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SOME REMARKS ON SEMI-GROUPS OF NONLINEAR OPERATORS

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1. Let X be a Banach space, and let X_0 be a subset of X. By a *contraction semi-group* on X_0 we mean a family $\{T(t);\ t\geqq 0\}$ of operators from X_0 into X_0 satisfying the following conditions:

$$(1.1)\qquad T(0)=I \text{ (the identity)},\ T(t+s)=T(t)T(s) \quad \text{for } t,s\geqq 0;$$

$$(1.2)\qquad \|T(t)x-T(t)y\|\leqq\|x-y\| \quad \text{for } t\geqq 0 \text{ and } x,y\in X_0;$$

$$(1.3)\qquad \lim_{t\to 0+} T(t)x=x \quad \text{for } x\in X_0.$$

We define the *infinitesimal generator* A_0 of $\{T(t);t\geqq 0\}$ by $A_0x=\lim_{h\to 0+} h^{-1}(T(h)x-x)$ and the *weak infinitesimal generator* A' by $A'x=\text{w-}\lim_{h\to 0+} h^{-1}(T(h)x-x)$ whenever the right sides exist.

We shall deal with multi-valued operators. By a multi-valued operator A in X we mean that A assigns to each $x\in D(A)$ a subset $Ax\neq\emptyset$ of X, where $D(A)=\{x\in X;\ Ax\neq\emptyset\}$. And $D(A)$ is called the domain of A, and the range of A is defined by $R(A)=\bigcup_{x\in D(A)} Ax$. We define $|||Ax|||=\inf\{\|x'\|;\ x\in Ax\}$ for $x\in D(A)$ and $A^0x=\{x'\in Ax;\ \|x\|=|||Ax|||\}$. A^0 is called the *canonical restriction* of A. A multi-valued operator A in X is said to be closed, if the graph $G(A)=\bigcup_{x\in D(A)}[x,Ax]$ is closed in the product space $X\times X$ where $[x,Ax]=\{[x,x']\in X\times X;\ x'\in Ax\}$ for $x\in D(A)$.

We now introduce the notion of dissipativity. Let X^* be the dual space of X and (x,x^*) denote the value of $x^*\in X^*$ at $x\in X$. A multi-valued operator A in X is said to be *dissipative* if for each $x,y\in D(A)$ and $x'\in Ax$, $y'\in Ay$ there exists a $\zeta^*\in F(x-y)$ such that

$$(1.4)\qquad \mathrm{Re}(x'-y',\zeta^*)\leqq 0,$$

where $F(x)=\{x^*\in X^*;\ (x,x^*)=\|x\|^2=\|x^*\|^2\}$ for $x\in X$ and Re (x,x^*) means the real part of (x, x^*). It is known that A is dissipative if and only if

$$\|x-y-\lambda(x'-y')\| \geqq \|x-y\| \tag{1.5}$$

for $\lambda>0$, $x, y \in D(A)$ and $x' \in Ax$, $y' \in Ay$ (see[5]).

Recently Crandall and Liggett [3] proved the following

THEOREM A. *If A is a dissipative operator satisfying*

$$(c_1) \qquad R(I-\lambda A) \supset D(A) \quad \textit{for} \quad \lambda>0,$$

then there exists a contraction semi-group $\{T(t);\ t \geqq 0\}$ on $\overline{D(A)}$ such that for each $x \in R \cap \overline{D(A)}$

$$T(t)x = \lim_{\lambda \to 0+} (I-\lambda A)^{-[t/\lambda]} x \tag{1.6}$$

uniformly on every bounded interval of $[0, \infty)$, and

$$\|T(t)x - T(s)x\| \leqq |||Ax|||\ |t-s| \quad \textit{for } x \in D(A) \textit{ and } t, s \geqq 0, \tag{1.7}$$

where $R = \bigcap_{\lambda>0} R(I-\lambda A)$ and $[\quad]$ denotes the Gaussian bracket.

In Section 2 we shall prove the following

THEOREM 1. *In addition to the assumption of* Theorem A, *suppose that A is closed. Let $\{T(t);\ t \geqq 0\}$ be the contraction semi-group on $\overline{D(A)}$ given by* Theorem A. *If $x \in \overline{D(A)}$ and if $T(t)x$ is strongly differentiable at $t_0>0$, then*

$$T(t_0)x \in D(A) \textit{ and } [(d/dt)T(t)x]_{t=t_0} \in AT(t_0)x.$$

This theorem has been proved in [3] under the condition

$$(c_2) \qquad R(I-\lambda A) \supset \mathrm{co}\, D(A) \quad \text{for} \quad \lambda>0,$$

where $\mathrm{co}\, D(A)$ denotes the convex hull of $D(A)$.

The proof of Theorem 1 is based on Lemma 1. By using the same lemma we have the following

THEOREM 3. *Let A be maximal dissipative in $\overline{D(A)}$ satisfying* (c_1), *and let $\{T(t);\ t \geqq 0\}$ be the contraction semi-group on $\overline{D(A)}$ given by* Theorem A. *Assume that A^0 is single valued.*

(i′) *If X is reflexive, then $D(A^0)=D(A)$, A^0 is the weak infinitesimal generator of $\{T(t)\,;\, t\geqq 0\}$ on $\overline{D(A)}$ and*

$$(\text{w-}D^+)T(t)x = A^0T(t)x \quad \textit{for} \quad x\in D(A) \quad \textit{and} \quad t\geqq 0.$$

(ii′) *If X is uniformly convex, then $D(A^0)=D(A)$, A^0 is the infinitesimal generator of $\{T(t)\,;\, t\geqq 0\}$ on $\overline{D(A)}$ and*

$$D^+T(t)x = A^0T(t)x \quad \textit{for} \quad x\in D(A) \quad \textit{and} \quad t\geqq 0.$$

Here $D^+T(t)x$ (or $(\text{w-}D^+)T(t)x$) denotes the strong (or weak) right derivative of $T(t)x$.

And it follows from Theorem 3 that if X and X^* are uniformly convex and if A is closed dissipative satisfying (c_1), then A^0 is single valued with $D(A^0)=D(A)$ and it is the infinitesimal generator of a unique contraction semi-group on $\overline{D(A)}$ (Corollary 2).

In Section 3 we shall deal with approximation of contraction semi-groups. And we may obtain the following

THEOREM 4. *Let $\{T(t)\,;\, t\geqq 0\}$ be a contraction semi-group on a closed convex set X_0, and put $E=\{x\in X_0\,;\, \|A^hx\|=O(1)$ as $h\to 0+\}$, where $A^h=h^{-1}(T(h)-I)$. Then for each $x\in\overline{E}$*

$$T(t)x = \lim_{(\lambda,h)\to(0,0)} (I-\lambda A^h)^{-[t/\lambda]}x \tag{1.8}$$

uniformly on every bounded interval of $[0,\infty)$.

For $\{T(t)\,;\, t\geqq 0\}$ in Theorem 4, we have also that for each $x\in\overline{E}$

$$T(t)x = \lim_{n\to\infty}\{(1-t)I + t\,T(1/n)\}^n x \tag{1.9}$$

uniformly in $t\in[0,1]$ (Corollary 3).

Theorem 4 is somewhat sharper than a theorem due to Neuberger [9], and (1.9) is well known in linear case (see [4, Theorem 10.4.3]).

REMARK. Theorem A can be extended to the following form (see [3]). If $A-\omega I$ is dissipative for some $\omega\geqq 0$ and $R(I-\lambda A)\supset D(A)$ for $\lambda\in(0,1/\omega)$, then there exists a semi-group $\{T(t)\,;\, t\geqq 0\}\in Q_\omega(\overline{D(A)})$ satisfying (1.6) with $R=\bigcap_{\lambda\in(0,1/\omega)} R(I-\lambda A)$ and

(1.7) $$\|T(t)x-T(s)x\| \leqq e^{\omega \max(t,s)} |||Ax||| \, |t-s|$$

for $x \in D(A)$ and $t, s \geqq 0$. And the results mentioned in Section 2 may be also extended to this type. Here by $\{T(t)\,;\, t \geqq 0\} \in Q_\omega(X_0)$ we mean that $T(t)$, $t \geqq 0$ are operators from X_0 into itself with the properties (1.1), (1.3) and

(1.2′) $$\|T(t)x-T(t)y\| \leqq e^{\omega t}\|x-y\| \text{ for } t \geqq 0 \text{ and } x, y \in X_0.$$

Our results in Section 3 also hold true for semi-groups of class $Q_\omega(X_0)$.

2. We define $< , >_s : X \times X \to (-\infty, \infty)$ by

$$<x, y>_s = \sup\{\mathrm{Re}(x, y^*)\,;\, y^* \in F(y)\}.$$

It is shown that $|<x, y>_s| \leqq \|x\|\,\|y\|$ and

(2.1) $< , >_s : X \times X \to (-\infty, \infty)$ is upper semicontinuous (see [3, Lemma 2.16]).

Let A be dissipative satisfying the condition

(c_1') $$R(I-\lambda A) \supset \overline{D(A)} \quad \text{for} \quad \lambda > 0.$$

Since A is dissipative one can define for each $\lambda > 0$ a single valued operator $J_\lambda = (I-\lambda A)^{-1} : R(I-\lambda A) \to D(A)$ such that

$$\|J_\lambda x - J_\lambda y\| \leqq \|x-y\| \quad \text{for} \quad x, y \in R(I-\lambda A).$$

We set $A_\lambda = \lambda^{-1}(J_\lambda - I)$ for $\lambda > 0$. The following properties of A_λ are well known:

(2.2) $$A_\lambda x \in AJ_\lambda x \quad \text{for} \quad x \in R(I-\lambda A);$$

(2.3) $$\|A_\lambda x\| \leqq |||Ax||| \quad \text{for} \quad x \in D(A) \quad \text{and} \quad \lambda > 0.$$

Theorem A shows that

(2.4) $$\lim_{\lambda \to 0+} J_\lambda^{[t/\lambda]} x \text{ exists for } x \in \overline{D(A)} \text{ and } t \geqq 0,$$

and if $T(t)x$ is defined as the limit in (2.4) then $\{T(t)\,;\, t \geqq 0\}$ is a contraction semi-group on $\overline{D(A)}$ satisfying

(2.5) $$\|T(t)x-T(s)x\| \leqq |||Ax||| \, |t-s| \quad \text{for} \quad x \in D(A) \quad \text{and} \quad t, s \geqq 0.$$

LEMMA 1. *Let A be dissipative satisfying* (c_1') and let $\{T(t)\,;\, t\geqq 0\}$ *be the contraction semi-group on* $\overline{D(A)}$ *defined by the limit in* (2.4). *If* $x\in\overline{D(A)}$ *and* $y_0\in Ax_0$, *then*

$$\sup_{\zeta^*\in F(x-x_0)}\ \limsup_{t\to 0+}\ \mathrm{Re}\left(\frac{T(t)x-x}{t}, \zeta^*\right)\leqq \langle y_0, x-x_0\rangle_s. \tag{2.6}$$

PROOF. Since $\|J_\lambda^{[t/\lambda]}x_0-x_0\|\leqq[t/\lambda]\|J_\lambda x_0-x_0\|\leqq t\,|||Ax_0|||$, we have

$$\|J_\lambda^{[t/\lambda]}x-x_0\|\leqq\|J_\lambda^{[t/\lambda]}x-J_\lambda^{[t/\lambda]}x_0\|+\|J_\lambda^{[t/\lambda]}x_0-x_0\|\leqq\|x-x_0\|+t\,|||Ax_0||| \tag{2.7}$$

for $\lambda>0$ and $t\geqq 0$. For each $\lambda>0$ and positive integer k,

$$y_{\lambda,k}\equiv\lambda^{-1}(J_\lambda^k x-J_\lambda^{k-1}x)=A_\lambda J_\lambda^{k-1}x\in AJ_\lambda^k x$$

by (2.2). Since A is dissipative, there is an $\eta^*\in F(J_\lambda^k x-x_0)$ such that

$$\mathrm{Re}(y_{\lambda,k}-y_0, \eta^*)\leqq 0. \tag{2.8}$$

Now

$$\begin{aligned}\mathrm{Re}(y_{\lambda,k}, \eta^*)&=\lambda^{-1}\,\mathrm{Re}(J_\lambda^k x-x_0-\{J_\lambda^{k-1}x-x_0\}, \eta^*)\\ &\geqq\lambda^{-1}(\|J_\lambda^k x-x_0\|^2-\|J_\lambda^{k-1}x-x_0\|\,\|J_\lambda^k x-x_0\|)\\ &\geqq(2\lambda)^{-1}(\|J_\lambda^k x-x_0\|^2-\|J_\lambda^{k-1}x-x_0\|^2);\end{aligned}$$

and hence

$$\begin{aligned}\|J_\lambda^k x-x_0\|^2-\|J_\lambda^{k-1}x-x_0\|^2&\leqq 2\lambda\,\mathrm{Re}(y_{\lambda,k}, \eta^*)\\ &=2\lambda\,\mathrm{Re}(y_{\lambda,k}-y_0, \eta^*)+2\lambda\,\mathrm{Re}(y_0, \eta^*)\\ &\leqq 2\lambda\,\mathrm{Re}(y_0, \eta^*)\quad(\text{by}(2.8))\\ &\leqq 2\lambda\langle y_0, J_\lambda^k x-x_0\rangle_s.\end{aligned}$$

Since $J_\lambda^{[\tau/\lambda]}x=J_\lambda^k x$ for $k\lambda\leqq\tau<(k+1)\lambda$,

$$\begin{aligned}&\|J_\lambda^k x-x_0\|^2-\|J_\lambda^{k-1}x-x_0\|^2\\ &\leqq 2\int_{k\lambda}^{(k+1)\lambda}\langle y_0, J_\lambda^{[\tau/\lambda]}x-x_0\rangle_s\,d\tau.\end{aligned} \tag{2.9}$$

Let $t\geqq\lambda$ and add (2.9) for $k=1, 2, \cdots, [t/\lambda]$. Then we have

$$\|J_\lambda^{[t/\lambda]}x - x_0\|^2 - \|x - x_0\|^2$$

$$\leqq 2\int_\lambda^{([t/\lambda]+1)\lambda} < y_0, J_\lambda^{[\tau/\lambda]}x - x_0 >_s d\tau.$$

Taking the lim sup as $\lambda \to 0+$ we have from (2.7), (2.1) and the Lebesgue convergence theorem that for $t \geqq 0$

$$\|T(t)x - x_0\|^2 - \|x - x_0\|^2 \leqq 2\int_0^t < y_0, T(\tau)x - x_0 >_s d\tau. \tag{2.10}$$

Since $\|T(t)x - x_0\|^2 - \|x - x_0\|^2 \geqq 2\,\mathrm{Re}(T(t)x - x, \zeta^*)$ for any $\zeta^* \in F(x - x_0)$, (2.10) yields

$$\mathrm{Re}(T(t)x - x, \zeta^*) \leqq \int_0^t < y_0, T(\tau)x - x_0 >_s d\tau \tag{2.11}$$

for $t \geqq 0$.

In view of (2.1) and the strong continuity of $T(\tau)x$ in $\tau \geqq 0$, $< y_0, T(\tau)x - x_0 >_s$ is upper semicontinuous in $\tau \geqq 0$. Thus for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$< y_0, T(\tau)x - x_0 >_s < < y_0, x - x_0 >_s + \varepsilon \text{ for } 0 \leqq \tau < \delta.$$

It follows from (2.11) that if $0 < t < \delta$ then

$$\mathrm{Re}\left(\frac{T(t)x - x}{t}, \zeta^*\right) \leqq < y_0, x - x_0 >_s + \varepsilon.$$

Consequently

$$\limsup_{t \to 0+} \mathrm{Re}\left(\frac{T(t)x - x}{t}, \zeta^*\right) \leqq < y_0, x - x_0 >_s$$

for any $\zeta^* \in F(x - x_0)$. This completes the proof.

PROOF OF THEOREM 1. We note that under assumptions of Theorem 1, (2.6) in Lemma 1 holds true. In fact, $R(I - \lambda A)$ is closed for each $\lambda > 0$ because A is closed; and hence (c_1) implies (c_1'). Then, by using the same method as in the proof of Theorem II in [3], we obtain the conclusion. Q. E. D.

REMARK. In [3], the condition (c_2) has been used only to prove Lemma 1 above.

Let A_i, $i=1, 2$ be multi-valued operators in X. A_2 is an extension of A_1, and A_1 is a restriction of A_2, in symbol $A_2 \supset A_1$, $A_1 \subset A_2$, if $D(A_1) \subset D(A_2)$ and $A_1 x \subset A_2 x$ for $x \in D(A_1)$. If S is a subset of X and A is a dissipative operator, we say that A is *maximal dissipative* in S if $D(A) \subset S$ and A has not any proper dissipative extension $\widetilde{A}$ such that $D(\widetilde{A}) \subset S$. Lemma 1 leads to the following

COROLLARY 1. *Let A be maximal dissipative in $\overline{D(A)}$ satisfying* (c_1), *and let $\{T(t)\,;\, t \geqq 0\}$ be the contraction semi-group on $\overline{D(A)}$ defined by the limit in* (2.4). (*Note that* (c_1') *is satisfied, since the maximality of A implies that A is closed.*)

(i) *If $x \in \overline{D(A)}$ and if $x' = \text{w-}\lim_{t_n \to 0+} t_n^{-1}(T(t_n)x - x)$, then $x \in D(A^0)$, $x \in A^0 x$ and*

$$\lim_{t_n \to 0+} t_n^{-1} \|T(t_n)x - x\| = \|x\| = |||Ax|||\,.$$

(ii) *If X is reflexive, then*

$$\{x \in \overline{D(A)}\,;\, \|T(t)x - x\| = O(t) \text{ as } t \to 0+\} = D(A) = D(A^0)$$

and for each x belonging to the set above

$$\lim_{t \to 0+} t^{-1} \|T(t)x - x\| = |||Ax|||\,.$$

PROOF. (i) We first note that there is a $y^* \in F(y)$ such that $\langle x, y \rangle_s = \text{Re}(x, y^*)$ since $F(y)$ is compact in the weak* topology of X^*.

Let $x_0 \in D(A)$ and let $y_0 \in Ax_0$. By Lemma 1,

$$\sup_{\zeta^* \in F(x - x_0)} \text{Re}(x, \zeta^*) \leqq \text{Re}(y_0, \eta^*) \text{ for some } \eta^* \in F(x - x_0)\,.$$

So that

$$\text{Re}(x' - y_0, \eta^*) \leqq 0 \quad \text{for some } \eta^* \in F(x - x_0)\,.$$

The maximal dissipativity of A implies that $x \in D(A)$ and $x' \in Ax$ (see [6, Lemma 3.4]). But, by (2.5), $\|T(t_n)x - x\| \leqq |||Ax|||\, t_n$. And hence

$$|||Ax||| \leqq \|x'\| \leqq \liminf_{t_n \to 0+} t_n^{-1} \|T(t_n)x - x\|$$

$$\leqq \limsup_{t_n \to 0+} t_n^{-1} \|T(t_n)x - x\| \leqq |||Ax|||.$$

Thus $\lim_{t_n \to 0+} t_n^{-1}(\|T(t_n)x - x\|) = \|x\| = |||Ax|||$, $x \in D(A^0)$ and $x' \in A^0 x$.

(ii) Clearly $\{x \in \overline{D(A)}; \|T(t)x - x\| = O(t)$ as $t \to 0+\} \supset D(A) \supset D(A^0)$ by (2.5). Let $x \in \overline{D(A)}$ and let $\|T(t)x - x\| = O(t)$ as $t \to 0+$. It follows from the reflexivity of X that every sequence $\{t_n\}$, $t_n \to 0+$ has a subsequence $\{t_{n_i}\}$ such that $\{t_{n_i}^{-1}(T(t_{n_i})x - x)\}$ is weakly convergent. Therefore, by (i), $x \in D(A^0)$ and $\lim_{t_{n_i} \to 0+} t_{n_i}^{-1} \|T(t_{n_i})x - x\| = |||Ax|||$. And the uniqueness of the limit shows

$$\lim_{t \to 0+} t^{-1} \|T(t)x - x\| = |||Ax|||.$$

Q. E. D.

Let us now consider the Cauchy problem

(2.12) $$(d/dt)u(t) \in Au(t) \text{ a. e. } t \in [0, \infty), \quad u(0) = x$$

where A is a given dissipative operator. A single valued mapping $u(t): [0, \infty) \to X$ is called a *solution* of (2.12) if $u(t)$ is Lipschitz continuous in $t \geqq 0$, $u(t)$ is strongly differentiable at a. e. $t \geqq 0$, $u(t) \in D(A)$ for a. e. $t \in [0, \infty)$ and $u(t)$ satisfies (2.12). It follows from the dissipativity of A that (2.12) has at most one solution (for example, see the proof of Theorem 3 in [8]).

In view of Theorem 1 we have the following

THEOREM 2. *Let A be closed dissipative satisfying* (c_1), *and let* $\{T(t); t \geqq 0\}$ *be the contraction semi-group on* $\overline{D(A)}$ *given by* Theorem A.

(a) *If $T(t)x$ with $x \in D(A)$ is strongly differentiable at a. e. $t \in [0, \infty)$ then it is a unique solution of* (2.12).

(b) *If X is reflexive then for each $x \in D(A)$ $T(t)x$ is a unique solution of* (2.12).

PROOF. By (1.7), $T(t)x$ is Lipschitz continuous in $t \geqq 0$ if $x \in D(A)$. Therefore (a) follows from Theorem 1. If X is reflexive, then every Lipschitz continuous X-valued function in $t \geqq 0$ is strongly differentiable at a. e. $t \in [0, \infty)$ (see [7, Appendix]). Hence (b) is obtained. Q. E. D.

REMARK. Let A be dissipative satisfying the condition (c_1), and let $x \in D(A)$. It has been proved by Brezis and Pazy [1] that if $u(t)$ is a solution of (2.12) then $u(t)=\lim_{\lambda\to 0+}(I-\lambda A)^{-[t/\lambda]}x$ uniformly on every bounded interval of $[0, \infty)$ and $(d/dt)u(t) \in A^0 u(t)$ for a. e. $t \in [0, \infty)$.

Theorem 2 (b) shows that if X is reflexive and if A is closed dissipative satisfying (c_1), then $\{T(t)\,;\, t \geqq 0\}$ difined by

$$T(t)x = \lim_{\lambda\to 0+}(I-\lambda A)^{-[t/\lambda]}x \text{ for } x \in \overline{D(A)} \text{ and } t \geqq 0 \tag{2.13}$$

is a unique contraction semi-group on $\overline{D(A)}$ such that for each $x \in D(A)$,

$$\|T(t)x-T(s)x\| \leqq |||Ax||| \, |t-s| \quad \text{for} \quad t, s \geqq 0$$

and

$$(d/dt)T(t)x \in AT(t)x \quad \text{a. e. } t \in [0, \infty)$$

(and hence $(d/dt)T(t)x \in A^0T(t)x$ a. e. $t \in [0, \infty)$).

Our next problem is to find the infinitesimal generator of this semi-group.

THEOREM 3. *Let A be maximal dissipative in $\overline{D(A)}$ satisfying (c_1), and let $\{T(t)\,;\, t \geqq 0\}$ be the contraction semi-group on $\overline{D(A)}$ defined by* (2.13). *Assume that A^0 is single valued. Then we have*

(i′) *if X is reflexive, then $D(A^0)=D(A)$, A^0 is the weak infinitesimal generator of $\{T(t)\,;\, t \geqq 0\}$ on $\overline{D(A)}$ and*

$$(\text{w-}D^+)T(t)x = A^0T(t)x \quad \textit{for} \quad x \in D(A) \quad \textit{and} \quad t \geqq 0, \tag{2.14}$$

(ii) *if X is uniformly convex, then $D(A^0)=D(A)$, A^0 is the infinitesimal generator of $\{T(t)\,;\, t \geqq 0\}$ on $\overline{D(A)}$ and*

$$D^+T(t)x = A^0T(t)x \quad \textit{for} \quad x \in D(A) \quad \textit{and} \quad t \geqq 0. \tag{2.15}$$

PROOF. (i) We proved already

$$\{x \in \overline{D(A)}\,;\, \|T(h)x-x\| = O(h) \text{ as } h \to 0+\} = D(A) = D(A^0)$$

(Corollary 1 (ii)). From this and $\|T(h)T(t)x-T(t)x\| \leqq \|T(h)x-x\|$, $T(t)x \in D(A^0)$ for $x \in D(A^0)$ and $t \geqq 0$. Thus it suffices to show that

$$\text{w-}\lim_{t\to 0+} t^{-1}(T(t)x - x) = A^0x \text{ for } x \in D(A^0)\,. \tag{2.16}$$

Let $x \in D(A^0)$ and let $\{t_n\}$ be a sequence such that $t_n \to 0+$ as $n \to \infty$. Since X is reflexive, there are an $x \in X$ and a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$x' = \operatorname*{w-lim}_{n_i \to \infty} t_{n_i}^{-1}(T(t_{n_i})x - x)\,.$$

By Corollary 1(i),

$$A^0 x = x = \operatorname*{w-lim}_{n_i \to \infty} t_{n_i}^{-1}(T(t_{n_i})x - x)\,.$$

From the uniqueness of the limit (2.16) follows.

(ii′) Since X is also reflexive, the conclusions in (i′) hold true. It is sufficient to show

$$\lim_{t \to 0+} t^{-1}(T(t)x - x) = A^0 x \quad \text{for} \quad x \in D(A^0)\,.$$

But this is obtained from the uniform convexity of X, (2.16) and $\lim_{t \to 0+} t^{-1}\|T(t)x - x\| = \|A^0 x\|$ for $x \in D(A^0)$ (Corollary 1 (ii)). Q. E. D.

COROLLARY 2. *Let X and X^* be uniformly convex, and let A be closed dissipative satisfying* (c_1). *Then*

(a) *A^0 is single valued with $D(A^0) = D(A)$,*

(b) *A^0 is the infinitesimal generator of a unique contraction semi-group on $\overline{D(A)}$.*

PROOF. Let $\widetilde{A}$ be a maximal dissipative operator in $\overline{D(A)}$ such that $\widetilde{A} \supset A$. Note that $\widetilde{A}^0$ is single valued with $D(\widetilde{A}^0) = D(\widetilde{A})$ (see [6, Lemma 3.10]). Since $D(A) \subset D(\widetilde{A}) \subset \overline{D(A)}$ and $R(I - \lambda A) \supset \overline{D(A)}$ for $\lambda > 0$, we have

$$R(I - \lambda \widetilde{A}) \supset \overline{D(\widetilde{A})} (= \overline{D(A)}) \quad \text{for} \quad \lambda > 0\,.$$

Put

$$T(t)x = \lim_{\lambda \to 0+} (I - \lambda \widetilde{A})^{-[t/\lambda]} x (= \lim_{\lambda \to 0+} (I - \lambda A)^{-[t/\lambda]} x)$$

for $x \in \overline{D(A)}$ and $t \geqq 0$. By Theorem 3 (ii), $\{T(t)\,;\, t \geqq 0\}$ is a unique contraction semi-group on $\overline{D(A)}$ with the infinitesimal generator $\widetilde{A}^0$.

On the other hand it is shown that $D(\widetilde{A}) = D(A) = D(A^0)$ and $\widetilde{A}^0 = A^0$ (see [10, Proposition 4.2]). This completes the proof.

3. Throughout this section it is assumed that X_0 is a closed convex subset of X. We start from the following

LEMMA 2. *Suppose that C is a contraction from X_0 into itself (i. e., $\|Cx-Cy\| \leqq \|x-y\|$ for $x, y \in X_0$), and put $C^h = h^{-1}(C-I)$ for $h > 0$.*

(i) *There exists a unique contraction semi-group $\{T(t;\, C-I);\, t \geqq 0\}$ on X_0 such that $(d/dt)\, T(t;\, C-I)x = (C-I)T(t;\, C-I)x$ for $x \in X_0$, $t \geqq 0$ and*

$$(3.1) \qquad \|T(m;\, C-I)x - C^m x\| \leqq \sqrt{m}\, \|(C-I)x\|$$

for $x \in X_0$, $m = 1, 2, \cdots$.

(ii) *For each $h > 0$ there exists a unique contraction semi-group $\{T(t;\, C^h);\, t \geqq 0\}$ on X_0 such that $(d/dt)T(t;\, C^h)x = C^h T(t;\, C^h)x$ for $x \in X_0$, $t \geqq 0$ and*

$$(3.2) \qquad \|T(t;\, C^h)x - C^{[t/h]}x\| \leqq (\sqrt{th} + h)\|C^h x\|$$

for $x \in X_0$, $t \geqq 0$.

PROOF. For the proof of (i), refer to [1, Lemma 2.4] or [8, Appendix]. (ii) is easily obtained from (i). In fact, $T(t;\, C^h) = T(t/h;\, C-I)$. By (3.1)

$$\|T([t/h]h;\, C^h)x - C^{[t/h]}x\| \leqq \sqrt{th}\, \|C^h x\|$$

for $t \geqq 0$, $x \in X_0$. Moreover

$$\|T(t;\, C^h)x - T([t/h]h;\, C^h)x\| \leqq \int_{[t/h]h}^{t} \|C^h T(s;\, C^h)x\|\, ds$$
$$\leqq h\|C^h x\| \quad \text{for} \quad t \geqq 0, x \in X_0\,.$$

From these inequalities (3.2) follows. Q. E. D.

Let $\{T(t);\, t \geqq 0\}$ be a contraction semi-group on X_0, and set

$$A^h = h^{-1}(T(h) - I) \quad \text{for} \quad h > 0\,.$$

Using Lemma 2 (ii) with $C = T(h)$ and $C^h = A^h$, we see that there is a unique contraction semi-group $\{T(t;\, A^h);\, t \geqq 0\}$ on X_0 such that

$$(3.3) \qquad (d/dt)T(t;\, A^h)x = A^h T(t;\, A^h)x \quad \text{for} \quad x \in X_0,\ t \geqq 0$$

and

$$\|T(t\,;\,A^h)x - T([t/h]h)x\| \leqq (\sqrt{th}+h)\|A^h x\| \tag{3.4}$$

for $x \in X_0$, $t \geqq 0$.

Next for any fixed $\xi \in [0, 1]$ and $h > 0$ we define $C(\xi, h)$ by

$$C(\xi, h) = \xi T(h) + (1-\xi)I.$$

Obviously $C(\xi, h)$ is a contraction from X_0 into itself. Put

$$A^h(\xi) = h^{-1}(C(\xi, h) - I).$$

In view of Lemma 2 (ii) (with $C = C(\xi, h)$ and $C^h = A^h(\xi)$), there is a unique contraction semi-group $\{T(t\,;\,A^h(\xi))\,;\,t \geqq 0\}$ on X_0 such that $(d/dt)\,T(t\,;\,A^h(\xi))x = A^h(\xi)T(t\,;\,A^h(\xi))x$ for $x \in X_0$, $t \geqq 0$ and

$$\|T(t\,;\,A^h(\xi))x - C(\xi, h)^{[t/h]}x\| \leqq (\sqrt{th}+h)\|A^h(\xi)x\| \tag{3.5}$$

for $x \in X_0$, $t \geqq 0$. Since $A^h(\xi) = \xi A^h$,

$$T(t\,;\,A^h(\xi)) = T(t\xi\,;\,A^h)$$

Combining this with (3.5)

$$\|T(t\xi\,;\,A^h)x - C(\xi, h)^{[t/h]}x\| \leqq (\sqrt{th}+h)\|A^h x\|$$

for $x \in X_0$, $t \geqq 0$. Setting $t = 1$ in the inequality above, we have

$$\|T(\xi\,;\,A^h)x - \{(1-\xi)I + \xi T(h)\}^{[1/h]}x\| \leqq (\sqrt{h}+h)\|A^h x\| \tag{3.6}$$

for $x \in X_0$, $\xi \in [0, 1]$ and $h > 0$.

Since $\|T(t)x - T([t/h]h)x\| \leqq \|T(t-[t/h]h)x - x\| \to 0$ uniformly in $t \geqq 0$, as $h \to 0+$, for any $x \in X_0$, (3.4) and (3.6) show the following

COROLLARY 3. *Set* $E = \{x \in X_0\,;\,\|A^h x\| = O(1)$ *as* $h \to 0+\}$.

(a) *For each* $x \in \overline{E}$

$$T(t)x = \lim_{h \to 0+} T(t\,;\,A^h)x$$

uniformly on every bounded interval of $[0, \infty)$.

(b) *For each* $x \in \overline{E}$

$$T(t)x = \lim_{h \to 0+} \{(1-t)I + tT(h)\}^{[1/h]}x$$

uniformly in $t \in [0, 1]$.

REMARK. Let A_0 be the infinitesimal generator of $\{T(t)\,;\, t \geqq 0\}$. Chambers [2] showed that the above (b) holds true for each $x \in \overline{D}_0$, where D_0 is a subset of $D(A_0)$ such that if $x \in D_0$ then $T(t)x \in D(A_0)$ for a. e. $t \geqq 0$.

It is easily shown that for each $h > 0$, A^h is dissipative and

$$R(I - \lambda A^h) \supset X_0 = D(A^h) \quad \text{for} \quad \lambda > 0. \tag{3.7}$$

We now consider the behavior of $(I - \lambda A^h)^{-[t/\lambda]}x$ as $(\lambda, h) \to (0, 0)$. An estimation by Crandall and Liggett [3, (1.9)] shows that

$$\begin{aligned} &\|(I - \lambda A^h)^{-[t/\lambda]}x - (I - \mu A^h)^{-[t/\mu]}x\| \\ &\leqq 2(\lambda^2 + t(\lambda - \mu))^{1/2}\|A^h x\| \end{aligned} \tag{3.8}$$

for $x \in X_0, t \geqq 0$ and $\lambda > \mu > 0$.

Note that $T(t\,;\, A^h)x = \lim_{\lambda \to 0+}(I - \lambda A^h)^{-[t/\lambda]}x$ for $x \in X_0$ and $t \geqq 0$. (For example, this follows from the remark after Theorem 2 because by (3.3) $T(t\,;\, A^h)x$ with $x \in X_0$ is a solution of the Cauchy problem $(d/dt)u(t) = A^h u(t), u(0) = x$.) Letting $\mu \to 0$ in (3.8), we have

$$\|(I - \lambda A^h)^{-[t/\lambda]}x - T(t\,;\, A^h)x\| \leqq 2(\lambda^2 + \lambda t)^{1/2}\|A^h x\|$$

for $x \in X_0$, $t \geqq 0$ and $\lambda > 0$. Combining this with (3.4),

$$\begin{aligned} &\|T([t/h]h)x - (I - \lambda A^h)^{-[t/\lambda]}x\| \\ &\leqq \{\sqrt{th} + h + 2(\lambda^2 + \lambda t)^{1/2}\}\, \|A^h x\| \end{aligned} \tag{3.9}$$

for $x \in X_0$, $t \geqq 0$, $\lambda > 0$ and $h > 0$.

Thus we obtain the following

THEOREM 4. *Let* $\{T(t)\,;\, t \geqq 0\}$ *be a contraction semi-group on a closed* *convex subset* X_0 *of* X, *and put* $E=\{x\in X_0\,;\ \|A^h x\|=O(1)$ *as* $h\to 0+\}$, *where* $A^h=h^{-1}(T(h)-I)$. *Then for each* $x\in\overline{E}$

$$T(t)x=\lim_{(\lambda,h)\to(0,0)}(I-\lambda A^h)^{-[t/\lambda]}x \tag{3.10}$$

uniformly on every bounded interval of $[0,\infty)$.

Added in Proof. Under the assumptions that X^* is uniformly convex and A is m-dissipative, the conclusion of Corollary 1 (ii) has been obtained by Brezis (On a problem of T. Kato, to appear).

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