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ON THE FUNCTIONAL EQUATION $\sum_{i=0}^{p} a_i f_i^{n_i} = 1$

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1. Let $a_0, \cdots, a_p$ $(p \geqslant 1)$ be $p+1$ meromorphic functions in $|z| < R(\leqslant \infty)$ and $n_0, \cdots, n_p$ positive integers. In this paper we consider whether the functional equation

$$\sum_{i=0}^{p} a_i f_i^{n_i} = 1$$

has holomorphic solutions $f_0, \cdots, f_p$ in $|z| < R$.

Recently, Yang [10] has proved the following

THEOREM A. *The functional equation*

$$a(z) f^m(z) + b(z) g^n(z) = 1 \tag{1}$$

$(a, b, f, g$ meromorphic in $|z| < \infty$, m and n integers $\geqslant 3)$ cannot hold, if

$$T(r, a) = o(T(r, f)), \quad T(r, b) = o(T(r, g)), \tag{2}$$

unless $m = n = 3$.

If $f(z)$ and $g(z)$ are entire and (2) holds, then (1) cannot hold, even if $m = n = 3$.

This is a generalization of the case $a \equiv b \equiv 1$ treated by Montel [7], Jategaonkar [5] and Gross [2, 3]. Further, Iyer [4], Jategaonkar and Gross considered many other cases.

We will generalize the latter half of Theorem A and give some consequences of the generalizations.

It is assumed that the reader is familier with the fundamental concepts of Nevanlinna's theory of meromorphic functions and the symbols $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $T(r, f)$, etc. (see [11]).

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2. First, we will treat the case $R=\infty$. The following lemma is a little sharpened form of Theorem 4 in [9].

LEMMA 1. *Let* $g_0, \cdots, g_p$ $(p \geqq 1)$ *be* $p+1$ *non-constant meromorphic functions in* $|z|<\infty$ *satisfying*

$$\sum_{i=0}^{p} a_i g_i = 1, \qquad a_0 \cdots a_p \neq 0, \tag{3}$$

with constant coefficients and $\delta(\infty, g_i)=1$ $(i=0, \cdots, p)$. *Then, we have*

$$\sum_{i=0}^{p} \theta_p(0, g_i) \leqq p.$$

Here,

$$\theta_p(0, g_i) = 1 - \limsup_{r\to\infty} \frac{N_p(r, 0, g_i)}{T(r, g_i)}$$

and

$$N_p(r, 0, g_i) = \sum_{a_k \neq 0} \log^+ \frac{r}{|a_k|} + \min(\rho_0, p) \log r$$

where the summation is taken over the different zeros $a_k (\neq 0)$ *of* g_i *counted* $\min(\rho_k, p)$ *times at* a_k, ρ_k*(resp.* ρ_0*) being the order of multiplicity of zero of* g_i *at* a_k*(resp.* 0*).*

PROOF. We proceed the proof in the same way as in the proof of Theorem 4 in [9].

1) The case when $g_0, \cdots, g_p$ are linearly independent. We note that the Wronskian $\Delta = \|g_0, \cdots, g_p\|$ is not identically zero in this case. By differentiating both sides of (3), we have

$$\sum_{i=0}^{p} a_i \frac{g_i^{(\mu)}}{g_i} g_i = 0, \ \mu = 1, \cdots, p. \tag{4}$$

From (3) and (4), we have

$$g_i = \frac{\tilde{\Delta}_i}{a_i \tilde{\Delta}},$$

where

$$\widetilde{\Delta} = \Delta / g_0 \cdots g_p \, (\not\equiv 0)$$

and

$$\widetilde{\Delta}_i = g_i \| g_0, \cdots, g_{i-1}, 1, g_{i+1}, \cdots, g_p \| / g_0 \cdots g_p .$$

Then,

$$\begin{aligned} m(r, g_i) &\leqslant m(r, \widetilde{\Delta}_i) + m(r, 1/\widetilde{\Delta}) + O(1) \\ &\leqslant m(r, \widetilde{\Delta}_i) + m(r, \widetilde{\Delta}) + N(r, \widetilde{\Delta}) + O(1) \\ &\leqslant \sum_{j=0}^{p} N_p(r, 0, g_j) + p \sum_{j=0}^{p} N(r, g_j) + S(r) , \end{aligned}$$

where $S(r)$ is independent of i and

$$S(r) = \begin{cases} O(1), \ (r \to \infty), & \text{when all } g_i \text{ are rational;} \\ o\left(\sum_{j=1}^{p} T(r, g_j)\right), & \text{with a possible exceptional set of } r \text{ of finite linear measure, in the other cases.} \end{cases}$$

Hence,

$$\begin{aligned} T(r) &\equiv \max_{0 \leqslant i \leqslant p} \{ m(r, g_i) + N(r, g_i) \} \\ &\leqslant \sum_{j=0}^{p} N_p(r, 0, g_j) + (p+1) \sum_{j=0}^{p} N(r, g_j) + S(r) . \end{aligned}$$

By the definition of $\theta_p(0, g_j)$ and $\delta(\infty, g_j)$, for an arbitrary given $\varepsilon > 0$,

$$N_p(r, 0, g_j) < (1 - \theta_p(0, g_j) + \varepsilon) T(r, g_j)$$

and

$$N(r, g_j) < (1 - \delta(\infty, g_j) + \varepsilon) T(r, g_j)$$

for $r \geqslant r_0$ and $j = 0, \cdots, p$. Using these inequalities and by the hypothesis, we have

$$T(r) \leqslant \sum_{j=0}^{p}(1-\theta_p(0, g_j)+\varepsilon)T(r)+(p+1)^2\varepsilon T(r)+S(r),$$

which implies

$$\sum_{j=0}^{p}\theta_p(0, g_j) \leqslant p.$$

2) The case when $g_0, \cdots, g_p$ are linearly dependent. Porceeding as in [9] and using the inequality

$$\theta_p(0, g_j) \leqslant \theta_s(0, g_j) \leqslant 1 \qquad \text{for} \quad 1 \leqslant s \leqslant p,$$

we have the desired

$$\sum_{j=0}^{p}\theta_p(0, g_j) \leqslant p.$$

Applying this lemma, we have the following

THEOREM 1. *Let* $f_0, \cdots, f_p$ $(p \geqslant 1)$ *be* $p+1$ *non-constant entire functions and let* $a_0, \cdots, a_p$ *be* $p+1$ *meromorphic functions* $(\not\equiv 0)$ *in* $|z| < \infty$ *such that*

$$T(r, a_i) = o(T(r, f_i)), \ (i = 0, \cdots, p). \tag{5}$$

Then, if the following functional equation

$$\sum_{i=0}^{p} a_i(z) f_i^{n_i}(z) = 1 \tag{6}$$

holds for some integers $n_0, \cdots, n_p (\geqslant 1)$, *it must be*

$$\sum_{i=0}^{p}\frac{1}{n_i} \geqslant \frac{1}{p}.$$

PROOF. We have for any meromorphic function f in $|z| < \infty$ and a positive integer n

$$T(r, f^n) \sim nT(r, f), \ (r \to \infty).$$

Therefore, by (5),

$$T(r, a_i f_i^{n_i}) \sim n_i T(r, f_i), \ (r \to \infty), i = 0, \cdots, p.$$

By the definition of $N_p(r, 0, g_i)$, we have

$$N_p(r, 0, a_i f_i^{n_i}) \leqslant N_p(r, 0, a_i) + p\overline{N}(r, 0, f_i),$$

so that

$$\limsup_{r\to\infty} \frac{N_p(r, 0, a_i f_i^{n_i})}{T(r, a_i f_i^{n_i})} \leqslant \frac{p}{n_i} \limsup_{r\to\infty} \frac{\overline{N}(r, 0, f_i)}{T(r, f_i)} \leqslant \frac{p}{n_i},$$

that is, by the definition of θ_p,

$$\theta_p(0, a_i f_i^{n_i}) \geqslant 1 - \frac{p}{n_i} \qquad (i = 0, \cdots, p).$$

Summing over these inequalities and using Lemma 1 and (5), we have

$$\sum_{i=0}^{p} \left(1 - \frac{p}{n_i}\right) \leqslant \sum_{i=0}^{p} \theta_p(0, a_i f_i^{n_i}) \leqslant p,$$

that is,

(7) $$\sum_{i=0}^{p} \frac{1}{n_i} \geqslant \frac{1}{p},$$

which is the desired.

As consequences of this theorem, we have generalizations of the latter half of Theorem A.

COROLLARY 1. *If*

$$\sum_{i=0}^{p} \frac{1}{n_i} < \frac{1}{p},$$

then the equation (6) *cannot hold.*

COROLLARY 2. *Let* p *be equal to* 1. *Then,* (6) *cannot hold for* $n_0 \geqslant 2$, $n_1 \geqslant 3$ *or* $n_0 \geqslant 3, n_1 \geqslant 2$.

This is an improvement of the latter half of Theorem A and of Lemma 2 in [5].

In [3], Gross considered the functional equation

$$\sum_{i=0}^{p} f_i^n(z) = 1$$

for non-constant entire functions f_i $(i = 0, \cdots, p)$ and a positive integer n. When $n = 3, p = 2$ ([6]) or $n = 4, p = 2$ ([3]), there is a solution Of course, these cases satisfy the inequality (7).

As an application of Theorem 1, we give some generalizations of Theorem 2 in [10] when f and g are entire.

THEOREM 2. *Let $f_0, \cdots, f_p$ $(p \geqq 1)$ be $p+1$ non-constant entire functions and let $n_0, \cdots, n_p$ be $p+1$ integers not less than one such that at least one of $f_i^{n_i}/f_j^{n_j}$ $(i \neq j)$ is transcendental and*

$$\sum_{i=0}^{p} \frac{1}{n_i} < \frac{1}{p}. \tag{8}$$

Then, for rational funcctions $R_i(z)$ $(\not\equiv 0)$ $(i = 0, \cdots, p)$

$$F(z) \equiv \sum_{i=0}^{p} R_i(z) f_i^{n_i}(z)$$

has infinitely many zeros or some of partial sums (one of which may be F) are equal to zero identically.

PROOF. Assume that the statement is false. Then $F(z)$ can be written as

$$F(z) = R(z)e^{f(z)}$$

where $R(z)$ is rational $(\not\equiv 0)$ and $f(z)$ is entire.

Let

$$-\frac{f(z)}{n_i} = g_i(z) \quad (i = 0, \cdots, p).$$

Then $g_i(z)$ is entire $(i = 0, \cdots, P)$ and

$$\sum_{i=0}^{p} R_i R^{-1}(f_i e^{g_i})^{n_i} = 1\,.$$

Since $R_i R^{-1}$ is rational, it holds that

$$T(r, R_i R^{-1}) = O(\log r),\ (r \to \infty),\ i = 0, \cdots, p\,.$$

By the hypothesis (8) and by using Theorem 1, at least one of $f_i e^{g_i}$ is polynomial. The number s of i such that $f_i e^{g_i}$ is polynomial is at most $p-1$. In fact, if $s \geqslant p$ we may suppose that $f_0 e^{g_0}, \cdots, f_{p-1} e^{g_{p-1}}$ are polynomial, so that

$$\sum_{i=0}^{p-1} R_i R^{-1}(f_i e^{g_i})^{n_i} \equiv A$$

is rational. This implies that

$$(f_p e^{g_p})^{n_p} = (1-A) R R_p^{-1}$$

is polynomial. Hence $f_p e^{g_p}$ is polynomial. This shows that $s = p+1$. In this case, any ratio

$$\frac{(f_i e^{g_i})^{n_i}}{(f_j e^{g_j})^{n_j}} = \frac{f_i^{n_i}}{f_j^{n_j}}$$

is rational, which is a contradiction. Therefore

$$1 \leqslant s \leqslant p-1\,.$$

We may suppose that $f_0 e^{g_0}, \cdots, f_{s-1} e^{g_{s-1}}$ are polynomial, so that

$$\sum_{i=0}^{s-1} R_i R^{-1}(f_i e^{g_i})^{n_i}$$

is rational and is not equal to 1 identically by the assumption. We have

$$\sum_{i=s}^{p} P_i (f_i e^{g_i})^{n_i} = 1\,,$$

where $f_i e^{g_i}$ $(i = s, \cdots, p)$ are transcendental and P_i $(\not\equiv 0)$ $(i = s, \cdots, p)$ are rational. Therefore, using

$$T(r, P_i) = o(T(r, f_i e^{g_i}))\quad (i = s, \cdots p)\,,$$

we have by Theorem 1

$$\sum_{i=0}^{p}\frac{1}{n_i}>\sum_{i=s}^{p}\frac{1}{n_i}\geqslant\frac{1}{p-s}>\frac{1}{p},$$

which contradicts (8). Thus we have our theorem.

3. In [10], it is remarked that the analogous result to Theorem A for some meromorphic functions in the unit disc can be proved as in Theorem A, but it is not noted whether analogous results to the type of Theorem 2 are valid or not. As we can prove some results concerning this type, we shall state these briefly in the following.

LEMMA 2. *Let $g(z)$ be meromorphic in $|z|<1$. Then for $1>r>r_0>0$ and $\lambda>0$*

$$\int_{r_0}^{r} m(t, g'/g)(1-t)^{\lambda-1}dt = O\left(\int_{r_0}^{r}\log^+ T(t, g)(1-t)^{\lambda-1}dt\right)$$

([8]).

Using this lemma, we can give a lemma analogous to Lemma 1.

LEMMA 3. *Let $g_0, \cdots, g_p$ $(p\geqslant 1)$ be $p+1$ functions meromorphic in $|z|<1$ such that*

$$\limsup_{r\to 1}\frac{T(r, g_i)}{\log\dfrac{1}{1-r}}=\infty, \quad (i=0, \cdots, p)$$

and $\delta(\infty, g_i)=1$ $(i=0, \cdots, p)$.

If, for non-zero constants α_i $(i=0, \cdots, p)$,

$$\sum_{i=0}^{p}\alpha_i g_i = 1,$$

then we have

$$\sum_{i=0}^{p}\theta_p(0, g_i)\leqslant p,$$

where

$$\theta_p(0, g_i) = 1 - \limsup_{r\to 1} \frac{N_p(r, 0, g_i)}{T(r, g_i)} .$$

We can prove this in the same way as in the proof of Lemma 1 by using Lemma 2 only when at least one of g_i is of order infinite.

THEOREM 3. *Let $f_0, \cdots, f_p$ $(p \geqslant 1)$ be $p+1$ holomorphic functions in $|z| < 1$ such that*

$$\limsup_{r\to 1} \frac{T(r, f_i)}{\log \frac{1}{1-r}} = \infty, \ (i = 0, \cdots, p)$$

and let $a_0, \cdots, a_p$ be $p+1$ meromorphic functions $(\not\equiv 0)$ in $|z| < 1$ such that

$$T(r, a_i) = o(T(r, f_i)), \ (r \to 1), \ i = 0, \cdots, p.$$

If the functional equation

$$\sum_{i=0}^{p} a_i f_i^{n_i} = 1$$

holds for integers n_i $(i = 0, \cdots, p)$ not less than one, then the following inequality must be valid:

$$\sum_{i=0}^{p} \frac{1}{n_i} \geqslant \frac{1}{p} .$$

We can prove this as in the proof of Theorem 1 by using Lemma 3 in place of Lemma 1.

REMARK. This is an improvement of Lemma 1′ in [5].

THEOREM 4. *Let $f_0, \cdots, f_p$ $(p \geqslant 1)$ be $p+1$ holomorphic functions in $|z| < 1$ and let $n_0, \cdots, n_p$ be $p+1$ integers not less than one such that at least one of $f_i^{n_i}/f_j^{n_j}$ $(i \neq j)$ satisfies*

$$\limsup_{r\to 1} \frac{T(r, f_i^{n_i}/f_j^{n_j})}{\log \frac{1}{1-r}} = \infty$$

and

$$\sum_{i=0}^{p}\frac{1}{n_i}<\frac{1}{p}.$$

Then, for meromorphic functions a_i $(\not\equiv 0)$ $(i=0,\cdots,p)$ *of bounded type in* $|z|<1$, *the zeros* $\{z_k\}$ *of*

$$F(z)\equiv\sum_{i=0}^{p}a_i(z)f_i^{n_i}(z) \tag{9}$$

satisfy

$$\sum_k(1-|z_k|)=\infty$$

or some of partial sums of $a_if_i^{n_i}$ *(one of which may be* F*) are equal to zero identically.*

PROOF. Assume that the statement is false. Let $P(z)$ be a canonical product of $\{z_k\}$. Then $P(z)$ is holomorphic, $|P(z)|<1$ in $|z|<1$ and $P(z_k)=0$. Let $Q(z)$ be a canonical product of the poles of $F(z)$. Then by the hypothesis, $Q(z)$ is holomorphic, $|Q(z)|<1$ in $|z|<1$. Put

$$F(z)=P(z)Q^{-1}(z)H(z).$$

Then $H(z)$ is holomorphic in $|z|<1$ and has no zeros in $|z|<1$. Let $g_i=H^{-1/n_i}$ be a branch in $|z|<1$. Then it is holomorphic in $|z|<1$ and has no zeros. We have from (9)

$$\sum_{i=0}^{p}a_iQP^{-1}(f_ig_i)_i^{n_i}=1,$$

where a_iQP^{-1} is of bounded type $(i=0,\cdots,p)$.

We can prove the rest as in the proof of Theorem 2 by using Theorem 3 in place of Theorem 1.

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