Tôhoku Math. Journ.
23(1971), 273-287.

RANDOM RECURRENCE TIME IN ERGODIC THEORY

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(Received October 30, 1970)

1. Introduction. For a measure preserving transformation on finite measure space, we shall define, in Section 2, a "random recurrence time" into a measurable set. It is, as it will be shown, a generalization of both ordinary recurrence time and successive recurrence time introduced by W. Roos [3].

In Section 3, we shall show some asymptotic properties of the random recurrence time in the sense of time means and as an application we shall prove a generalized integral theorem of Kac. The idea in our theorems is due to S. Tsurumi [4], but the pattern of the proof is slightly different from his.

In the same section, we shall further remark that the random recurrence time is nothing but a "stopping time of dynamical system" introduced by J. Neveu [2], that is, we shall make an analysis of the stopping time from a different point of view.

2. Definition of random recurrence time. Let $(X, \mathfrak{B}, \mu)$ be a finite measure space. We begin with a definition of the sojourn time.

DEFINITION 1. Let T be an invertible measure preserving transformation on the space $(X, \mathfrak{B}, \mu)$ and D any measurable set of X with a positive measure. For the set D and the transformation T, we call "sojourn time" the sequence of measurable functions $\{\varphi_n(x)\}_{n=0}^{\infty}$ on X defined as follows;

For any $x \in \bigcup_{i=0}^{\infty} T^{-i}D$, put by induction

$$\varphi_0(x) = 0, \quad \varphi_1(x) = 1_D(Tx),$$
$$\varphi_{n+1}(x) = \varphi_n(x) + \varphi_1(T^n x) \qquad (n \geqq 1)$$

where 1_D is the indicator of the set D, and for any $x \in X \setminus \bigcup_{i=0}^{\infty} T^{-i}D$, put

$$\varphi_n(x) = 0$$

for all $n \geqq 0$.

From the definition we can see easily

$$\varphi_n(x) = \sum_{k=1}^{n} 1_D(T^k x) \qquad (n \geqq 1)$$

and

$$\varphi_{n+m}(x) = \varphi_n(x) + \varphi_m(T^n x) \qquad (m, n \geqq 0)$$

for any $x \in X$.

Poincaré's recurrence theorem states that

$$\varphi_n(x) \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty$$

for any $x \in \bigcup_{i=0}^{\infty} T^{-i} D$ a. e.

The ordinary recurrence time $r(x)$ of the transformation T in the set D is defined a. e. in X as follows

$$r(x) = \begin{cases} \text{min. } \{l > 0 ;\ T^l x \quad D\} & \text{if } x \in D, \\ 0 & \text{if } x \in X \backslash D \end{cases}$$

and the successive recurrence time $\{r_n(x)\}_{n=0}^{\infty}$ of the transformation T in the set D is defined a. e. in X inductively

$$r_0(x) = 0, \qquad r_1(x) = r(x)$$

$$r_{n+1}(x) = r_n(x) + r_1(T^{r_n(x)} x) \qquad (n \geqq 1)$$

By the property of the sojourn time, we can alternate equivalent definition of $r(x)$ and $\{r_n(x)\}_{n=0}^{\infty}$ a. e. in X, that is,

$$r(x) = \text{min. } \{l \geqq 0 ;\ \varphi_l(x) = 1_D(x)\}$$

and

$$r_n(x) = \text{min. } \{l \geqq 0 ;\ \varphi_l(x) = n 1_D(x)\}.$$

DEFINITION 2. Let $n(x)$ be a non-negative integer valued measurable function such that $n(x)$ is positive on D and zero on $X\backslash D$. For any $x \in X$ a. e., put

$$r^*(x) = \min. \{l \geqq 0 ; \varphi_l(x) = n(x)\} .$$

If the family of measurable sets

$$[T^l \{x ; r^*(x) = l\}]_{l=0}^{\infty}$$

is a partition of X up to a set of measure zero, $r^*(x)$ is said to be a random recurrence time of the transformation T in the D.

It is clear that the random recurrence time is a generalization of both ordinary and successive recurrence times.

It is also easy to show that the transformation $T^{r^*(\cdot)}$:

$$T^{r^*(x)}x = T^l x \qquad \text{if } r^*(x) = l$$

is invertible and measure preserving.

Using the notation of the successive recurrence time, we can write for any $x \in X$ a. e.

$$r^*(x) = r_{n(x)}(x)$$

where

$$r_{n(x)}(x) = r_n(x) \qquad \text{if } n(x) = n .$$

The family of measurable sets

$$[T^{r_n(\cdot)} \{x ; n(x) = n\}]_{n=0}^{\infty}$$

is a partition of X up to a set of measure zero.

3. **Generalized integral theorem.** Before we investigate the asymptotic properties of the random recurrence time, we state Tsurumi's theorem which our theorem depends on.

THEOREM 1 (S. Tsurumi). *Let T be a measure preserving transformation on X and $r(x)$ an ordinary recurrence time of T in any measurable set D with a positive measure. Then*

$$\tilde{r}(x) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} r(T^k x) = \begin{cases} 1 & \text{if } x \in \bigcup_{n=0}^{\infty} T^{-n}D \\ 0 & \text{if } x \in X \setminus \bigcup_{n=0}^{\infty} T^{-n}D \end{cases}$$

a. e.

PROOF. We give an alternative proof.

By the individual ergodic theorem, the limit function

$$\lim_{m\to\infty}\frac{1}{m}\sum_{\kappa=1}^{m} 1_D(T^k x)$$

exists for a. e. $x \in X$, that is,

$$\lim_{n\to\infty}\frac{1}{m}\varphi_m(x)$$

exists for a. e. $x \in X$.

For the successive recurrence time $\{r_n(x)\}$ of T in D, if we put

$$m(x) = r_n(x)$$

then $m(x)$ is a non-negative integer valued measurable function and for each n and for a. e. x

$$\varphi_m(x) = n$$

where $m = m(x)$. By the definition of the successive recurrence time, $m \to \infty$ as $n \to \infty$ for a. e. $x \in \bigcup_{n=0}^{\infty} T^{-n}D$. Since

$$\frac{1}{n} r_n(x) = \frac{1}{\frac{1}{m}\varphi_m(x)},$$

the limit function

$$\lim_{n\to\infty}\frac{1}{n} r_n(x)$$

exists for a. e. $x \in \bigcup_{n=0}^{\infty} T^{-n}D$.

First we shall show the result for $x \in D$.

For any sufficiently large integer n, we can find a positive integer N such that $r_{N-1}(x) \leqq n < r_N(x)$, then

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} r(T^k x) &= \frac{1}{n}\sum_{j=0}^{N-1} r(T^{r_j(x)}x) \\ &= \frac{1}{n}\sum_{j=0}^{N-1}(r_{j+1}(x) - r_j(x)) \\ &= \frac{1}{n} r_N(x) .\end{aligned}$$

Therefore for each $x \in D$

$$1 < \frac{1}{n} r_N(x) \leqq r_N(x)/r_{N-1}(x) .$$

Since $N \to \infty$ as $n \to \infty$, we write

$$1 \leqq \lim_{n\to\infty} \frac{1}{n} r_N(x) \leqq \lim_{N\to\infty}\left(\frac{1}{N} r_N(x) \Big/ \frac{1}{N-1} r_{N-1}(x)\right)\frac{N}{N-1} .$$

As we have shown, the limit function $\lim_{N\to\infty} \frac{1}{N} r_N(x)$ exists a. e. in D, and we get

$$\tilde{r}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} r(T^k x) = 1$$

a. e. in D.

For any x in the set $\bigcup_{n=1}^{\infty} T^{-n}D$, we can find an positive integer n_0 such that

$$T^{n_0}x \in D, \quad T^m x \in X \backslash D \qquad (\text{for all } m < n_0) .$$

If we put

$$y = T^{n_0} x ,$$

then for any sufficiently large integer n, we can find a positive integer N such that

$$r_{N-1}(y)+n_0 \leqq n < r_N(y)+n_0 .$$

By the discussion similar to the case $x \in D$

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} r(T^k x) &= \frac{1}{n}\sum_{k=n_0}^{n-1} r(T^k x) \\ &= \frac{1}{n}\sum_{i=0}^{n-n_0-1} r(T^i y) \\ &= \frac{1}{n}\sum_{j=0}^{N-1} r(T^{r_j(y)} y) \\ &= \frac{1}{n}\sum_{j=0}^{N-1} (r_{j+1}(y)-r_j(y)) \\ &= \frac{1}{n} r_N(y)\end{aligned}$$

and

$$\frac{r_N(y)}{r_N(y)+n_0} < \frac{1}{n} r_N(y) \leqq \frac{r_N(y)}{r_{N-1}(y)+n_0} .$$

So for a. e. $x \in \bigcup_{n=1}^{\infty} T^{-n} D$

$$\tilde{r}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} r(T^k x) = \lim_{n\to\infty} \frac{1}{n} r_N(y) = 1 .$$

Thus we get

$$\tilde{r}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} r(T^k x) = 1$$

for a. e. $x \in \bigcup_{n=0}^{\infty} T^{-n} D$.

Finally if $x \in X \setminus \bigcup_{n=0}^{\infty} T^{-n} D$, then

$$T^k x \in X \setminus D$$

for all integers $k \geqq 0$, that is,

$$r(T^k x) = 0$$

and

$$\widetilde{r}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r(T^k x) = 0.$$

This completes the proof.

In the proof of Theorem 1, we did not make an explicit use of the integrability of the recurrence time $r(x)$. The integrability of a non-negative measurable function $f(x)$ on a finite measure space X is deduced from the integrability of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$$

where T is a measure preserving transformation on X ([4], Lemma). Therefore from Theorem 1, the ordinary recurrence time is an integrable function.

Using Theorem 1, we shall prove some results of the same type for the random recurrence time as for the ordinary recurrence time. We need the following lemmas. The idea in Lemma 1 is due to J. Neveu (Theorem 8 in [2]).

LEMMA 1. *Let $r^*(x) \equiv r_{n(x)}(x) \equiv r_{n(x)}(x, D)$ be a random recurrence time of an invertible measure preserving transformation T in a measurable set D with a positive measure. If we put*

$$m_1(x) = \begin{cases} n(x) - 1 & \text{if } x \in D, \\ 0 & \text{if } x \in X \backslash D, \end{cases}$$

then $r_{m_1(x)}(x)$ is also a random recurrence time of the transformation T in the set $D_1 = \{x\,;\, m_1(x) > 0\}$, that is, there exists a non-negative integer valued measurable function $n_1(x)$ such that

$$r_{m_1(x)}(x) \equiv r_{m_1(x)}(x, D) = r_{n_1(x)}(x, D_1)$$

for any $x \in X$ and the family of measurable sets

$$[T^{r_n(\cdot, D_1)}\{x\,;\, n_1(x) = n\}]_{n=0}^{\infty}$$

is a partition of X up to a set of measure zero, where the notation $r(\cdot, D_1)$ is assigned to the recurrence time of the transformation T in the set D_1.

PROOF. If $m_1(x)=0$, that is, if x belongs to $X\backslash D_1$,

$$r_{m_1(x)}(x) = r_0(x) = 0\,.$$

Then we put for any $x \in X\backslash D_1$

$$n_1(x) = 0\,.$$

Since D includes D_1, for any $x \in D_1$ such that $m_1(x)=m$ there exists a positive integer n such that

$$r_m(x, D) = r_n(x, D_1)\,.$$

Then we put for any $x \in D_1$

$$n_1(x) = n\,.$$

The function $n_1(x)$ is thus defined for every $x \in X$. We show that $r_{n_1(x)}(x,D_1)$ is a random recurrence time of the transformation T in the set D_1.

Since $r_{n(x)}(x)$ is a random recurrence time of the transformation T in the set D, the family

$$[T^{r_n(\cdot)}\{x\,;\ n(x) = n\}]_{n=0}^{\infty}$$

is a partition of X up to a set of measure zero, and

$$[T^{r_{n-1}(\cdot)}\{x\,;\ m_1(x) = n-1\},\ X\backslash D]_{n=1}^{\infty}$$

is also a partition of X up to a set of measure zero. Because $T^{r_1(\cdot)}$ is the invertible measure preserving transformation on X. Then $T^{r_{m_1(\cdot)}(\cdot)}$ is also an invertible measure preserving transformation on X. Since $r_{m_1(x)}(x, D) = r_{n_1(x)}(x, D_1)$, the family

$$[T^{r_{n_1(\cdot)}(\cdot,D_1)}\{x\,;\ n_1(x) = n\}]_{n=0}^{\infty}$$
$$= [T^{r_n(\cdot,D_1)}\{x\,;\ n_1(x) = n\}]_{n=0}^{\infty}$$

is a partition of X up to a set of measure zero. Lemma is proved.

By Lemma 1, we can obtain a sequence of the random recurrence times inductively. We define

$$r_{n_0(x)}(x, D_0) = r_{n(x)}(x), \qquad D_0 \equiv D$$

and $r_{n_1(x)}(x, D_1)$ as in Lemma 1. Suppose that $r_{n_i}(x, D_i)$ is defined for some integer i. We define $r_{n_{i+1}(x)}(x, D_{i+1})$ as follows: Put

$$m_{i+1}(x) = \begin{cases} n_i(x) - 1 & \text{if } x \in D_i . \\ 0 & \text{if } x \in X \backslash D_i . \end{cases}$$

and

$$D_{i+1} = \{x ; m_{i+1}(x) > 0\} ,$$

then by Lemma 1, there exists $n_{i+1}(x)$ such that $r_{n_{i+1}(x)}(x, D_{i+1})$ is a random recurrence time of the transformation T in the set D_{i+1}.

The next lemma is well known as a probabilistic interpretation of the individual ergodic theorem. (see, p. 351 in [1])

LEMMA 2. *Let T be a measure preserving transformation on a finite measure space X. For any integrable function f*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = M[f \mid \mathfrak{S}](x) \qquad a. e. \text{ in } X$$

where $\mathfrak{S}$ is the σ-algebra of T-invariant measurable sets up to a set of measure zero and $M[f \mid \mathfrak{S}]$ is the conditional mean of f with respect to $\mathfrak{S}$.

Now let us prove a limit theorem for the random recurrence time. We use the same notations about the sequence of the random recurrence times in the remark of Lemma 1.

THEOREM 2. *Let $r^*(x) \equiv r_{n(x)}(x)$ be a random recurrence time of the transformation T in the set D. Then for each $i \geqq 0$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r^*(T^k x) = \begin{cases} 0 & \text{if } x \in X \backslash \bigcup_{n=0}^{\infty} T^{-n} D_0 \\ i+1 & \text{if } x \in \bigcup_{n=0}^{\infty} T^{-n} D_i \backslash \bigcup_{n=0}^{\infty} T^{-n} D_{i+1} \end{cases}$$

a. e.

PROOF. We begin with the case $i=0$. By the property of the successive recurrence time and by Lemma 1

$$r^*(x) \equiv r_{n(x)}(x) = r_{m_1(x)}(x) + r_1(T^{r_{m_1(x)}(x)}x)$$
$$= r_{n_1(x)}(x, D_1) + r_1(T^{r_{n_1(x)}(x, D_1)}x)$$

for any $x \in X$. We consider the limit function

$$\bar{r}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r_1(T^{r_{n_1(\cdot)}(\cdot, D_1)} T^k x)$$

for any $x \in X$. We denote by S for the sake of simplicity the transformation $T^{r_{n_1(\cdot)}(\cdot, D_1)}$ and by $\mathfrak{S}$ the σ-algebra of T-invariant measurable sets up to a set of measure zero. Then $S\mathfrak{S}$ is the σ-algebra of STS^{-1}-invariant measurable sets up to a set of measure zero. Because, if E is any T-invariant measurable set then

$$STS^{-1}(SE) = STE = SE$$

up to a set of measure zero and if F is any STS^{-1}-invariant measurable set then

$$T(S^{-1}F) = S^{-1}(STS^{-1})F = S^{-1}F$$

up to measure zero. Further any T-invariant measurable set is obviously S-invariant, that is,

$$S\mathfrak{S} = \mathfrak{S}$$

If we put for a. e. $x \in X$

$$y = Sx$$

$$\bar{r}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r_1(ST^k x)$$
$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r_1(ST^k S^{-1} y)$$
$$= M[r_1 \mid S\mathfrak{S}](y)$$
$$= M[r_1 \mid \mathfrak{S}](y)$$
$$= \widetilde{r}(Sx) \qquad \text{a. e.}$$

We have applied the individual ergodic theorem to r_1 with respect to the transformation STS^{-1}.

By Theorem 1, $\tilde{r}$ is a T-invariant function, that is, S-invariant function. Therefore

$$\bar{r}(x) = \tilde{r}(Sx) = \tilde{r}(x)$$

for a. e. $x \in X$. Then for a. e. $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r^*(T^k x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r_{n_1(\cdot)}(T^k x, D_1) + \tilde{r}(x) .$$

For any $x \in \bigcup_{n=0}^{\infty} T^{-n} D_0 \setminus \bigcup_{n=0}^{\infty} T^{-n} D_1$

$$r_{n_1(\cdot)}(T^k x, D_1) = 0$$

and for any $x \in X \setminus \bigcup_{n=0}^{\infty} T^{-n} D_0$

$$r^*(T^k x) = 0 .$$

Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r^*(T^k x) = \begin{cases} 0 & \text{if } x \in X \setminus \bigcup_{n=0}^{\infty} T^{-n} D_0 , \\ 1 & \text{if } x \in \bigcup_{n=0}^{\infty} T^{-n} D_0 \setminus \bigcup_{n=0}^{\infty} T^{-n} D_1 . \end{cases} \quad \text{a. e.}$$

For the case $i > 0$, we use the discussion of the case $i = 0$ inductively. Namely by the equation for any $i > 0$

$$\begin{aligned} r_{n_i}(x, D_i) &= r_{m_{i+1}}(x, D_i) + r_1(T^{r_{m_{i+1}(x)}(x, D_i)} x, D_i) \\ &= r_{n_{i+1}}(x, D_{i+1}) + r_1(T^{r_{n_{i+1}(x)}(x, D_{i+1})} x, D_i) . \end{aligned}$$

We can show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r_1(T^{r_{n_{i+1}(x)}(\cdot, D_{i+1})} T^k x, D_i)$$

$$= \begin{cases} 1 & \text{if } x \in \bigcup_{n=0}^{\infty} T^{-n} D_i \\ 0 & \text{if } x \in X \setminus \bigcup_{n=0}^{\infty} T^{-n} D_i . \quad \text{a. e.} \end{cases}$$

Therefore the proof is completed.

Now we can prove a generalized integral theorem of Kac. We use the same notations as in Theorem 2.

COROLLARY. *For a random recurrence time $r^*(x)$ of the transformation T in the set D,*

$$\int_X r^*(x) d\mu(x) = \sum_{i=0}^{\infty} \mu\left(\bigcup_{n=0}^{\infty} T^{-n} D_i \right),$$

in the sense that, if the right hand side is finite so is the left hand side and they are equal, and if the right hand side is infinite so is the left hand side.

PROOF. If the right hand side is finite, by the individual ergodic theorem

$$\begin{aligned} \int r^*(x) d\mu(x) &= \int_X \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} r^*(T^k x) d\mu(x) \\ &= \sum_{i=0}^{\infty} (i+1) \mu\left(\bigcup_{n=0}^{\infty} T^{-n} D_i \setminus \bigcup_{n=0}^{\infty} T^{-n} D_{i+1} \right) \\ &= \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \mu\left(\bigcup_{n=0}^{\infty} T^{-n} D_j \setminus \bigcup_{n=0}^{\infty} T^{-n} D_{j+1} \right) \\ &= \sum_{i=0}^{\infty} \mu\left(\bigcup_{n=0}^{\infty} T^{-n} D_i \right). \end{aligned}$$

If the right hand is infinite, then we put for a fixed integer N,

$$\begin{aligned} r^*_{(N)}(x) = r_{n_{N+1}(x)}(x, D_{N+1}) &+ r_1(T^{r_{n_{N+1}(x)}(x, D_{N+1})} x, D_N) \\ &+ \cdots\cdots + r_1(T^{r_{n_1(x)}(x, D_1)} x, D_0) . \end{aligned}$$

Obviously $r^*_{(N)}(x)$ increases to $r^*(x)$ as $N \to \infty$. By the monotone convergence theorem

$$\int_X r^*(x)d\mu(x) = \lim_{N\to\infty}\int_X r^*_{(N)}(x)d\mu(x)$$
$$\geqq \lim_{N\to\infty}\sum_{i=0}^{N}\mu\left(\bigcup_{i=0}^{\infty}T^{-n}D_i\right) = \infty\,.$$

We remark that $r^*_{(N)}(x)$ is not always a random recurrence time because the requirement about the partition in Definition 2 is uncertain. Finally we illustrate the random recurrence time by simple example.

Let $X\equiv[0, 1]$ be the unit interval with the ordinary Lebesgue measure μ and T an invertible measure preserving transformation on X such that

$$T\left[0,\frac{1}{3}\right]=\left[\frac{1}{3},\frac{2}{3}\right], T\left[\frac{1}{3},\frac{2}{3}\right]=\left[\frac{2}{3},1\right].$$

Put

$$D_0 = \left[0,\frac{2}{3}\right]$$

and we consider the recurrence time of T in D_0.

Obviously the ordinary recurrence time in D_0 is following;

$$r(x, D_0) = \begin{cases} 1 & \text{if} \quad x\in\left[0,\frac{1}{3}\right] \\ 2 & \text{if} \quad x\in\left[\frac{1}{3},\frac{2}{3}\right] \\ 0 & \text{if} \quad x\in\left[\frac{2}{3},1\right] \end{cases}$$

Next we define a random recurrence time of T in D_0 as follows; Put

$$n(x) = \begin{cases} 1 & \text{if} \quad x\in\left[0,\frac{1}{3}\right] \\ 3 & \text{if} \quad x\in\left[\frac{1}{3},\frac{2}{3}\right], \end{cases}$$

then $r^*(x) = r_{n(x)}(x, D_0)$ is a random recurrence time. Indeed,

$$r_{n(x)}(x, D_0) = \begin{cases} r_1(x, D_0) = 1 & \text{if } x \in \left[0, \frac{1}{3}\right] \\ r_3(x, D_0) = 5 & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right] \end{cases}$$

and the family

$$\left[\left[\frac{2}{3}, 1\right], T^{r_1(\cdot)}\left[0, \frac{1}{3}\right], T^{r_3(\cdot)}\left[\frac{1}{3}, \frac{2}{3}\right]\right]$$

$$= \left[\left[\frac{2}{3}, 1\right], T^1\left[0, \frac{1}{3}\right], T^5\left[\frac{1}{3}, \frac{2}{3}\right]\right]$$

$$= \left[\left[\frac{2}{3}, 1\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[0, \frac{1}{3}\right]\right]$$

is a partition of X.

Further Lemma 1 states the following with respect to the example. Put

$$D_1 = \{x \,;\, n(x) - 1 > 0\} = \left[\frac{1}{3}, \frac{2}{3}\right]$$

and

$$n_1(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{1}{3}, \frac{2}{3}\right] \\ 0 & \text{if } x \in \left[0, \frac{1}{3}\right], \left[\frac{2}{3}, 1\right]. \end{cases}$$

Then

$$r_{m_1(x)}(x, D_0) = r_{n_1(x)}(x, D_1)$$

and

$$r_{n(x)}(x, D_0) = r_1(x, D_1) + r_1(T^{r_1(x, D_1)}x, D_0)$$

About Theorem 1 and Corollary, since

$$\bigcup_{n=0}^{\infty} T^{-n}D_0 = X, \quad \bigcup_{n=0}^{\infty} T^{-n}D_1 = X,$$

we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} r^*(T^k x) = 1+1 = 2 \qquad \text{a. e. in } X,$$

and

$$\int_X r^*(x)d\mu(x) = \mu\left(\bigcup_{n=0}^{\infty} T^{-n}D_0\right) + \mu\left(\bigcup_{n=0}^{\infty} T^{-n}D_1\right)$$

$$= 2\mu(x) = 2.$$

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