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THE EIGENFUNCTION EXPANSION OF THE SYMMETRIC OPERATORS ASSOCIATED WITH GELFAND TRIPLET

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Since the work of F. Mautner [10], the integral representation of a spectral measure is successfully established by Gårding, F. E. Browder and others in the field of the theory of partial differential operators (see, e. g., [5]). Gelfand and Kostyuchenko connected naturally these results with the generalized eigenfunction expansion of selfadjoint operators and proved abstract expansion theorem [6]. There, they considered the problem in the rigged Hilbert space or so-called Gelfand triplet [7, 9]. This space is now widely applied; the theory and its proofs have been extended and clarified by many research workers [2]. Our paper is along with Foias [3, 4]. In contrast with the other papers the main part of ours is to prove the eigenfunction expansion theorem without going through the spectral representation or the diagonalization of a selfadjoint operator. It may well be said that our proof is analogous to that of Maurin for the diagonal form [8]. We shall state the subject of the present study more precisely in the following section.

1. First, a few notations. Let Φ be a nuclear locally convex vector space and at the same time pre-Hilbert space under the scalar product $(\, , \,)$. We denote by H the completion of Φ under the norm induced by $(\, , \,)$. We suppose that embedding p from Φ to H is continuous. Then for each element h of H, the anti-linear functional which maps e of Φ to (h, pe) is continuous. Denoting the transpose of p by p^*, we write this relation in the following way:

$$(h, pe) = \langle p^*h, e\rangle, = \langle h^*, e\rangle, \text{ for any } e \text{ of } \Phi,$$

where h^* is an element of Φ', anti-dual space of Φ. We identify h to h^*. Thus

$$\Phi' \supset H \supset \Phi .$$

Those three spaces we call Gelfand triplet.

Let $\| \ \|_\alpha$ be a continuous norm on Φ and satisfies the formula $\|e\| = (e, e)^{1/2} \leqq \|e\|_\alpha$ for all e of Φ. We suppose for simplicity that two norms $\| \ \|, \| \ \|_\alpha$ are in concordance (see Definition 2). Then Φ_α, the completion of Φ under $\| \ \|_\alpha$, is naturally embedded into H. This embedding mapping we denote by p_α. Similarly to the above relation

$$(h, p_\alpha e) = <p_\alpha{}^* h, e>_\alpha = <h^*, e>_\alpha ,$$

where h^* is an element of $\Phi_\alpha{}' \subset \Phi'$. When e is an element of $\Phi \subset \Phi_\alpha$, last value is equal to $<h^*, e>$.

$$\Phi' \supset \Phi_\alpha{}' \supset H' = H \supset \Phi_\alpha \supset \Phi .$$

By the hypothsis that Φ is nuclear, $\| \ \|_\alpha$ can always be taken such that p_α is a nuclear mapping. In the following, the nuclearity of Φ is not essential, but the existense of such p_α is.

Let A be a continuous linear operator on Φ. Through this paper A is fixed and we assume that A is formally selfadjoint with respect to the scalar product in H, that is,

$$(Ae, f) = (e, Af), \text{ for } e, f \text{ of } \Phi ;$$

exactly

$$(pAe, pf) = (pe, pAf) .$$

But we shall suppress p or p^* when no confusion is likely to result.

We denote the transpose of A by A'. Then $A' \supset A$ in Φ'. The smallest closed extension of A in H, which we shall denote by A_0, is also symmetric in H. Therefore by the spectral theory of the symmetric operator in the Hilbert space there exists a semi-spectral measure $\{F(\sigma)\}$ (generelized resolution of identity) defined on Borel field Σ of real line R satisfying the following relations:

(1) $$(A_0 e, f) = \int_R \lambda d(F(\lambda)e, f), \text{ for } e \in D_{A_0}, \text{ domain of } A_0,$$

and $f \in H$,

(2) $$\|A_0 e\|^2 = \int_R |\lambda|^2 d(F(\lambda)e, e), \text{ for } e \in D_{A_0} .$$

$\{F(\sigma)\}$ is uniquely determined only when A_0 is maximally symmetric [1].

Now we can describe the expansion theorem which will be proved by our method.

THEOREM. *Suppose that* $\| \ \|_\alpha$ *is Hilbert norm and mapping* p_α *is of Hilbert-Schmidt type. Then there exist a bounded positive measure on* (R, Σ), $\mu(\sigma)$ *and one-parameter continuous linear operators from* Φ_α *to* Φ_α', $\mathcal{X}(\lambda)$ *which satisfy the properties*:

(a) $(F(\sigma)e, f) = \int_\sigma <\mathcal{X}(\lambda)e, f> d\mu(\lambda)$, *for any* $e, f \in \Phi$, $\sigma \in \Sigma$,

(b) $<\mathcal{X}(\lambda)e, e> \geqq 0$, *for* $e \in \Phi$,

(c) $A'\mathcal{X}(\lambda) = \lambda\mathcal{X}(\lambda)$, μ-*a. e.*

$\mathcal{X}(\lambda)$ *is almost everywhere with respect to* μ (*abbreviated*: μ-*a. e.*) *uniquely determined for* $\{F(\sigma)\}$.

Moreover $\mathcal{X}(\lambda)$ *has* μ-*a. e. the following representation with* $\{e_n^*(\lambda)\}$, *an orthonormal basis in the Hilbert space* Φ_α.

(d) $\mathcal{X}(\lambda)e = \sum_{n=1}^{\infty} t_n(\lambda) <\overline{e_n^*(\lambda), e}> e_n^*(\lambda)$, $t_n(\lambda) \geqq 0, \sum_n t_n(\lambda) < \infty$, *for any* $e \in \Phi$.

(e) $A'e_n^*(\lambda) = \lambda e_n^*(\lambda), n = 1, 2, \cdots$

$\{e_n^*(\lambda)\}$ is the eigenfunction in the sense that it satisfies the eigen-equation for the extended operator of A.

REMARK 1. We call $\mathcal{X}(\lambda)$ an eigen-operator. $\mathcal{X}(\lambda)$ is of rank$\leqq n$ μ-a. e. if and only if Neumark extension of $F(\sigma)$ is of multiplicity$\leqq n$. Let $F'(\sigma)$ be another semi-spectral measure and the corresponding eigen-operator be $\mathcal{X}'(\lambda)$. Then Neumark extensions of $F(\sigma)$ and $F'(\sigma)$ are unitary equivalent if and only if $\mathcal{X}(\lambda)$ and $\mathcal{X}'(\lambda)$ are of the same rank μ-a. e. [3].

Foias proved above theorem for such a norm $\| \ \|_\alpha$ that p_α is nuclear. When A_0 is a selfadjoint operator this case is proved by Maurin using the decomposition into the direct integral representatation relative to A_0. Above theorem insists on that when $\| \ \|_\alpha$ is a quadratic norm the condition for p_α is weakened to be of Hilbert-Schmidt type. We remark in section 3 that this is considered to be strictly a generalization.

2. Let X and Y be separable Banach spaces, Y' be the Banach space of continuous anti-linear functional on Y under the uniform topology, $B(X, Y')$, be the Banach space consisting of bounded linear operators from X to Y' with usual norm topology, and Σ be a Borel field whose element is a subset of a set S. Let m be a vector-valued measure on (S, Σ) with values in $B(X, Y')$, that is, m maps Σ into $B(X, Y')$, and for every element σ of Σ, $<m(\sigma)e,f>$ is a usual measure on Σ for each $e \in X$ and $f \in Y$.

DEFINITION 1. The vector-valued measure m is said to be of bounded variation if for all finite partition $\{\sigma_i\}$ of S, $\sup_{\{\sigma_i\}} \sum_i \|m(\sigma_i)\|$ has a finite value.

Let m be of bounded variation. Then we can define the indefinite total variation ν of m by

$$\nu(\sigma) = \sup_{\{\sigma_i\}} \sum_i \|m(\sigma_i)\| ,$$

where $\{\sigma_i\}$ denote any partition of σ. ν is a bounded positive measure on Σ.

With above notations we can describe our main tool.

THEOREM 1. *If the vector-valued measure m on (S, Σ) is of bounded variation, there exists $B(X, Y')$-valued function $\chi(\lambda)$ which is ν-a.e. uniquely determined on S and satisfies the following formula.*

$$(3) \qquad <m(\sigma)e,f> = \int_\sigma <\chi(\lambda)e, f> d\nu(\lambda), \textit{ for any } e \in X, f \in Y \textit{ and } \sigma \in \Sigma .$$

And $\|\chi(\lambda)\| = 1$, *ν-a. e.*

Full proof is mentioned in [3]. Here we sketch only its outline.

By Radon-Nikodym theorem there exists ν-integrable function $\chi_{e,f}(\lambda)$ which satisfies

$$(4) \qquad <m(\sigma)e, f> = \int_\sigma \chi_{e,f}(\lambda) d\nu(\lambda), \quad \sigma \in \Sigma .$$

If $\varphi(\lambda)$ is an element of $L^1(S, \Sigma, \nu)$, i.e., $|\varphi(\lambda)|$ is ν-integrable, it is also integrable with respect to $<m(\sigma)e,f>$ and

$$\left| \int_S \varphi(\lambda) d<m(\lambda)e, f> \right| \leqq \|e\| \|f\| \int_S |\varphi(\lambda)| d\nu(\lambda) .$$

Thus $<m(\sigma)e,f>$ defines a linear functional on L^1, whose norm is bounded by $\|e\|\|f\|$. Therefore

$$(5) \qquad |\chi_{e,f}(\lambda)| \leqq \|e\|\|f\|,$$

for any λ except for some ν-null set $N(e,f)$. But when we set $\chi_{e,f}(\lambda)=0$ for λ of $N(e,f)$, above formulas (4), (5) are also satisfied.

Let X_0, Y_0 be countable dense sets of X, Y. Then for finite linear combination of their elements with rational complex, $\chi_{e,f}(\lambda)$ is linear w. r. t. e and f ν-a. e. We set $<\chi_0(\lambda)e,f>=\chi_{e,f}(\lambda)$. Since $\chi_0(\lambda)$ is bounded by (5), it can be extended for all of X and Y. Thus there exist $\chi(\lambda)\in B(X,Y')$ for which

$$<\chi(\lambda)e, f> = \chi_{e,f}(\lambda), \text{ for all } e\in X, f\in Y.$$

By (4) for $e\in X_0, f\in Y_0$ the formula (3) is satisfied, therefore, by Lebesgue theorem, (3) is always satisfied ν-a. e.

Other part is easily proved.

Before passing to the next proposition, we note some properties about the mapping p_α, which we shall use in Theorem 2.

Let $\| \ \|_0$ and $\| \ \|_1$ be the norms on linear space Φ and satisfy the inequality $\|f\|_0 \leqq \|f\|_1$ for an arbitrary f of Φ. It will suffice to suppose that there exists a $M>0$ such that $\|f\|_0 \leqq M\|f\|_1$. We denote the completion of Φ under $\| \ \|_i$ by Φ_i. From identity mapping on Φ we derive naturally a continuous linear mapping on Φ_1 into Φ_0 when both spaces are provided with the norms $\| \ \|_1$, $\| \ \|_0$ respectively. We denote it by p_1 and its transpose by $p_1{}^*$.

DEFINITION 2. Two norms are said to be in concordance with each other, when any sequence in Φ which is Cauchy for both norms, and which converges to zero for one of the norms, necessarily converges to zero for the other.

When above two norms $\| \ \|_0$, $\| \ \|_1$ are in concordance, correspondence of equivalent Cauchy sequences in both norms is one-to-one. Therefore Φ_1 is naturally identified to a subset of Φ_0.

LEMMA 1. *The following three conditions are equivalent.*

(6) $\| \ \|_0$ *and* $\| \ \|_1$ *in* Φ *are in concordance.*

(7) p_1 *is one-to-one.*

(8) $p_1{}^*(\Phi_0')$ *is total over* Φ_1.

PROOF. We have already seen that p_1 is one-to-one when (6) is satisfied.

(7)→(8). Suppose that for any e^* of Φ_1' and some f of Φ_1 $<p_1^*e^*, f>_1 = 0$. Then $<e^*, p_1 f>_0 = <p_1^*e^*, f>_1 = 0$, whence $p_1 f = 0$. By the assumption $f = 0$, which implies that $p_1^*(\Phi_0')$ is total.

(8)→(6). Suppose that the sequence $\{f_n\}$ in Φ converges to zero in Φ_0 and converges to f in Φ_1. Then, by the equation $<p_1^*e^*, f>_1 = <e^*, p_1 f>_0$, for any e^* of Φ_0' $<p_1^*e^*, f>_1 = 0$. Therefore by the assumption $f = 0$.

3. We now return to our problem.

LEMMA 2. *Let Φ_α be a separable Hilbert space. (When Φ is nuclear, Φ is separable, so Φ_α is separable.) If p_α is of Hilbert-Schmidt type, $p_\alpha^*F(\sigma)p_\alpha$ is of bounded variation.*

PROOF. p_α is represented in the following form

$$p_\alpha e = \sum_n \lambda_n <\overline{e_n, e}>_\alpha h_n, \quad \lambda_n \geqq 0, \quad \sum_n \lambda_n^2 < \infty ,$$

where $\{e_n\}$ and $\{h_n\}$ are orthonormal basis in Φ_α and H respectively. As $(p_\alpha^*F(\sigma)p_\alpha)$ is regarded as a positive bounded linear operator on Φ_α by the identification of Φ_α' to Φ_α, it possesses a positive square root $(p_\alpha^*F(\sigma)p_\alpha)^{1/2}$ on Φ_α, for which

$$\sum_n ((p_\alpha^*F(\sigma)p_\alpha)^{1/2}e_n, (p_\alpha^*F(\sigma)p_\alpha)^{1/2}e_n)_\alpha = \sum_n <p_\alpha^*F(\sigma)p_\alpha e_n, e_n>_\alpha$$
$$= \sum_n (F(\sigma)p_\alpha e_n, p_\alpha e_n) = \sum_n \lambda_n^2 (F(\sigma)h_n, h_n) \leqq \sum_n \lambda_n^2 < \infty ,$$

where $(\ ,\ )_\alpha$ denotes the scalar product in Φ_α. Therefore $(p_\alpha^*F(\sigma)p_\alpha)^{1/2}$ is of Hilbert-Schmidt type; the value of the first formula is independent of the choices of the orthonormal basis $\{e_n\}$. Especially when we take $\{e_n\}$ the eigenvectors of $(p_\alpha^*F(\sigma)p_\alpha)$, its value is the sum of eigenvalues which are all non-negative, whence it is not smaller than the norm of $p_\alpha^*F(\sigma)p_\alpha$. From this fact we derive

$$\|p_\alpha^*F(\sigma)p_\alpha\|_\alpha \leqq \sum_n \lambda_n^2 (F(\sigma)h_n, h_n) .$$

consequently for any partition $\{\sigma_i\}$ of σ

$$\sum_i \|p_\alpha^*F(\sigma_i)p_\alpha\|_\alpha \leqq \sum_i \sum_n \lambda_n^2 (F(\sigma_i)h_n, h_n) = \sum_n \lambda_n^2 \sum_i (F(\sigma_i)h_n, h_n)$$
$$= \sum_n \lambda_n^2 (F(\sigma)h_n, h_n) \leqq \sum_n \lambda_n^2 < \infty .$$

which was to be proved.

REMARK 2. Above lemma is valid when $p_\alpha^* F(\sigma) p_\alpha$ is of nuclear type. In fact such $p_\alpha^* F(\sigma) p_\alpha$ satisfies the following inequality; for the orthonormal basis $\{e_n\}$ in Φ_α

$$\|p_\alpha^* F(\sigma) p_\alpha\|_\alpha \leqq \sum_n \langle p_\alpha^* F(\sigma) p_\alpha e_n, e_n \rangle_\alpha \quad < \infty ,$$

where the second member represents the trace norm of $p_\alpha^* F(\sigma) p_\alpha$ [7]. Therefore the latter part of the above proof is similarily applicable. When $F(\sigma)^{1/2} p_\alpha$ is of Hilbert-Schmidt type, $p_\alpha^* F(\sigma) p_\alpha$ is nuclear. So above lemma is a special case of this.

Foias proved the above lemma for the Banach space Φ_α under the condition that p_α is nuclear. The following theorem shows that this case is reduced to that for Hilbert space, which may be interest in itself.

THEOREM 2. *Let* $\| \ \|_\alpha$ *be a general norm and* p_α *is the mapping of nuclear type. Then there exists a Hilbert space* Φ_β *such that*

$$H \supset \Phi_\beta \supset \Phi_\alpha$$

and embedding p_β *from* Φ_β *into* H *is of Hilbert-Schmidt type.*

PROOF. Since p_α is nuclear, its transpose $p_\alpha^* : H \to \Phi_\alpha'$ is also nuclear, i. e., there exist $\{h_n\} \subset H$ and $\{e_n^*\} \subset \Phi_\alpha'$ such that

$$p_\alpha^* h = \sum_n \lambda_n (\overline{h_n, h}) e_n^*, \ \|h_n\| = 1, \ \|e_n^*\|_\alpha = 1, \ \lambda_n \geqq 0 \text{ and } \sum_n \lambda_n < \infty .$$

We equip H with scalar product

$$(g, h)_\gamma = \sum_n \lambda_n (\overline{h_n, g})(h_n, h) \text{ for each } g, h \in H$$

and denote this pre-Hilbert space by H_γ.

$$\|p_\alpha h\|_\alpha \leqq K^{1/2} \|h\|_\gamma \leqq K \|h\|, \text{ for } h \in H, \text{ where } K = \sum_n \lambda_n .$$

Identity mapping $p_\gamma : H \to H_\gamma$ is of Hilbert-Schmidt type. Because, for any orthonormal basis $\{g_i\}$ in H,

$$\sum_i (g_i, g_i)_\gamma = \sum_i \sum_n \lambda_n \overline{(h_n, g_i)}(h_n, g_i) = \sum_n \lambda_n \sum_i |(h_n, g_i)|^2$$
$$= \sum_n \lambda_n \|h_n\|^2 = \sum_n \lambda_n < \infty .$$

Norms $\| \ \|$ and $\| \ \|_\gamma$ are in concordance but in general $\| \ \|_\alpha$ and $\| \ \|_\gamma = (\ , \ )_\gamma^{1/2}$ are not in concordance. But if we take $\|h\|_\beta$ for each $h \in H$ to be the infimum of $\|h\|_\gamma = \left(\sum \lambda_n |\alpha_n|^2\right)^{1/2}$ for all representation of $p_\alpha{}^* h$ for fixed $\{e_n{}^*\}$, $\| \ \|_\beta$ are also in concordance.

In fact, let $L_\lambda{}^2$ be the Hilbert space consisting of all sequences $\alpha = \{\alpha_n\}$ such that $\sum_n \lambda_n |\alpha_n|^2 < \infty$ under the scalar product of weight sequence $\{\lambda_n\}$. We denote by P the mapping of $\alpha = \{\alpha_n\} \in L_\lambda{}^2$ to $\varphi_\alpha = \sum_n \lambda_n \alpha_n e_n{}^*$. Since for the same K as above the inequality

$$\|\varphi_\alpha\| \leqq K^{1/2} \|\alpha\|_{L_\lambda{}^2}$$

holds, P is continuous mapping from $L_\lambda{}^2$ into Φ_α'. Evidently for any element φ of H, there exists $\alpha \in L_\lambda{}^2$ such that $\varphi = \varphi_\alpha$. Thus the image of P contains H. Let M be the orthogonal complement of the null kernel of P in $L_\lambda{}^2$ and P_0 be the mapping of M into Φ_α' induced by P. Then by the identification of M into Φ_α with P_0,

$$\Phi_\alpha' \supset M \supset H .$$

Since the norms in M is equal to $\| \ \|_\beta$ for the element of H (this shows, by the way, that $\| \ \|_\beta$ defines an inner product), $\| \ \|_\alpha$ and $\| \ \|_\beta$ are in concordance.

Let $\Phi_\beta{}^*$ be the completion of H under $\| \ \|_\beta$. Then, a fortiori, embedding $p_\beta{}^* : H \to \Phi_\beta{}^*$ is of Hilbert-Schmidt type. When we regard, as in section 1, the anti-dual space of Hilbert space $\Phi_\beta{}^*$ as the subset of H, it is Φ_β which we seeked for. In fact, $H \supset \Phi_\beta$ holds by Lemma 1 since H is dense in $\Phi_\beta{}^*$. On the other hand as $p_\beta{}^*$ is one-to-one and H is dense in Φ_α', $\Phi_\beta{}^*$ is dense in Φ_α'. Therefore by Lemma 1 $\Phi_\beta \supset \Phi_\alpha$. Then p_β is obtained as the transpose of $p_\beta{}^*$.

Under the condition of Lemma 2, applying Theorem 1 to it, we obtain the following theorem. There we shall be able to take as $\mu(\sigma)$ any positive bounded measure with respect to which the indefinite total variation determined by $p_\alpha^* F(\sigma) p_\alpha$ is absolutely continuous. In the sequel we shall fix one of them.

THEOREM 3. *$B(\Phi_\alpha, \Phi_\alpha')$-valued function $\chi(\lambda)$ on R which satisfies the* *following relations is μ-a. e. uniquely determined.*

(a) $(F(\sigma)p_\alpha e, p_\alpha f) = <p_\alpha^* F(\sigma) p_\alpha e, f>_\alpha = \int_\sigma <\chi(\lambda)e, f>_\alpha d\mu(\lambda),$
for all $e, f \in \Phi_\alpha, \sigma \in \sum,$

(b) $<\chi(\lambda)e, e>_\alpha \geqq 0, \ e \in \Phi_\alpha,$
$\|\chi(\lambda)\|_\alpha$ is *μ-integrable,*

(a′) $(p_\alpha e, p_\alpha f) = <p_\alpha^* p_\alpha e, f>_\alpha = \int_R <\chi(\lambda)e, f>_\alpha d\mu(\lambda).$

It is easily proved that conversely if for $B(\Phi_\alpha, \Phi_\alpha')$-valued function $\chi(\lambda)$ the conditions (b), (a′) hold, there exists μ-a. e. uniquely determined semi-spectral measure $\{F(\sigma)\}$ for which the relation (a) is satisfied.

4. Now we are in the position to prove the main theorem.

THEOREM 4. *Under the same condition as Theorem* 3,

(c) $A\chi(\lambda) = \lambda\chi(\lambda)$ *μ-a. e.,*

and for any $e \in \Phi$,

(d) $\chi(\lambda)e = \sum_n t_n(\lambda) <e_n^*(\lambda), e> e_n^*(\lambda), \ t_n(\lambda) \geqq 0, \sum_n t_n(\lambda) < \infty$ μ-a. e.,
where $A e_n^*(\lambda) = \lambda e_n^*(\lambda), \ n = 1, 2, \cdots$

Here we can select $\{e_n^*(\lambda)\}$ *such that for some* $\{e_n(\lambda)\}$

$$e_n^*(\lambda) = \chi(\lambda) e_n(\lambda), \quad <e_m^*(\lambda), e_n(\lambda)>_\alpha = \delta_{m,n}$$

and $\{e_n^*(\lambda)\}$ *is an orthonormal basis in the Hilbert space* Φ_α'.

PROOF. By (1) and (2) in section 1, $\chi(\lambda)$ which we get in Theorem 3 satisfies the following two relations.

(9)
$$(A_0 p_\alpha e, p_\alpha f) = \int_R \lambda <\chi(\lambda)e, f> d\mu(\lambda),$$

$$\|A_0 p_\alpha e\|^2 = \int_R \lambda^2 <\chi(\lambda)e, e> d\mu(\lambda), \text{ for all } e, f \in \Phi.$$

On the other hand

$$(F(\sigma)p_\alpha Ae, p_\alpha f) = \int_\sigma <\chi(\lambda)Ae, f> d\mu(\lambda), \text{ for all } e, f \in \Phi .$$

Since $\chi(\lambda)Ae$ is $B(\Phi_\alpha, C')$-valued function, where C denote the space of complex numbers, by Theorem 1, $\chi(\lambda)Ae$ is μ-a. e. uniquely determined. First we show that

$$(F(\sigma)A_0 p_\alpha e, p_\alpha f) = \int_\sigma \lambda <\chi(\lambda)e, f> d\mu(\lambda), \text{ for } e, f \in \Phi ,$$

from which, since $A_0 p_\alpha$ is equal to $p_\alpha A$ on Φ with our identification, it follows that

$$\int_\sigma <\chi(\lambda)Ae - \lambda\chi(\lambda)e, f> d\mu(\lambda) = 0, \text{ for any } f \in \Phi \text{ and } \sigma \in \sum ;$$

consequently

$$\chi(\lambda)Ae = \lambda\chi(\lambda)e, \quad \mu\text{-a. e} .$$

Let the spectral measure $\{E(\sigma)\}$ on a Hilbert space K be the Neumark extension of $\{F(\sigma)\}$ [1]. For

$$h = \sum_i E(\sigma_i)p_\alpha e_i, \quad e_i \in \Phi$$

put

$$h(\lambda) = \sum_i \varphi_{\sigma_i}(\lambda)e_i ,$$

where φ_{σ_i} denote the characteristic function of σ_i. We define an anti-linear functional $\varphi(h)$ on a dense subset of K which maps h to

$$\int_R \lambda <\chi(\lambda)e, h(\lambda)> d\mu(\lambda) .$$

Then

$$\left|\int_R \lambda <\chi(\lambda)e, \sum_i \varphi_{\sigma_i}(\lambda)e_i> d\mu(\lambda)\right|$$

$$\leqq \left|\int_R |\lambda| <\chi(\lambda)e, e>^{1/2} <\chi(\lambda) \sum_i \varphi_{\sigma_i} e_i, \sum_i \varphi_{\sigma_i} e_i>^{1/2} d\mu(\lambda)\right|$$

$$\leqq \left\{\int_R \lambda^2 <\chi(\lambda)e, e> d\mu(\lambda)\right\}^{1/2} \left\{\sum_{i,j} \int_{\sigma_i \cap \sigma_j} <\chi(\lambda)e_i, e_j> d\mu(\lambda)\right\}^{1/2},$$

since $p_\alpha^* E(\sigma) p_\alpha = p_\alpha^* F(\sigma) p_\alpha$ and by (9)

$$= \|A_0 p_\alpha e\| \left(\sum_{i,j} (E(\sigma_i \cap \sigma_j) p_\alpha e_i, p_\alpha e_j)\right)^{1/2}$$

$$= \|A_0 p_\alpha e\| \left(\sum_i E(\sigma_i) p_\alpha e_i, \sum_j E(\sigma_j) p_\alpha e_j\right)^{1/2}$$

$$= \|A_0 p_\alpha e\| \|h\|_K .$$

This show that φ is continuous, so we extend it on all of K. There exists h^* of K such that

$$(h^*, h)_K = \int_R \lambda <\chi(\lambda)e, h(\lambda)> d\mu(\lambda) .$$

In particular, putting $h = p_\alpha f$, by (9)

$$(h^*, p_\alpha f)_K = \int_R \lambda <\chi(\lambda)e, f> d\mu(\lambda) = (A_0 p_\alpha e, p_\alpha f),\ f \in \Phi .$$

Thus $Ph^* = A_0 p_\alpha e$, where P denotes the projection from K to H. Since by the above inequality $\|h^*\| \leqq \|A_0 p_\alpha e\|$, $h^* = A_0 p_\alpha e$, which was to be proved.

From the separability of Φ_α, it follows that

$$\chi(\lambda) A = \lambda \chi(\lambda) \quad \mu\text{-a. e} .$$

Since $\chi(\lambda)$ are positive by (5) and λ is real, for any e and f of Φ,

$$<\chi(\lambda)f, Ae> = \overline{<\chi(\lambda)Ae, f>} = \overline{<\lambda\chi(\lambda)e, f>} = <\lambda\chi(\lambda)f, e> .$$

Here we note that the above relation is valid for all $e \in \Phi, f \in \Phi_\alpha$ and $< \ , \ >_\alpha$.

Hence

$$A\,\mathcal{X}(\lambda)=\lambda\mathcal{X}(\lambda)\quad \mu\text{-a. e}.$$

Next we shall prove the rest part of the theorem. By Theorem 3,

$$(p_\alpha e, p_\alpha f)=\int_R <\mathcal{X}(\lambda)e, f>_\alpha d\mu(\lambda),\ e, f\in\Phi_\alpha .$$

Let $\{e_n\}$ be an orthonormal basis in Φ_α, then following value is finite by the assumption that p_α is of Hilbert-Schmidt type.

$$\sum_n \|p_\alpha e_n\|^2=\sum_n (p_\alpha e_n, p_\alpha e_n)=\sum_n \int_R <\mathcal{X}(\lambda)e_n, e_n>_\alpha d\mu(\lambda).$$

By Beppo-Levi's theorem

$$\sum_n <\mathcal{X}(\lambda)e_n, e_n>_\alpha \quad <\infty,\quad \mu\text{-a. e}.$$

Identifying Φ_α' to Φ_α, we regard $\mathcal{X}(\lambda)$ as the bounded positive operator on Φ_α. Above relation shows that the square root of $\mathcal{X}(\lambda)$ is of Hilbert-Schmidt type, i. e.,

$$\sum_n \|\mathcal{X}(\lambda)^{1/2}e_n\|_\alpha^2<\infty .$$

We denote its spectral decomposition by

$$\mathcal{X}(\lambda)e=\sum_n t_n(\lambda)(e_n(\lambda), e)_\alpha e_n(\lambda),$$

where eigenvectors are orthonormal in Φ_α, and $\sum_n t_n(\lambda)<\infty$. When we regard $e_n(\lambda)$ as the element of Φ_α', we denote it by $e_n^*(\lambda)$. Thus the relation $A'e_n^*(\lambda)=\lambda e_n^*(\lambda)$ can be got by putting e by $e_n(\lambda)$ in the formula $A'\mathcal{X}(\lambda)e=\lambda\mathcal{X}(\lambda)e$.

When Φ_α is a Banach space and p_α is nuclear, by Theorem 2, this theorem also holds: the only exception concerns the last description; in this case the space Φ_α' must be replaced by Φ_β' of Theorem 2.

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