots of unity in K. Then it is known

$$L_1(1)L_2(1) = \lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{n_0}\pi^{n_0}Rh_1h_0}{w\sqrt{d}}$$

and

$$L_2(1) = \lim_{s\to 1}(s-1)\zeta_{K_0}(s) = \frac{2^{n_0}R_0h_0}{2\sqrt{d_0}}$$

Then it holds

$$L_1(1) = \frac{2^{n_0}\pi^{n_0}h_1\sqrt{d_0}}{qw\sqrt{d}}$$

or

$$h_1 = \frac{qw\sqrt{d}}{2^{n_0}\pi^{n_0}\sqrt{d_0}} L_1(1)$$

$$\geqq \frac{1}{2^{n_0}\pi^{n_0}} d^{1/4} L_1(1).$$

1) $L_1(1)$ is positive real. See section 1.

LEMMA 1. *Let k be the conductor of K, i.e., the smallest positive integer such that K is contained in the field of the k-th roots of unity. Then it holds*

$$d \geqq k^{n_0}.$$

PROOF. Let p be a prime divisor of k. Then d is divisible by p. The p-part d_p of d satisfies an inequality

$$d_p = N\mathfrak{D}_p \geqq N\prod_{\mathfrak{p}|p} \mathfrak{p}^{e-1} = p^{n(1-1/e)} \geqq p^{n_0},$$

where $\mathfrak{D}_p$ is the p-part of the different of K and $\mathfrak{p}$ is a divisor of p in K with ramification index $e \geqq 2$. If k is just divisible by p^s and $s \geqq 2$, K contains a cyclic subfield K_p whose degree is a power of p and conductor is divisible by p^s. Then the conductor-discriminant formula [2, Chap. VI. §4. 4] shows that the discriminant of K_p is divisible by $p^{st/2}$, where t is the degree of K_p. Then it holds

$$d_p \geqq (p^{st/2})^{n/t} = p^{n_0 s}.$$

Therefore

$$d \geqq \prod_p p^{n_0 s} = (\prod_p p^s)^{n_0} = k^{n_0}.$$

Landau's estimate for $L(1,\chi)$ shows [3]

$$|L(1,\chi)|^{-1} < C \log k$$

for non-real character χ, where C is a computable constant. Real character χ corresponds to a quadratic subfield of K. Siegel's theorem [5] shows that for any $\varepsilon > 0$

$$L(1,\chi) > k^{-\varepsilon}$$

holds for almost all χ. Let ε be equal to 1/5, and let

$$b = \mathrm{Min}(\prod L(1,\chi), 1)$$

where the product is taken over finite χ not satisfying the above inequality. Then

$$h_1 > \left(\frac{k^{1/4}}{2\pi C \log k \cdot k^{1/5}}\right)^{n_0} \cdot b = \left(\frac{k^{1/20}}{2\pi C \log k}\right)^{n_0} \cdot b,$$

and the right hand side becomes large with k. This proves Theorem 1. Tatuzawa's theorem [8] shows that there exists a computable constant $C(\varepsilon)$ for any $\varepsilon>0$ such that

$$L(1,\chi)>C(\varepsilon)k^{-\varepsilon}$$

holds for any (with at most one exception) real character χ. As

$$L(1,\chi)>\frac{1}{\sqrt{k}}$$

for any real character χ, it holds

$$h_1>\left(\frac{C(\varepsilon)k^{1/4-\varepsilon}}{2\pi C\log k}\right)^{n_0}\frac{1}{\sqrt{k}}.$$

If $n_0\geqq 3$, we take $\varepsilon<1/12$. Then h_1 becomes large with k, and C and $C(\varepsilon)$ are computable. In the case that K is cyclic of degree 4, there exist no real character χ with $\chi(-1)=-1$. So it holds

$$h_1>\left(\frac{k^{1/4}}{2\pi C\log k}\right)^{n_0}.$$

This completes the proof of Theorem 1′. Next we assume the Proposition and deduce Theorem 2. If $l\equiv 3 \pmod 4$, K_l contains an imaginary quadratic subfield whose class number is greater than 1 if $l>163$. Then the following lemma shows that $h_1>1$ if $l>163$.

LEMMA 2. *Let K be an imaginary subfield of K_l. If $h_1=1$, so is the first factor of the class number of K.*

PROOF. Let $K_{l,0}$ and K_0 be maximal real subfields of K_l and K respectively. Let E be HCF(Hilbert class field) of $K_{l,0}$, and let F be HCF of K. By assumption $E\cdot K_l$ is HCF of K_l. So $F_1=F\cdot K_l$ is contained in $E\cdot K_l$. As the Galois group of $E/K_{l,0}$ and that of $E\cdot K_l/K_l$ are isomorphic, there exists unique subfield F_2 of E corresponding to F_1. F_1 is normal over K_0, as F is normal over K_0. The Galois group of F_1/K is abelian which is isomorphic to the product of the Galois group of F/K and that of K_l/K. F_2 is totally real as a subfield of E. As $[F_1: F_2]=2$, F_2 is normal over K_0. Then the Galois group of F_2/K_0 is abelian which is isomorphic to that of F_1/K. Let F_3 be inertia field of l in F_2/K_0. Then F_3 is HCF of K_0 and $F=K\cdot F_3$. This proves the lemma.

We now assume $l\equiv 1 \pmod 4$. In this case,

$$h_1 = 2l(2\pi)^{(1-l)/2}l^{(l-1)/4}L_1(1) > \frac{2l}{215(l-2)}\left(\frac{\sqrt{l}}{2\pi\log l}\right)^{(l-1)/2}$$

$$> \frac{1}{108}\left(\frac{\sqrt{l}}{2\pi\log l}\right)^{(l-1)/2}.$$

Then $h_1 > 1$, if

$$\sqrt{l} > 2\pi\cdot 108^{2/(l-1)}\log l\,.$$

This inequality holds for $l = 2417$, as

$$2\pi\cdot 108^{1/1208}\log 2417 < 6.284\times 1.004\times 7.791 < 49.16 < \sqrt{2417}\,.$$

Then it holds for $l \geqq 2417$. As 2417 is the least prime number over 2400 such that $l\equiv 1 \pmod 4$, this proves Theorem 2.

2. Lemmas. The rest of this paper is devoted to the proof of Proposition. Techniques of the proof are almost equal to those of Landau [4]. But complete proofs are given for the convenience of reader. In this section K denotes an imaginary abelian number field with conductor k.

LEMMA 3. (Landau [5, Hilfssatz]). *Let s_0 be a complex number. Let $f(z)$ be a holomorphic function on $|z-s_0|\leqq r$* such that $f(z)\neq 0$ *if* $\Re z > \Re s_0$. *If*

$$\left|\frac{f(z)}{f(s_0)}\right| \leqq e^M$$

in this circle for some positive constant M, we have

$$-\Re\frac{f'}{f}(s_0) \leqq \frac{4M}{r}\,.$$

Moreover if $f(z)$ has zero points in the circle $|z-s_0|\leqq \frac{r}{2}$, we have

$$-\Re\frac{f'}{f}(s_0) \leqq \frac{4M}{r} - \sum_{\rho}\Re\frac{1}{s_0-\rho},$$

where ρ *runs over zero points in the circle* $|z-s_0|\leqq\frac{r}{2}$ *with their multiplicities.*

PROOF. We put

$$g(z)=f(z)/\prod(z-\rho).$$

Then $g(z)$ is holomophic over $|z-s_0|\leqq r$ and has no zero in $|z-s_0|\leqq\frac{r}{2}$. If $|z-s_0|=r$, *it* holds

$$\left|\frac{g(z)}{g(s_0)}\right|=\left|\frac{f(z)}{f(s_0)}\right|\cdot\left|\frac{\Pi(s_0-\rho)}{\Pi(z-\rho)}\right|\leqq\left|\frac{f(z)}{f(s_0)}\right|\leqq e^{M}.$$

Then this inequality also holds in $|z-s_0|\leqq\frac{r}{2}$ by Maximum Principle. There exists a holomorphic function $h(z)$ over $|z-s_0|\leqq\frac{r}{2}$ such that

$$e^{h(z)}=\frac{g(z)}{g(s_0)},\ h(s_0)=0,\ \Re h(z)\leqq M.$$

If we put

$$\phi(z)=\frac{h(z)}{2M-h(z)},$$

it holds

$$|\phi(z)|\leqq 1,\quad \phi(s_0)=0$$

for $|s-s_0|\leqq\frac{r}{2}$. By Schwarz's theorem it holds

$$|\phi'(s_0)|=|h'(s_0)/2M|\leqq\frac{2}{r}.$$

As

$$h'(s_0)=\frac{g'}{g}(s_0)=\frac{f'}{f}(s_0)-\sum_{\rho}\frac{1}{s_0-\rho},$$

and as $\Re\frac{1}{s_0-\rho}\geqq 0$, Lemma follows at once from above inequality.

LEMMA 4 (Landau [4, Hilfssatz]). *Let* $s_0=1+\varepsilon$, *where* ε *is positive real such that* $\varepsilon<\frac{r}{1000}$. *Let* $f(z)$ *be holomorphic over* $|z-s_0|\leqq r$ *such that* $f(z)\neq 0$ *for* $\Re z\geqq 1$. *We assume* $\left|\frac{f(z)}{f(s_0)}\right|\leqq e^M$ *in this circle. Let* $b=\operatorname{Max}\Re\frac{1}{s_0-\rho}$ *where* ρ *runs over all zeros of* $f(z)$ *such that* $|\rho-s_0|\leqq\frac{r}{2}$. *If we put*

$$s=1+x\varepsilon,\quad 0\leqq x\leqq 1,$$

we have

$$\Re\frac{f'}{f}(s)<\frac{1}{1+b(x-1)\varepsilon}\sum_\rho\Re\frac{1}{s_0-\rho}+\frac{4.05}{r}M.$$

PROOF. Let $h(z)$ and $\phi(z)$ be as in the proof of Lemma 3. Then by Schwarz's theorem we have

$$|\phi(s)|\leqq\frac{2\varepsilon}{r}<\frac{1}{500}.$$

Hence

$$|h(s)|=\left|\frac{2M\phi(s)}{1+\phi(s)}\right|<\frac{2M}{499}.$$

Therefore

$$\Re(h(z)-h(s))<\frac{501}{499}M<1.005\,M.$$

If we put

$$\psi(z)=\frac{h(z)-h(s)}{2.01M-h(z)+h(s)},$$

$\psi(z)$ is holomorphic over $|z-s_0|\leqq\frac{r}{2}$ and

$$|\psi(z)|<1,\quad \psi(s)=0$$

By Schwarz's theorem

$$|\psi'(s)|=\left|\frac{h'(s)}{2.01\,M}\right|<\frac{1}{r/2-\varepsilon}<\frac{2.01}{r}.$$

Then

$$|h'(s)| < \frac{4.05}{r} M$$

and

$$\Re \frac{f'}{f}(s) < \sum_{\rho} \Re \frac{1}{s-\rho} + \frac{4.05}{r} M.$$

Hence it suffices to show

$$\Re \frac{1}{s-\rho} \leqq \frac{b_\rho}{1 + b_\rho (x-1)\varepsilon}$$

for any root ρ, where

$$b_\rho = \Re \frac{1}{s_0-\rho} \leqq b\,.$$

If we put

$$\rho = \sigma + it\,, \quad \sigma < 1\,,$$

it holds

$$\Re \frac{1}{s-\rho} = \frac{s-\sigma}{(s-\sigma)^2+t^2} = \frac{s-\sigma}{(s-\sigma)^2-(s_0-\sigma)^2+(s_0-\sigma)^2+t^2}$$

$$= \frac{s-\sigma}{2(s-s_0)(s-\sigma)-(s-s_0)^2+(s_0-\sigma)/b_\rho}$$

$$= \frac{b_\rho(s-\sigma)}{(2b_\rho(s-s_0)+1)(s-\sigma)-b_\rho(s-s_0)^2+s_0-s}\,.$$

This attains the greatest value when $\sigma = s_0 - 1/b_\rho$ and $t = 0$. Hence it holds

$$\Re \frac{1}{s-\rho} \leqq \frac{b_\rho}{b_\rho(s-s_0)+1} = \frac{b_\rho}{b_\rho(x-1)\varepsilon+1}\,.$$

LEMMA 5. *For any real* $s > 1$,

(1) $$\log \zeta(s) + \Re \log L_1(s) + \Re \log L_2(s) \geqq 0\,,$$

and

(2) $$-\frac{\zeta'}{\zeta}(s)-\Re\frac{L_1'}{L_1}(s)-\Re\frac{L_2''}{L_2}(s)\geqq 0.$$

PROOF. For any character χ (including the case $\chi=1$),

$$\log L(s,\chi) = -\sum_p \log\left(1-\frac{\chi(p)}{p^s}\right)=\sum_p\left(\frac{\chi(p)}{p^s}+\frac{\chi^2(p)}{2p^{2s}}+\cdots\right)$$

and

$$-\frac{L'}{L}(s,\chi)=\sum_p \log p\left(\frac{\chi(p)}{p^s}+\frac{\chi^2(p)}{p^{2s}}+\cdots\right)$$

hold for any real $s>1$, where sums are taken over all prime numbers. Inequalities (1) and (2) are obtained from above equalities by summing up for all characters.

LEMMA 6. *Let the conductor k be greater than* 410. *Let $s_0>1$ be real. Let $L(s,\chi)$ be an L-function with non-trivial character χ. Then for any complex number s such that $|s-s_0|\leqq\frac{2}{3}$, it holds*

$$|L(s,\chi)|<2k^{2/3}.$$

PROOF. Let $s=\sigma+it$ be such that $|s-s_0|\leqq\frac{2}{3}$. We put $S(n)=\sum_{r=1}^{n}\chi(r)$. Then it holds Max $|S(n)|\leqq\frac{k}{2}$. It holds

$$\begin{aligned}|L(s)|&<\left|\sum_{n=1}^{k}\frac{1}{n^\sigma}\right|+\left|\sum_{n=k+1}^{\infty}S(n)\cdot s\int_n^{n+1}\frac{dx}{x^{s+1}}\right|\\&<\sum_{n=1}^{k}n^{-1/3}+\frac{k}{2}\cdot\frac{|s|}{\sigma}\sum_{n=k+1}^{\infty}\left(\frac{1}{n^\sigma}-\frac{1}{(n+1)^\sigma}\right)\\&<\frac{3}{2}k^{2/3}+\frac{k^{1-\sigma}}{2}\cdot\frac{|s|}{\sigma}.\end{aligned}$$

Desired inequality is obtained if we show

$$\frac{|s|}{\sigma}k^{1/3-\sigma}\leqq 1,$$

or

$$k^{2\sigma-2/3}-\frac{\sigma^2+t^2}{\sigma^2}\geqq 0\,.$$

This inequality is shown by replacing t^2 by $\frac{4}{9}-(\sigma-s_0)^2$ and examining its derivative with respect to σ, because $\log k>6$.

LEMMA 7. *For any non-trivial character χ, it holds*

$$|L(s,\chi)|<\log k$$

for any real $s\geqq 1$.

PROOF. As $\sum_{n=1}^{k}\chi(n)=0$, it holds

$$\begin{aligned}
|L(s,\chi)| &\leqq \left|\sum_{n=1}^{k}\chi(n)\left(\frac{1}{n^s}-\left(\frac{2}{k}\right)^s\right)\right|+\left|\sum_{n=k+1}^{\infty}S(n)\left(\frac{1}{n^s}-\frac{1}{n^{s+1}}\right)\right| \\
&\leqq \sum_{n=1}^{[k/2]}\left(\frac{1}{n^s}-\left(\frac{2}{k}\right)^s\right)+\sum_{[k/2]+1}^{k}\left(\left(\frac{2}{k}\right)^s-\frac{1}{n^s}\right)+\frac{k}{2}\cdot\frac{1}{k^s} \\
&\leqq \sum_{n=1}^{[k/2]}\left(\frac{1}{n^s}-\frac{1}{(k-n+1)^s}\right)+\frac{1}{2} \\
&< 1+\sum_{n=2}^{[k/2]}\left(\frac{1}{n}-\frac{1}{k-n+1}\right)+\frac{1}{2} \\
&< 1+\frac{1}{2}+\left(\log\frac{k}{2}-\log 2\right)-\left(\log k-\log\frac{k+3}{2}\right)+\frac{1}{2} \\
&< 2-3\log 2+\log(k+3) \\
&< \log k \text{ (This holds for } k>50)\,.
\end{aligned}$$

LEMMA 8 (Landau [4]). *For any real $s>1$*

$$\frac{1}{s-1}<\zeta(s)<1+\frac{1}{s-1}$$

and

$$-\frac{\zeta'}{\zeta}(s)<s-1+\frac{1}{s-1}.$$

PROOF. They are easily seen by

$$\frac{1}{s-1}=\int_1^\infty \frac{dx}{x^s}<\zeta(s)=\sum_n \frac{1}{n^s}<1+\int_1^\infty \frac{dx}{x^s}=1+\frac{1}{s-1}$$

and

$$-\zeta'(s)=\sum_n \frac{\log n}{n^s}<\frac{\log 2}{2}+\frac{\log 3}{3}+\int_1^\infty \frac{\log x}{x^s}dx$$

$$<1+\frac{1}{(s-1)^2}.$$

3. Proof of Proposition. We now apply above lemmas for $L_1(s)$ and $L_2(s)$ From now on we put

$$r=\frac{2}{3},\ \varepsilon=\frac{1}{a\log l}\ \text{and}\ s_0=1+\frac{1}{a\log l}.$$

We assume

$$l>410\ \text{and}\ a\geqq 250.$$

Then $L_1(s)$ and $L_2(s)$ satisfy conditions of Lemmas 3, 4 and 6, as the conductor $k=l$ in the situation of the Proposition. We now calculate corresponding M. Lemmas 5(1), 6, 7 and 8 give

$$(3)\qquad M_1=\operatorname{Max}\log\left|\frac{L_1(s)}{L_1(s_0)}\right|<\frac{l-1}{2}\log 2+\frac{l-1}{3}\log l+\frac{1}{a\log l}$$

$$+\log a+\frac{l-1}{2}\log\log l.$$

If we put

$$M_2=\operatorname{Max}\log\left|\frac{L_2(s)}{L_2(s_0)}\right|,$$

it holds

$$(4)\qquad M_1+M_2=\operatorname{Max}\log|L_1(s)|+\operatorname{Max}\log|L_2(s)|-\Re\log L_1(s_0)-\Re\log L_2(s_0)$$

$$<(l-2)\log 2+\frac{2}{3}(l-2)\log l+\frac{1}{a\log l}+\log a+\log\log l$$

Now we estimate $L_1(1)$. Lemmas 3, 5(2) and 8 give

$$(5)\qquad \sum_{\rho} \Re \frac{1}{s_0-\rho} \leqq \Re \frac{L_1'}{L_1}(s_0) + 6M_1 \leqq -\frac{\zeta'}{\zeta}(s_0) - \Re \frac{L_2'}{L_2}(s_0) + 6M_1$$

$$< \frac{1}{a\log l} + a\log l + 6(M_1+M_2),$$

where ρ runs over all zeros of $L_1(s)$ such that $|\rho - s_0| \leqq \frac{1}{3}$. We put

$$A = \frac{1}{a\log l} + a\log l + 6(M_1+M_2).$$

If ρ is a zero of $L_1(s)$, i. e., a zero of some $L(s, \chi)$. $\bar{\rho}$ also is a zero of $L_1(s)$, as it is a zero of $L(s, \bar{\chi})$. As $l \equiv 1 (\mathrm{mod}\ 4)$, $\chi \neq \bar{\chi}$ for any χ such that $\chi(-1) = -1$. As every $\Re \frac{1}{s_0-\rho}$ is positive, and as $\Re \frac{1}{s_0-\rho} = \Re \frac{1}{s_0-\bar{\rho}}$, it holds

$$\Re \frac{1}{s_0-\rho} < \frac{A}{2}$$

for every ρ. By Lemma 4 and (5)

$$\Re \frac{L_1'}{L_1}(s) < \frac{2aA\log l}{2a\log l + A(x-1)} + 6.1M_1$$

$$= \frac{2a\log l}{2a\log l/A + x - 1} + 6.1M_1$$

for

$$s = 1 + \frac{x}{a\log l}, \quad 0 \leqq x \leqq 1.$$

For $x = 0$, the right hand side of the above inequality takes the smallest value near $a = 4(1+\sqrt{2})(l-2)$. If we put

$$a = 4(1+\sqrt{2})(l-2),$$

$$A < \frac{7}{a\log l} + a\log l + 4(l-2)\log l + 6(l-2)\log 2 + 6\log 4(1+\sqrt{2})$$

$$+ 6\log(l-2) + 6\log\log l$$

$$< 4(2+\sqrt{2})(l-2)\log l + 4.16(l-2) + 7.8\log l + 13.61$$

$$< 13.66(l-2)\log l + 0.694(l-2)\log l + 10.07\log l$$

$$< 14.4(l-2)\log l$$

and

$$M_1 < \frac{l-1}{2}\log 2 + \frac{l-1}{3}\log l + \frac{1}{a\log l} + \log 4(1+\sqrt{2})$$

$$+ \log l + \frac{l-1}{2}\log\log l$$

$$< \frac{l-2}{3}\log l + 0.15(l-2)\log l + 0.347(l-2) + \frac{4}{3}\log l$$

$$+ 0.15\log l + 0.347 + 2.268 + 0.001$$

$$< 0.484(l-2)\log l + 0.058(l-2)\log l + 1.93\log l$$

$$< 0.547(l-2)\log l\,.$$

In the above estimate we use

$$\log l > 6 \text{ and } \log\log l < 0.3\log l$$

for $l > 410$. Therefore

$$\Re\frac{L_1'}{L_1}(s) < \frac{2a\log l}{0.34+x} + 3.4(l-2)\log l\,.$$

Then

$$-\log L_1(1) = -\Re\log L_1(s_0) + \int_1^{s_0}\Re\frac{L_1'}{L_1}(s)ds$$

$$< \frac{1}{a\log l} + \log 4(1+\sqrt{2}) + \log(l-2) + \frac{l-1}{2}\log\log l$$

$$+ 2\log\frac{1.34}{0.34} + \frac{3.4}{4(1+\sqrt{2})}$$

$$< \log(l-2) + \frac{l-1}{2}\log\log l + 0.001 + 2.268$$

$$+ 2 \times 1.374 + 0.353$$

$$= \log(l-2) + \frac{l-1}{2} \log \log l + 5.37$$

Therefore

$$\frac{1}{L_1(1)} < 215(l-2)(\log l)^{(l-1)/2}.$$

This completes the proof.

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