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ON THE C_p-CLASSES IN THE MAXIMAL CCR IDEAL OF A VON NEUMANN ALGEBRA

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1. **Introduction.** Let H be a Hilbert space and $B(H)$ the algebra of all bounded operators on H and $C(H)$ the uniformly closed ideal of all completely continuous operators on H. If a is an element of $C(H)$, then the positive operator $|a| = (a^*a)^{1/2}$ is also a non-negative self adjoint operator of $C(H)$. The eigenvalues $\mu_1, \mu_2 \cdots$ of $|a|$, arranged in decreasing order and repeated according to the multiplicities form a sequence of numbers approaching to zero. These numbers are called the characteristic numbers of the operator a, and the n-th characteristic number of a is written $\mu_n(a)$ [3]. Furthermore, Dunford-Schwartz defines the classes C_p by the following; $C_p = \left\{ a \in C(H);\ |a|_p = \left\{ \sum_{n=1} \mu_n(a)^p \right\}^{1/p} < \infty \right\}$. We show the extension of C_p-classes to type I von Neumann algebras. Let M be a type I von Neumann algebra with the center Z and let $C_\infty(M)$ be the uniformly closed ideal in M generated by all abelian projections in M. Then, $C_\infty(M)$ is a CCR-ideal in M and is the natural analogue in M of the ideal of completely continuous operators on a Hilbert space. By the above consideration, H. Halpern [6] has showed that every positive element a in $C_\infty(M)$ may be written in the form $a = \sum_{i=1}^{\infty} a_i e_i$ where $\{e_i\}$ is a sequence of mutually orthogonal abelian projections such that $e_1 \succsim e_2 \succsim \cdots$ and $\{a_i\}$ is a sequence of positive central elements such that $a_1 \geqq a_2 \geqq \cdots$ and $\lim a_i = 0$. In this note, we shall define the characteristic operators and argue some properties of the characteristic operators. Furthermore, we shall set the classes $C_p(M)$ in a type I von Neumann algebra M by using the characteristic operators and consider the dual spaces of the classes $C_p(M)$ by using the center (Z)-module linear functionals.

2. **Spectral decomposition of positive elements in $C_\infty(M)$ and characteristic operators.** Let M be a type I von Neumann algebra with the center Z and let X be the spectrum of Z. For each $\zeta \in X$, define $[\zeta]$ to be the closed ideal given by

$$[\zeta] = \text{the uniform closure of } \left\{ \sum_{i=1}^{n} a_i z_i \,;\ a_i \in M,\ z_i \in Z \text{ and } z_i{}^{\wedge}(\zeta) = 0 \right\}.$$

There is for each $\zeta \in X$ an irreducible representation π_ζ of M whose kernel is $[\zeta]$ on the Hilbert space $H(\zeta)$. We denote the image of a in M under π_ζ by $a(\zeta)$. Then the function $\zeta \to ||a(\zeta)||$ of X into the positive real numbers is a continuous function. The image of $C_\infty(M)$ under π_ζ is the ideal of all completely continuous operators of $H(\zeta)$. We need the following result, which has been showed by H. Halpern [6].

THEOREM 1 (H. Halpern). *Let M be a von Neumann algebra of type I with the center Z and $C_\infty(M)$ the ideal generated by all abelian projections in M. Let a be a positive element in $C_\infty(M)$. Then, there exist an at most countable set $\{e_i\}_{i=1}^{r}$ of mutually orthogonal abelian projections such that $e_1 \succsim e_2 \succsim \cdots$ and at most countable set $\{a_i\}_{i=1}^{r}$ of positive elements in Z such that $a_1 \geqq a_2 \geqq \cdots$ and such that $\lim_{i\to\infty} a_i = 0$ if $\{a_i\}_{i=1}^{r}$ is infinitely many with the property $a = \sum_{i=1}^{r} a_i e_i$ in the uniform topology.*

Furthermore, the representation obtained in the above situation is unique. That is, we have; let a be a positive element in $C_\infty(M)$ and let $\{a_i\}_{i=1}^{m}$ (resp. $\{b_i\}_{i=1}^{n}$) be a set of positive central elements and $\{e_i\}_{i=1}^{m}$ (resp. $\{f_i\}_{i=1}^{n}$) be a set of orthogonal abelian projections with the following properties: (1) $a_i \neq 0$ (resp. $b_i \neq 0$) for all i; (2) $a_1 \geqq a_2 \geqq \cdots$ (resp. $b_1 \geqq b_2 \geqq \cdots$); (3) if X is the spectrum of Z, then $\{\zeta \in X | e_i(\zeta) \neq 0\}$ = closure of $\{\zeta \in X | a_i{}^{\wedge}(\zeta) \neq 0\}$ (resp., $\{\zeta \in X | f_i(\zeta) \neq 0\}$ = closure of $\{\zeta \in X | b_i{}^{\wedge}(\zeta) \neq 0)$ for every i; (4) if $m = +\infty$ (resp., $n = +\infty$), then $\lim_{i\to\infty} a_i = 0$ (resp., $\lim_{i\to\infty} b_i = 0$); (5) $\sum_{i=1}^{m} a_i e_i = a$ (resp., $\sum_{i=1}^{n} b_i f_i = a$). Then $m = n$ and $a_i = b_i$ for every i.

PROOF. See [6; Theorem 2.2 and 2.3].

By Theorem 1, we set the following definition.

DEFINITION 1. We call the representation for positive element a of $C_\infty(M)$ in Theorem 1 as a spectral representation for a and element of $C_\infty(M)$ as completely continuous element. Furthermore, for any element a in $C_\infty(M)$, we have the spectral representation of $|a|$; $|a| = \sum_{i=1}^{\infty} a_i e_i$, then we define the n-th characteristic operator $\mu_n(a)$ of a to be a_n.

Then we can extend the properties of characteristic number for completely continuous operator on a Hilbert space that is seen in [3].

LEMMA 1. *For any element a in $C_\infty(M)$ and any element ζ of X, we have the equality $\mu_n(a)^\wedge(\zeta) = \mu_n(a(\zeta))$ where $\mu_n(a(\zeta))$ is the n-th characteristic number of the operator $a(\zeta)$ on $H(\zeta)$.*

PROOF. By the definition of characteristic operetor, $\mu_n(a)^\wedge(\zeta) = a_n{}^\wedge(\zeta)$. Furthermore, we have;

$$|a(\zeta)| = (a(\zeta)^*a(\zeta))^{1/2} = ((a^*a)(\zeta))^{1/2} = (a^*a)^{1/2}(\zeta)$$

and

$$(a^*a)^{1/2}(\zeta) = \sum_{i=1}^{\infty} a_i{}^\wedge(\zeta)e_i(\zeta)\,.$$

The projection e_i is an abelian projection, so $e_i(\zeta)$ is a one-dimensional projection on $H(\zeta)$. Furthermore, the sequence $\{a_i{}^\wedge(\zeta)\}_{i=1}^{\infty}$ is a monotone decreasing sequence and $|a(\zeta)| = \sum_{i=1}^{\infty} a_i{}^\wedge(\zeta)e_i(\zeta)$. Therefore, $\mu_n(a(\zeta)) = a_n{}^\wedge(\zeta) = \mu_n(a)^\wedge(\zeta)$.

LEMMA 2. *The characteristic operators of completely continuous elements a and b in M satisfy the inequality*

$$\mu_{n+m+1}(a+b) \leqq \mu_{n+1}(a) + \mu_{m+1}(b)$$

and

$$\mu_{n+m+1}(ab) \leqq \mu_{n+1}(a) \cdot \mu_{m+1}(b)\,.$$

PROOF. For each $\zeta \in X$, we have the equality $(a^*a)^{1/2}(\zeta) = \sum_{i=1}^{\infty} a_i{}^\wedge(\zeta)e_i(\zeta)$. Since $e_i(\zeta)$ is a one-dimensional projection on $H(\zeta)$, by [3; p. 1089], we have

$$\begin{aligned}\mu_{n+m+1}(a+b)^\wedge(\zeta) &= \mu_{n+m+1}(a(\zeta) + b(\zeta)) \\ &\leqq \mu_{n+1}(a(\zeta)) + \mu_{m+1}(b(\zeta)) \\ &= \mu_{n+1}(a)^\wedge(\zeta) + \mu_{m+1}(b)^\wedge(\zeta)\,.\end{aligned}$$

Since $\zeta \in X$ is any element, we have: $\mu_{n+m+1}(a+b) \leqq \mu_{n+1}(a) + \mu_{m+1}(b)$.

By using the same argument, we can show the second inequality.

LEMMA 3. *For $a,b \in C_\infty(M)$, we have;*

(a) $$\|\mu_n(a)-\mu_n(b)\| \leqq \|a-b\|;$$

(b) $$\mu_n(at) \leqq \mu_n(a)\|t\| \quad \text{and} \quad \mu_n(ta) \leqq \|t\|\mu_n(a) \quad \text{for } t \in M;$$

(c) $$\mu_n(au)=\mu_n(a) \quad \text{if} \quad 1=\|u\|=\|u^{-1}\| \quad \text{and} \quad u \in M.$$

PROOF. At first, we show the assertion (a). For each $\zeta \in X$. By the Lemma 2, we have;

$$|\mu_n(a)^\wedge(\zeta)-\mu_n(b)^\wedge(\zeta)| = |\mu_n(a(\zeta))-\mu_n(b(\zeta))|.$$

Since $\mu_n(a(\zeta))$ is the n-th characteristic number of completely continuous operator $a(\zeta)$ on $H(\zeta)$, we can adopt the fact in [3; p. 1090] to yield that

$$\begin{aligned} |\mu_n(a)^\wedge(\zeta) - \mu_n(b)^\wedge(\zeta)| &= |\mu_n(a(\zeta)) - \mu_n(b(\zeta))| \\ &\leqq \|a(\zeta) - a(\zeta)\| \\ &= \|(a-b)(\zeta)\| \\ &\leqq \|a-b\|. \end{aligned}$$

which proves the assertion (a).

Similarly, for all $\zeta \in X$,

$$\mu_n(at)^\wedge(\zeta) = \mu_n(a(\zeta)t(\zeta)) \leqq \mu_n(a(\zeta))\|t(\zeta)\| \leqq \mu_n(a)^\wedge(\zeta)\|t\|$$

and

$$\mu_n(ta)^\wedge(\zeta) = \mu_n(t(\zeta)a(\zeta)) \leqq \mu_n(a(\zeta))\|t(\zeta)\| \leqq \mu_n(a)^\wedge(\zeta)\|t\|.$$

Therefore we have: $\mu_n(at) \leqq \mu_n(a)\|t\|$ and $\mu_n(ta) \leqq \|t\|\mu_n(a)$.

We can similarly show the last assertion (c).

3. The classes $C_p(M)$ of completely continuous elements in the type I von Neumann algebra M. In this section, we shall define the classes $C_p(M)$ of completely continuous elements in M and show that these $C_p(M)$ are Banach algebras.

In terms of the characteristic operators, we may define the various norms for any class of completely continuous elements.

DEFINITION 2. We set the following definition.

(a) For each $a \in C_\infty(M)$, $\|a\|_p = \left\| \left\{ \sum_{n=1}^{\infty} \mu_n(a)^p \right\}^{1/p} \right\|$; $\infty > p \geqq 1$;

(b) $C_p(M)$ is the set of all completely continuous elements a such that $\|a\|_p$ is finite.

In particular, we provide $C_\infty(M)$ with the uniform operator norm.

The following result states some useful elementary property of the spaces $C_p(M)$.

PROPOSITION 1.

(a) *We have* $C_p(M) \subset C_{p'}(M)$ *if* $p \leqq p'$, $\|a\|_p \downarrow$ *for* $p \uparrow$;

(b) *If* a, b *are in* $C_p(M)$, *then* $a+b$ *is in* $C_p(M)$ *and* $\|a+b\|_p \leqq \|a\|_p + \|b\|_p$;

(c) *If* a *is* $C_p(M)$ *and* b *is in* $C_q(M)$, *then* ab *is in* $C_1(M)$, *where* $1/p + 1/q = 1$. *Moreover* $\|ab\|_1 \leqq \|a\|_p \|b\|_q$;

(d) *If* a *is in* $C_p(M)$ *and* t *is in* M, *then* at *and* ta *are in* $C_p(M)$; *moreover,* $\|at\|_p \leqq \|a\|_p \|t\|$ *and* $\|ta\|_p \leqq \|t\| \cdot \|a\|_p$.

The proof of Proposition 1 can be easily showed by using lemma 9 and lemma 14 in [3 ; p. 1098] and our Lemma 2, so we shall omit the proof.

By Proposition 1, the classes $C_p(M)$ are normed algebras. Furthermore, we show in the following theorem that the normed algebras $C_p(M)$ are complete with respect to this norm, that is, the classes $C_p(M)$ are Banach algebras.

THEOREM 2. *If* $\{a_n\}_{n=1}^{\infty}$ *is a sequence in* $C_p(M)$ *such that* $\|a_n - a_m\|_p \to 0$ *as* $m, n \to \infty$, *there exists a completely continuous element* a *of* $C_p(M)$ *such that* $\|a_n - a\|_p \to 0$ *as* $n \to \infty$.

PROOF. By the fact $\|a\| \leqq \|a\|_p$ for each $a \in C_p(M)$ and the fact that $C_\infty(M)$ is closed in the uniform topology of operator, there exists a completely continuous elements a such that $\|a_n - a\| \to 0$ as $n \to \infty$. Thus, by Lemma 3, $\|\mu_k(a_n - a_m) - \mu_k(a_n - a)\| \to 0$ as $m \to \infty$. It follows that, for each $\zeta \in X$ and each positive integer N,

$$\left\{ \sum_{k=1}^{N} \mu_k(a_n - a)^p \right\}^{1/p\wedge}(\zeta) = \left\{ \sum_{k=1}^{N} \mu_k(a_n(\zeta) - a(\zeta))^p \right\}^{1/p}$$

$$\leqq \limsup_{m\to\infty} \left\{ \sum_{k=1}^{\infty} \mu_k(a_n(\zeta) - a_m(\zeta))^p \right\}^{1/p}$$

$$\leqq \limsup_{m\to\infty} \|a_n - a_m\|_p \text{ for all } n.$$

Therefore, it follows that

$$\left\{\sum_{k=1}^{N} \mu_k(a_n - a)^p\right\}^{1/p} \leqq \limsup_{m\to\infty} \|a_n - a_m\|_p \text{ for all } n.$$

Therefore, letting $N\to\infty$, we fined

$$\|a_n - a\|_p \leqq \limsup_{m\to\infty} \|a_n - a_m\|_p$$

so that

$$\lim_{n\to\infty} \|a_n - a\|_p \leqq \lim_{m,n\to\infty} \|a_n - a_m\|_p = 0.$$

Thus the theorem is proved.

In stead of considering the operators of finite rank in a Hilbert space, we shall consider the subset F in $C(M)$ defined by $F=\left\{a\in C_\infty(M)\,;\ |a| = \sum_{i=1}^{N} a_i e_i, N<\infty\right\}$ where $\sum_{i=1}^{N} a_i e_i$ is the spectral representation of $|a|$.

Then the following lemma will be useful in the sequel.

LEMMA 4. For each $a\in F$ and $b\in M$, ab and ba are elements in F.

PROOF Since a is an element in $C_\infty(M)$, ab and ba are elements in $C_\infty(M)$, We shall show that ba is an element in F. Let $\sum_{i=1}^{N} a_i e_i$ (resp., $\sum_{j=1}^{r} c_j p_j$) be a spectral representation for a (resp., ba) where $N<\infty$ and $r\leqq\infty$. Let $u|a|$ be the polar decomposition for a.

If $r=\infty$, then there exists an element ζ_0 in X and positive integer $s>N$ such that $c_j{}^\wedge(\zeta_0)\neq 0$ and $p_j(\zeta_0)\neq 0$ for each $s\geqq j\geqq 1$. For each $\zeta\in X$, we have

$$\begin{aligned}(ba)^*(ba)(\zeta) &= a^*(\zeta)b^*(\zeta)b(\zeta)a(\zeta)\\ &= a^*(\zeta)b^*(\zeta)b(\zeta)u(\zeta)\,|a|(\zeta)\\ &= a^*(\zeta)b^*(\zeta)b(\zeta)\left(\sum_{j=1}^{N} a_j{}^\wedge(\zeta)u(\zeta)e_j(\zeta)\right).\end{aligned}$$

Therefore, the dimension of the range of $(ba)^*(ba)(\zeta)$ is smaller than N for each $\zeta \in X$. On the other hand, we have

$$(ba)^*(ba)(\zeta_0) = \sum_{j=1}^{\infty} c_j{}^{\wedge}(\zeta_0)p_j(\zeta_0)$$

so that the dimension of the range of $(ba)^*(ba)(\zeta_0)$ is larger than N. This is a contradiction. Therefore, ba is an element in F. By the same argument, we can show the fact that ab is an element in F.

PROPOSITION 2. *For each $a \in C_\infty(M)$, there exists a sequence $\{b_n\}_{n=1}^{\infty}$ in F such that*

(a) *$b_n \to a$ in the uniform topology as $n \to \infty$;*
(b) *$\|b_n - a\|_p \to 0$ as $n \to \infty$ if $a \in C_p(M)$;*
(c) *$\|b_n\|_p \to \|a\|_p$ as $n \to \infty$ if $a \in C_p(M)$.*

PROOF. Let $|a| = \sum_{i=1}^{\infty} a_i e_i$ be the spectral representation of $|a|$. Put $f_n = \sum_{i=1}^{n} e_i$, $f_n' = 1 - f_n$, $b_n = af_n$ and $b_n' = af_n'$. Then we have

$$\begin{aligned} b_n^* b_n &= (af_n)^*(af_n) = f_n a^* a f_n \\ &= f_n\left(\sum_{i=1}^{\infty} a_i^2 e_i\right) f_n = \sum_{i=1}^{\infty} a_i^2 f_n e_i f_n \\ &= \sum_{i=1}^{n} a_i^2 e_i = \sum_{i=1}^{\infty} a_i^2 e_i f_n = a^* a f_n \end{aligned}$$

and

$$\begin{aligned} |b_n| &= (b_n^* b_n)^{1/2} = \sum_{i=1}^{n} a_i e_i = \left(\sum_{i=1}^{\infty} a_i e_i\right) f_n \\ &= (a^* a)^{1/2} f_n = |a| f_n . \end{aligned}$$

Therefore, it follows that $\|b_n - a\|_p = \left\| \left\{ \sum_{i=n+1}^{\infty} a_i^p \right\}^{1/p} \right\| \to 0$ as $n \to \infty$ and $\|b_n - a\| = \left\| \sum_{i=n+2}^{\infty} a_i e_i \right\| = \|a_{n+1}\| \to 0$ as $n \to \infty$. This proves (a) and (b).

Since $b_n^* b_n = a^* a f_n$, it is plain that $\mu_m(b_n) = \mu_m(a)$ if $m \leqq n$ and $\mu_m(b_n) = 0$

if $m>n$. Therefore we have

$$|\|b_n\|_p-\|a\|_p| = \left|\left\|\left\{\sum_{i=1}^{n} a_i^p\right\}^{1/p}\right\| - \left\|\left\{\sum_{i=1}^{\infty} a_i^p\right\}^{1/p}\right\|\right|$$
$$\leqq \left\|\left\{\sum_{i=n+1}^{\infty} a_i^p\right\}^{1/p}\right\| \to 0 \text{ as } n\to\infty .$$

This proves the assertion (c).

4. Duality of the classes $C_p(M)$ ($\infty \geqq p \geqq 1$) of the type I von Neumann algebra M. For a positive element a in $C_1(M)$ with the spectral representation $\sum_{i=1}^{\infty} a_i e_i$ we define the trace $Tr(a)$ to be $\sum_{i=1}^{\infty} a_i$. Then, if a is a positive element in $C_1(M)$, then $Tr(a)^\wedge(\zeta) = Tr(a(\zeta))$ for each ζ in X where $Tr(a(\zeta))$ is the semifinite trace on $B(H(\zeta))$. If a is an element in $C_1(M)$, then there exist the positive elements $\{a_n\}_{n=1}^{4}$ in $C_1(M)$ such that $a = a_1 - a_2 + i(a_3 - a_4)$. Thus, we can define $Tr(a) = Tr(a_1) - Tr(a_2) + i(Tr(a_3) - Tr(a_4))$ so that Tr is a linear operator of $C_1(M)$ into Z. The trace thus defined on $C_1(M)$ has the following properties; (1) if a, b are elements of $C_1(M)$ and c, d are elements of Z then $Tr(ca+db) = cTr(a) + dTr(b)$; (2) if $a \in C_1(M)$, then $Tr(u^*au) = Tr(a)$ for every unitary operator u in M; (3) if $a \in C_1(M)$, the function $\varphi(b) = Tr(ba)$ is continuous on M. The classes $C_p(M)$ are Banach algebras and may be considered as the spaces module over Z. Therefore, a functional φ of $C_p(M)$ into Z may be called a Z-linear functional if $\varphi(ca+db) = c\varphi(a) + d\varphi(b)$ for all $c, d \in Z$ and $a, b \in C_p(M)$. In this section, we shall consider the duality of $C_p(M)$ in the sense of the above notation. At first, we show the following fact.

THEOREM 3. *For each $\infty > p > 1$ and each $a \in C_p(M)$, we have the equality;*

$$\|a\|_p = \sup_{b\in F} \frac{\|Tr(ab)\|}{\|b\|_q} \cdot \qquad \text{where } \frac{1}{p}+\frac{1}{q}=1 .$$

PROOF. For each $\zeta \in X$, we have;

$$\sup_{b\in F}\frac{\|Tr(ab)\|}{\|b\|_q} = \sup_{b\in F}\frac{\sup_{\zeta\in X}|Tr(ab)^\wedge(\zeta)|}{\sup_{\zeta\in X}\left\{\sum_{i=1}^{N} b_i{}^\wedge(\zeta)^q\right\}^{1/q}}$$

$$\leqq \sup_{b\in F}\sup_{\zeta\in X}\frac{|Tr(a(\zeta)b(\zeta))|}{\left\{\sum_{i=1}^{N} b_i{}^{\wedge}(\zeta)^q\right\}^{1/q}}$$

$$= \sup_{\zeta\in X}\sup_{b\in F}\frac{|Tr(a(\zeta)b(\zeta))|}{\left\{\sum_{i=1}^{N} b_i{}^{\wedge}(\zeta)^q\right\}^{1/q}}$$

$$= \sup_{\zeta\in X}\|a(\zeta)\|_p = \|a\|_p .$$

Therefore, $\sup_{b\in F}\frac{\|Tr(ab)\|}{\|b\|_q}\leqq \|a\|_p$.

Next, we show the converse inequality. At first, we suppose that a is an element in F and let $a=u|a|$ is the polar decomposition of a and $\sum_{i=1}^{N} a_i e_i$ is the spectral representation of $|a|$. Put $b=\left(\sum_{i=1}^{N} a_i{}^{p-1}e_i\right)u$ then, by Lemma 4, b is an element in F. Thus, we have;

$$ab=\sum_{i=1}^{N} a_i{}^p e_i,\ \|Tr(ab)\| = \left\|\sum_{i=1}^{N} a_i{}^p\right\| \ and\ \|b\|_q = \left\|\left\{\sum_{i=1}^{N} a_i{}^p\right\}^{1/q}\right\|.$$

Since $\left\{\sum_{i=1}^{N} a_i{}^p\right\}$ and $\left\{\sum_{i=1}^{N} a_i{}^p\right\}^{1/q}$ attain the maximum at the same point, we have

$$\frac{\|Tr(ab)\|}{\|b\|_q} = \left\|\left\{\sum_{i=1}^{N} a_i{}^p\right\}^{1/p}\right\| = \|a\|_p .$$

That is,

$$\|a\|_p \leqq \sup_{b\in F}\frac{\|Tr(ab)\|}{\|b\|_q} \quad \text{for } a\in F .$$

For each $a\in C_p(M)$, there exists a sequence $\{a_n\}_{n=1}^{\infty}$ in F such that $\|a_n-a\|\to 0$ as $n\to\infty$, $\|a_n-a\|_p\to 0$ as $n\to\infty$ and $\|a_n\|_p\to\|a\|_p$ as $n\to\infty$ by Proposition 2. Thus, we have

$$\|a\|_p = \lim_{n\to\infty}\|a_n\|_p \leqq \lim_{n\to\infty}\sup_{b\in F}\frac{\|Tr(a_n b)\|}{\|b\|_q} = \sup_{b\in F}\frac{\|Tr(ab)\|}{\|b\|_q} .$$

Therefore, $\|a\|_p = \sup_{b \in F} \frac{\|Tr(ab)\|}{\|b\|_q}$. This completes the proof of Theorem 3.

For the duality for the classes $C_\infty(M)$ and $C_1(M)$, H. Halpern has showed that $C_\infty(M)^* = C_1(M)$ and $C_1(M)^* = M$ (see [6]; Theorem 4.8 and Theorem 4.9). Furthermore, by using Theorem 3, we can show that for $p > 1$ the dual space of the class $C_p(M)$ is $C_q(M)$ where $1/p + 1/q = 1$.

THEOREM 4. *For each* $a_0 \in C_q(M)$ $(\infty > q > 1)$, *the* Z*-linear functional* $\varphi(a) = Tr(aa_0)$ *for* $a \in C_p(M)$ *is a continuous* Z*-linear functional of* $C_p(M)$ *into* Z *where* p *is the dual number of* q. *Furthermore, if* φ *is a continuous* Z*-linear functional of* $C_p(M)$ $(\infty > p > 1)$ *into* Z, *there exists a unique* $a_\varphi \in C_q(M)$ *such that* $\varphi(a) = Tr(aa_\varphi)$ *for all* $a \in C_p(M)$ *and* $\|\varphi\| = \|a_\varphi\|_q$ *where* q *is the dual number of* p.

PROOF. Let $a_0 \in C_q(M)$ and $a \in C_p(M)$. Then we have

$$\begin{aligned}\|Tr(aa_0)\| &= \|Tr(u|aa_0|)\| \leqq \|Tr(|aa_0|)\| \\ &= \|aa_0\|_1 \leqq \|a\|_p \|a_0\|_q\end{aligned}$$

where $u|aa_0|$ is the polar decomposition of aa_0. The last inequality due to Proposition 1. This completes the proof of the first.

Let φ be a continuous Z-linear functional of $C_p(M)$ into Z, then Proposition 1(a), the restriction of φ to $C_1(M)$ is a continuous Z-linear functional of $C_1(M)$ into Z. Therefore, by ([6], Theorem 4.9), there exists an element $a_\varphi \in M$ such that $\varphi(a) = Tr(aa_\varphi)$ for each a $C_1(M)$. For each $a \in C_p(M)$, there exists a sequence $\{a_n\}_{n=1}^\infty$ in F such that $\|a_n - a\|_p \to 0$, $\|a_n\|_p \to \|a\|_p$ and $\|a_n - a\| \to 0$. Each a_n is an element of $C_1(M)$ so that $\varphi(a_n) = Tr(a_n a_\varphi)$. Since φ is a continuous Z-linear functional of $C_p(M)$, $\varphi(a) = \lim_{n\to\infty} \varphi(a_n) = \lim_{n=\infty} Tr(a_n a_\varphi) = Tr(aa_\varphi)$. That is, $\varphi(a) = Tr(aa_\varphi)$ for all $a \in C_p(M)$.

Furthermore, by Theorem 4 in [4] and the properties of Tr, we can show that a_φ is an element of $C_\infty(M)$.

Next, we shall show the equality $\|\varphi\| = \|a_\varphi\|_q$.

By Theorem 3, we have

$$\begin{aligned}\|\varphi\| &= \sup_{a \in C_p(M)} \frac{\|\varphi(a)\|}{\|a\|_p} = \sup_{a \in C_p(M)} \frac{\|Tr(aa_\varphi)\|}{\|a\|_p} \\ &\geqq \sup_{a \in F} \frac{\|Tr(aa_\varphi)\|}{\|a\|_p} = \|a_\varphi\|_q .\end{aligned}$$

Therefore, we have the inequality $\|\varphi\| \geqq \|a_\varphi\|_q$ so that a_φ is an element of $C_q(M)$. We have showed the converse inequality in the first place in this proof. That is, $\|\varphi\| = \|a_\varphi\|_q$. This completes the proof of Theorem 4.

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